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Original Study

Special functions with general kernel: Properties and applications to fractional partial differential equations

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Abstract

In this paper, we reconstruct gamma and beta functions using a general kernel function in their integral representations. We also reconstruct the Gauss and confluent hypergeometric functions using the beta function with general kernel in their series representations. The general kernel function we use here can be chosen as any special function such as the exponential function, Mittag-Leffler function, Wright function, Fox-Wright function, Kummer function or M-series. Using different choices of this general kernel function, various of the generalized gamma, beta, Gauss hypergeometric and confluent hypergeometric functions in literature can be obtained. In this paper, we first obtain the integral representations, functional relations, summation, derivative and transformation formulas and double Laplace transforms of the special functions we construct. Furthermore, we compute the solutions of some fractional partial differential equations involving special functions with general kernel via the double Laplace transform and graph some of the solutions for specific values. Finally, we obtain the incomplete beta function with general kernel by defining the beta distribution with general kernel.

Keywords: Laplace transform, Fractional derivatives and integrals, Fractional partial differential equations, Gamma and Beta functions, Gauss and Confluent hypergeometric functions.
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1 Introduction

Fractional calculus has been developed by many scientists from the past to the present and various fractional integral and derivative operators have been defined, see Baleanu et al. [1], Hilfer [2], Miller and Ross [3], Samko et al. [4], Podlubny [5] and Kilbas et al. [6]. Later, scientists were interested in fractional order differential equations and obtained their solutions by integral transformations, see Tanriverdi et al. [7], Ata and Kıymaz [8], Luchko et al. [9], Ata and Kıymaz [10] and Lin and Lu [11].

Special functions have an important role in many scientific fields such as physics, mathematics and engineering. Some of these special functions are gamma, beta, Gauss hypergeometric and confluent hypergeometric functions and we give these special functions below.

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The gamma function Andrews et al. [12] for $\Re(\sigma) > 0$ is defined by

$$\Gamma(\sigma) = \int_0^\infty \omega^{\sigma-1} \exp(-\omega)\, d\omega.$$

The beta function Andrews et al. [12] for $\Re(\sigma) > 0$ and $\Re(\tau) > 0$ is given by

$$B(\sigma,\tau) = \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1} d\omega.$$

The Gauss hypergeometric function Kilbas et al. [6] for $\Re(\vartheta_3) > \Re(\vartheta_2) > 0$ is defined by

$${}_2F_1(\vartheta_1,\vartheta_2;\vartheta_3;z) = \sum_{k=0}^\infty (\vartheta_1)_k \frac{B(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \frac{z^k}{k!}, \quad \text{for } |z|<1.$$

The confluent hypergeometric function Kilbas et al. [6] for $\Re(\vartheta_3) > \Re(\vartheta_2) > 0$ is given by

$$\Phi(\vartheta_2;\vartheta_3;z) = \sum_{k=0}^\infty \frac{B(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \frac{z^k}{k!}.$$

Here denotes $(\cdot)_k$ is known as the Pochhammer symbol Andrews et al. [12] and defined by

$$(\vartheta)_k = \vartheta(\vartheta+1)\cdots(\vartheta+k-1) \quad \text{and} \quad (\vartheta)_0 \equiv 1.$$

Scientists have obtained various generalizations of these special functions by working on the special functions mentioned above. We let $\Re(p) > 0$, $\Re(q) > 0$, $\Re(\kappa) > 0$, $\Re(\mu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\sigma) > 0$, $\Re(\tau) > 0$, $\Re(\vartheta_3) > \Re(\vartheta_2) > 0$ unless otherwise stated.

The generalized gamma functions were introduced by Parmar [13] and Şahin et al. [14] respectively as follows

$$\Gamma_p^{(\alpha,\beta;\kappa)}(\sigma) = \int_0^\infty \omega^{\sigma-1} {}_1F_1\left(\alpha;\beta;-\omega-\frac{p}{\omega^\kappa}\right) d\omega, \tag{1}$$

$$\Gamma_{p,q}^{(\kappa,\mu)}(\sigma) = \int_0^\infty \omega^{\sigma-1} \exp\left(-\frac{\omega^\kappa}{p}-\frac{q}{\omega^\mu}\right) d\omega. \tag{2}$$

The generalized beta functions were introduced by Khan and Husain [15], Çetinkaya et al. [16] and Şahin et al. [14] respectively as follows

$$B_{\alpha,\beta}^{p,\kappa,\mu}(\sigma,\tau) = \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1} E_{\alpha,\beta}\left(-\frac{p}{\omega^\kappa(1-\omega)^\mu}\right) d\omega, \tag{3}$$

$$B_{p,q}^{(\alpha,\beta;\kappa,\mu)}(\sigma,\tau) = \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{\omega^\kappa}-\frac{q}{(1-\omega)^\mu}\right) d\omega, \tag{4}$$

$$B_{p,q}^{(\kappa,\mu)}(\sigma,\tau) = \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1} \exp\left(-\frac{p}{\omega^\kappa}\right) \exp\left(-\frac{q}{(1-\omega)^\mu}\right) d\omega. \tag{5}$$

The generalized Gauss and confluent hypergeometric functions were introduced by Khan and Husain [15] and Şahin et al. [14] respectively as follows

$$F_{\alpha,\beta}^{p,\kappa,\mu}(\vartheta_1,\vartheta_2;\vartheta_3;z)=\sum_{k=0}^{\infty}(\vartheta_1)_k\frac{B_{\alpha,\beta}^{p,\kappa,\mu}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!},\quad \text{for } |z|<1, \tag{6}$$

$$\Phi_{\alpha,\beta}^{p,\kappa,\mu}(\vartheta_2;\vartheta_3;z)=\sum_{k=0}^{\infty}\frac{B_{\alpha,\beta}^{p,\kappa,\mu}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!} \tag{7}$$

and

$$F_{p,q}^{(\kappa,\mu)}(\vartheta_1,\vartheta_2;\vartheta_3;z)=\sum_{k=0}^{\infty}(\vartheta_1)_k\frac{B_{p,q}^{(\kappa,\mu)}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!},\quad \text{for } |z|<1, \tag{8}$$

$$\Phi_{p,q}^{(\kappa,\mu)}(\vartheta_2;\vartheta_3;z)=\sum_{k=0}^{\infty}\frac{B_{p,q}^{(\kappa,\mu)}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}. \tag{9}$$

Also for other studies that can be found in the specific literature, see Abubakar [17], Abubakar [18], Al-Gonah and Mohammed [19], Ata and Kıymaz [20], Ata [21], Ata and Kıymaz [22], Atash et al. [23], Chaudhry and Zubair [24], Chaudhry et al. [25], Chaudhry et al. [26], Choi et al. [27], Goswami et al. [28], Goyal et al. [29], Kulip et al. [30], Lee et al. [31], Mubeen et al. [32], Özergin et al. [33], Rahman et al. [34], Rahman et al. [35], Shadab et al. [36], Ata and Kıymaz [37], Srivastava et al. [38], Ata [39], Kıymaz et al. [40], Srivastava et al. [41] and Kıymaz et al. [42].

The motivation of this paper is to introduce special functions with general kernel that generate generalized gamma, beta, Gauss hypergeometric and confluent hypergeometric functions and to obtain solutions of fractional partial differential equations involving special functions with general kernel.

The remainder of this paper is organized as follows: In Section 2, we provide the basic information needed throughout the paper. In Section 3, we describe the special functions with general kernel and show that they generate other special functions. In Section 4, we give some properties of the special functions with general kernel. In Section 5, we obtain solutions of fractional partial differential equations involving special functions with general kernel and then we present graphs for some specific values. In Section 6, we give the beta distribution with general kernel and introduce the incomplete beta function with general kernel. Finally, we give conclusions and remarks in Section 7.

2 Preliminaries

The Laplace and inverse Laplace transforms are obtained from the Fourier integral formula in Debnath and Bhatta [43]. They are also very powerful tools for solving ordinary, partial and fractional differential equations. Since the special functions with general kernel that we define in this paper are multi-parameters, it is more convenient to apply the double Laplace transform to them. Therefore, we use this transformation in this paper. In this section, we now give the basic materials needed throughout the paper.

Definition 2.1 (Anwar et al. [44])**.** The partial fractional Caputo derivative is given by

$${}_{q}^{c}D_{0^+}^{\varepsilon_2}\,{}_{p}^{c}D_{0^+}^{\varepsilon_1}f(p,q)=\frac{1}{\Gamma(m-\varepsilon_1)}\frac{1}{\Gamma(n-\varepsilon_2)}\int_0^q\int_0^p(p-x)^{m-\varepsilon_1-1}(q-y)^{n-\varepsilon_2-1}\frac{\partial^{m+n}f(x,y)}{\partial y^n\partial x^m}dxdy,$$

where $m-1<\Re(\varepsilon_1)\le m$, $n-1<\Re(\varepsilon_2)\le n$, $m,n\in\mathbb{N}$.

Definition 2.2 (Debnath [45])**.** The double Laplace and inverse Laplace transforms respectively are defined by

$$\mathfrak{L}_q\mathfrak{L}_p\Big[f(p,q)\Big](s_1,s_2)=\int_0^\infty\int_0^\infty\exp(-s_1p)\exp(-s_2q)f(p,q)dpdq \tag{10}$$

and

$$\mathfrak{L}_q^{-1}\mathfrak{L}_p^{-1}\left[\mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2)\right](p,q) = f(p,q) = \frac{1}{(2\pi i)^2}\int\limits_{c-i\infty}^{c+i\infty}\int\limits_{d-i\infty}^{d+i\infty}\exp(s_1p)\exp(s_2q) \times \mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2)ds_1ds_2, \qquad (11)$$

where $\Re(s_1) \geqq c$ and $\Re(s_2) \geqq d$.

We give the double Laplace transforms of partial fractional Caputo derivatives below.

Theorem 2.1 (Anwar et al. [44])**.** Let $\Re(\varepsilon_1), \Re(\varepsilon_2) > 0$ and $m-1 < \Re(\varepsilon_1) \leq m$, $n-1 < \Re(\varepsilon_2) \leq n$ for $m,n \in \mathbb{N}$. Then, we have

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}_q^cD_{0^+}^{\varepsilon_2}\,{}_p^cD_{0^+}^{\varepsilon_1}f(p,q)\right](s_1,s_2) = s_1^{\varepsilon_1}s_2^{\varepsilon_2}\left(\mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2) - \sum_{i=0}^{m-1}s_1^{-1-i}\mathfrak{L}_q\left[\frac{\partial^i f(0,q)}{\partial p^i}\right](s_2) - \sum_{j=0}^{n-1}s_2^{-1-j}\mathfrak{L}_p\left[\frac{\partial^j f(p,0)}{\partial q^j}\right](s_1) + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}s_1^{-1-i}s_2^{-1-j}\frac{\partial^{i+j}f(0,0)}{\partial q^j\partial p^i}\right). \qquad (12)$$

3 Special functions with general kernel

In this section, we introduce the gamma, beta, Gauss hypergeometric and confluent hypergeometric functions with general kernel and present some of their basic properties. We also show that they generate the special functions given in Section 1.

Definition 3.1. The gamma function with general kernel is defined by

$$^{K}\widehat{\Gamma}(\sigma) := \int\limits_{0}^{\infty}\omega^{\sigma-1}\,K(\omega,\mathbf{X})\,d\omega, \qquad (13)$$

$$\left(\Re(p)>0, \Re(q)>0, \Re(\kappa)>0, \Re(\mu)>0, \Re(\sigma)>0\right),$$

where K is the general kernel and $\mathbf{X} = \mathbf{X}(p,q,\kappa,\mu)$ is a multi-parameter variable.

Definition 3.2. The beta function with general kernel is defined by

$$^{K}\widehat{B}(\sigma,\tau) := \int\limits_{0}^{1}\omega^{\sigma-1}(1-\omega)^{\tau-1}\,K(\omega,\mathbf{X})\,d\omega, \qquad (14)$$

$$\left(\Re(p)>0, \Re(q)>0, \Re(\kappa)>0, \Re(\mu)>0, \Re(\sigma)>0, \Re(\tau)>0\right),$$

where K is the general kernel and $\mathbf{X} = \mathbf{X}(p,q,\kappa,\mu)$ is a multi-parameter variable.

Definition 3.3. The Gauss hypergeometric function with general kernel is defined by

$$^{K}\widehat{F}(\vartheta_1,\vartheta_2;\vartheta_3;z) := \sum_{k=0}^{\infty}(\vartheta_1)_k\frac{^{K}\widehat{B}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}, \quad \text{for } |z|<1, \qquad (15)$$

$$\left(\Re(p)>0, \Re(q)>0, \Re(\kappa)>0, \Re(\mu)>0, \Re(\vartheta_3)>\Re(\vartheta_2)>0\right),$$

where K is the general kernel and $\mathbf{X} = \mathbf{X}(p,q,\kappa,\mu)$ is a multi-parameter variable.

Definition 3.4. The confluent hypergeometric function with general kernel is defined by

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right) := \sum_{k=0}^{\infty} \frac{^{K}\widehat{B}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \frac{z^k}{k!}, \tag{16}$$

$$\left(\Re(p)>0,\Re(q)>0,\Re(\kappa)>0,\Re(\mu)>0,\Re(\vartheta_3)>\Re(\vartheta_2)>0\right),$$

where K is the general kernel and $\mathbf{X}=\mathbf{X}(p,q,\kappa,\mu)$ is a multi-parameter variable.

Remark 3.1. We note that the general kernel function K can be any special function such as an exponential function, Kummer function, Mittag-Leffler function, Wright function, Fox-Wright function or M-series.

Throughout this paper, we will take $\Re(p)>0$, $\Re(q)>0$, $\Re(\kappa)>0$, $\Re(\mu)>0$, $\Re(\vartheta_3)>\Re(\vartheta_2)>0$, $\Re(\sigma)>0$, $\Re(\tau)>0$ unless otherwise stated. We now show that they generate the special functions given in Section 1. We also note that they generate the other special functions, which can be found in literature.

Taking the general kernels as follows

$$K\left(\omega,\mathbf{X}\right) := K\left(-\omega-\frac{p}{\omega^{\kappa}}\right), \tag{17}$$

$$K\left(\omega,\mathbf{X}\right) := K\left(-\frac{\omega^{\kappa}}{p}-\frac{q}{\omega^{\mu}}\right). \tag{18}$$

Using Eqs. (17) and (18) in Eq. (13), respectively we have

$$^{K}\widehat{\Gamma}(\sigma) := \int_{0}^{\infty} \omega^{\sigma-1}\, K\left(-\omega-\frac{p}{\omega^{\kappa}}\right) d\omega, \tag{19}$$

$$^{K}\widehat{\Gamma}(\sigma) := \int_{0}^{\infty} \omega^{\sigma-1}\, K\left(-\frac{\omega^{\kappa}}{p}-\frac{q}{\omega^{\mu}}\right) d\omega. \tag{20}$$

- If we take the K in Eq. (19) as the function ${}_1F_1$, we obtain Eq. (1).
- If we take the K in Eq. (20) as the function exp, we obtain Eq. (2).

Taking the general kernels as follows

$$K\left(\omega,\mathbf{X}\right) := K\left(-\frac{p}{\omega^{\kappa}(1-\omega)^{\mu}}\right), \tag{21}$$

$$K\left(\omega,\mathbf{X}\right) := K\left(-\frac{p}{\omega^{\kappa}}-\frac{q}{(1-\omega)^{\mu}}\right), \tag{22}$$

$$K\left(\omega,\mathbf{X}\right) := K\left(-\frac{p}{\omega^{\kappa}}\right), \tag{23}$$

$$K\left(\omega,\mathbf{X}\right) := K\left(-\frac{q}{(1-\omega)^{\mu}}\right). \tag{24}$$

Using Eqs. (21) and (22) in Eq. (14), respectively we have

$$^{K}\widehat{B}(\sigma,\tau) := \int_{0}^{1} \omega^{\sigma-1}(1-\omega)^{\tau-1}\, K\left(-\frac{p}{\omega^{\kappa}(1-\omega)^{\mu}}\right) d\omega, \tag{25}$$

$$^{K}\widehat{B}(\sigma,\tau) := \int_{0}^{1} \omega^{\sigma-1}(1-\omega)^{\tau-1}\, K\left(-\frac{p}{\omega^{\kappa}}-\frac{q}{(1-\omega)^{\mu}}\right) d\omega. \tag{26}$$

Multiplying (23) by (24) and then writing in Eq. (14), we have

$$^{K}\widehat{B}(\sigma,\tau) := \int_{0}^{1} \omega^{\sigma-1}(1-\omega)^{\tau-1}\, K\left(-\frac{p}{\omega^{\kappa}}\right) K\left(-\frac{q}{(1-\omega)^{\mu}}\right) d\omega. \tag{27}$$

- If we take the K in Eq. (25) as the function $E_{\alpha,\beta}$, we obtain Eq. (3).
- If we take the K in Eq. (26) as the function ${}_1F_1$, we obtain Eq. (4).
- If we take the K in Eq. (27) as the function exp, we obtain Eq. (5).

Finally,

- If we use Eq. (25) in Eqs. (15) and (16), and take the K as the function $E_{\alpha,\beta}$, we obtain Eqs. (6) and (7).
- If we use Eq. (27) in Eqs. (15) and (16), and take the K as the function exp, we obtain Eqs. (8) and (9).

4 Fundamental properties of special functions with general kernel

In this section, we give fundamental properties of special functions with general kernel.

Theorem 4.1. We have the following formula

$$^{K}\widehat{\Gamma}(\sigma)\ ^{K}\widehat{\Gamma}(\tau) = 4\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty} r^{2(\sigma+\tau)-1}\cos^{2\sigma-1}(\theta)\sin^{2\tau-1}(\theta)\, K\left(r^{2}\cos^{2}(\theta),\mathbf{X}\right) K\left(r^{2}\sin^{2}(\theta),\mathbf{X}\right) dr d\theta.$$

Proof. Substituting $\omega = u^{2}$ in Eq. (13), we get

$$^{K}\widehat{\Gamma}(\sigma) = 2\int_{0}^{\infty} u^{2\sigma-1}\, K\left(u^{2},\mathbf{X}\right) du.$$

Therefore,

$$^{K}\widehat{\Gamma}(\sigma)\ ^{K}\widehat{\Gamma}(\tau) = 4\int_{0}^{\infty}\int_{0}^{\infty} u^{2\sigma-1}v^{2\tau-1}\, K\left(u^{2},\mathbf{X}\right) K\left(v^{2},\mathbf{X}\right) du dv.$$

Taking $u = r\cos(\theta)$ and $v = r\sin(\theta)$ yields

$$^{K}\widehat{\Gamma}(\sigma)\ ^{K}\widehat{\Gamma}(\tau) = 4\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty} r^{2(\sigma+\tau)-1}\cos^{2\sigma-1}(\theta)\sin^{2\tau-1}(\theta)\, K\left(r^{2}\cos^{2}(\theta),\mathbf{X}\right) K\left(r^{2}\sin^{2}(\theta),\mathbf{X}\right) dr d\theta.$$

Theorem 4.2. We have the following integral representations

$$^{K}\widehat{B}(\sigma,\tau) = 2\int_{0}^{\frac{\pi}{2}} \sin^{2\sigma-1}(\theta)\cos^{2\tau-1}(\theta)\, K\left(\sin^{2}(\theta),\mathbf{X}\right) d\theta,$$

$$^{K}\widehat{B}(\sigma,\tau) = \int_{0}^{\infty} \frac{t^{\sigma-1}}{(1+t)^{\sigma+\tau}}\, K\left(\frac{t}{1+t},\mathbf{X}\right) dt,$$

$$^{K}\widehat{B}(\sigma,\tau) = (b-a)^{1-\sigma-\tau}\int_{a}^{b} (t-a)^{\sigma-1}(b-t)^{\tau-1}\, K\left(\frac{t-a}{b-a},\mathbf{X}\right) dt.$$

Proof. Taking $\omega = \sin^2(\theta)$, $\omega = \frac{t}{1+t}$ and $\omega = \frac{t-a}{b-a}$ in Eq. (14), respectively, completes the proof.

Theorem 4.3. We have the following functional relation

$$ {}^{K}\widehat{B}(\sigma,\tau+1) + {}^{K}\widehat{B}(\sigma+1,\tau) = {}^{K}\widehat{B}(\sigma,\tau). $$

Proof. Using Eq. (14), we have

$$ \begin{aligned} {}^{K}\widehat{B}(\sigma,\tau+1) + {}^{K}\widehat{B}(\sigma+1,\tau) &= \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau}\, K(\omega,\mathbf{X})\, d\omega + \int_0^1 \omega^{\sigma}(1-\omega)^{\tau-1}\, K(\omega,\mathbf{X})\, d\omega \\ &= \int_0^1 \left(\omega^{\sigma-1}(1-\omega)^{\tau} + \omega^{\sigma}(1-\omega)^{\tau-1}\right) K(\omega,\mathbf{X})\, d\omega \\ &= \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1}\, K(\omega,\mathbf{X})\, d\omega \\ &= {}^{K}\widehat{B}(\sigma,\tau). \end{aligned} $$

Theorem 4.4. We have the following summation relation

$$ {}^{K}\widehat{B}(\sigma,1-\tau) = \sum_{k=0}^{\infty} \frac{(\tau)_k}{k!}\, {}^{K}\widehat{B}(\sigma+k,1), \quad \text{for } \Re(1-\tau) > 0. $$

Proof. From Eq. (14), we have

$$ {}^{K}\widehat{B}(\sigma,1-\tau) = \int_0^1 \omega^{\sigma-1}(1-\omega)^{-\tau}\, K(\omega,\mathbf{X})\, d\omega. \tag{28} $$

The binomial series [12] is defined by

$$ (1-\omega)^{-\tau} = \sum_{k=0}^{\infty} (\tau)_k \frac{\omega^k}{k!}, \quad \text{for } |\omega| < 1. \tag{29} $$

Using Eq. (29) in Eq. (28), completes the proof.

Theorem 4.5. We have the following integral representations

$$ {}^{K}\widehat{F}(\vartheta_1,\vartheta_2;\vartheta_3;z) = \frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \int_0^1 \omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\, K(\omega,\mathbf{X})\, d\omega, \tag{30} $$

$$ {}^{K}\widehat{F}(\vartheta_1,\vartheta_2;\vartheta_3;z) = \frac{2}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \int_0^{\frac{\pi}{2}} \sin^{2\vartheta_2-1}(\theta)\cos^{2\vartheta_3-2\vartheta_2-1}(\theta)\left(1-z\sin^2(\theta)\right)^{-\vartheta_1} K\left(\sin^2(\theta),\mathbf{X}\right) d\theta, \tag{31} $$

$$ {}^{K}\widehat{F}(\vartheta_1,\vartheta_2;\vartheta_3;z) = \frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \int_0^{\infty} t^{\vartheta_2-1}(1+t)^{\vartheta_1-\vartheta_3}\left(1+t(1-z)\right)^{-\vartheta_1} K\left(\frac{t}{1+t},\mathbf{X}\right) dt, \tag{32} $$

$$ {}^{K}\widehat{F}(\vartheta_1,\vartheta_2;\vartheta_3;z) = \frac{(b-a)^{1-\vartheta_3}}{B(\vartheta_2,\vartheta_3-\vartheta_2)} \int_a^{b} (t-a)^{\vartheta_2-1}(b-t)^{\vartheta_3-\vartheta_2-1}\left(1-\frac{z(t-a)}{b-a}\right)^{-\vartheta_1} K\left(\frac{t-a}{b-a},\mathbf{X}\right) dt. \tag{33} $$

Proof. Rewriting Eq. (15), we have

$$^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\sum_{k=0}^{\infty}(\vartheta_1)_k\frac{^{K}\widehat{B}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}. \tag{34}$$

Using Eqs. (14) and (29) in Eq. (34), we have

$$^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1\omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\,K\left(\omega,\mathbf{X}\right)d\omega, \tag{35}$$

which is Eq. (30). Then we take $\omega=\sin^2(\theta)$, $\omega=\frac{t}{1+t}$, and $\omega=\frac{t-a}{b-a}$ in Eq. (35), we obtain Eqs. (31), (32) and (33) respectively.

Theorem 4.6. We have the following integral representations

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1\omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}\exp(z\omega)\,K\left(\omega,\mathbf{X}\right)d\omega, \tag{36}$$

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1 t^{\vartheta_3-\vartheta_2-1}(1-t)^{\vartheta_2-1}\exp\left(z(1-t)\right)\,K\left(1-t,\mathbf{X}\right)dt, \tag{37}$$

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\frac{2}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{\frac{\pi}{2}}\sin^{2\vartheta_2-1}(\theta)\cos^{2\vartheta_3-2\vartheta_2-1}(\theta)\exp\left(z\sin^2(\theta)\right)\,K\left(\sin^2(\theta),\mathbf{X}\right)d\theta, \tag{38}$$

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{\infty}t^{\vartheta_2-1}(1+t)^{-\vartheta_3}\exp\left(\frac{zt}{1+t}\right)\,K\left(\frac{t}{1+t},\mathbf{X}\right)dt, \tag{39}$$

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\frac{(b-a)^{1-\vartheta_3}}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_a^{b}(t-a)^{\vartheta_2-1}(b-t)^{\vartheta_3-\vartheta_2-1}\exp\left(\frac{z(t-a)}{b-a}\right)\,K\left(\frac{t-a}{b-a},\mathbf{X}\right)dt. \tag{40}$$

Proof. Rewriting Eq. (16), we have

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\sum_{k=0}^{\infty}\frac{^{K}\widehat{B}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}. \tag{41}$$

Using Eqs. (14) and (29) in Eq. (41), we have

$$^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1\omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}\exp(z\omega)\,K\left(\omega,\mathbf{X}\right)d\omega, \tag{42}$$

which is Eq. (36). Then we take $\omega=1-t$, $\omega=\sin^2(\theta)$, $\omega=\frac{t}{1+t}$, and $\omega=\frac{t-a}{b-a}$ in Eq. (42), we obtain Eqs. (37), (38), (39) and (40) respectively.

Theorem 4.7. We have the following derivative formulas

$$\frac{d^r}{dz^r}\left\{^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)\right\}=\frac{(\vartheta_1)_r(\vartheta_2)_r}{(\vartheta_3)_r}\,^{K}\widehat{F}\left(\vartheta_1+r,\vartheta_2+r;\vartheta_3+r;z\right), \tag{43}$$

$$\frac{d^r}{dz^r}\left\{^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right)\right\}=\frac{(\vartheta_2)_r}{(\vartheta_3)_r}\,^{K}\widehat{\Phi}\left(\vartheta_2+r;\vartheta_3+r;z\right). \tag{44}$$

Proof. Differentiating Eq. (15), we have

$$\begin{aligned}\frac{d}{dz}\left\{{}^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)\right\} &= \frac{d}{dz}\left\{\sum_{k=0}^{\infty}(\vartheta_1)_k\frac{{}^{K}\widehat{B}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}\right\}\\ &= \sum_{k=1}^{\infty}(\vartheta_1)_k\frac{{}^{K}\widehat{B}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^{k-1}}{(k-1)!}.\end{aligned}$$

Writing $k \to k+1$ and then using formulas $B(\vartheta_2,\vartheta_3-\vartheta_2)=\frac{\vartheta_3}{\vartheta_2}B(\vartheta_2+1,\vartheta_3-\vartheta_2)$ for $\Re(\vartheta_3)>\Re(\vartheta_2)>0$ and $(\vartheta_1)_{n+1}=\vartheta_1(\vartheta_1+1)_n$, we get

$$\begin{aligned}\frac{d}{dz}\left\{{}^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)\right\} &= \frac{(\vartheta_1)(\vartheta_2)}{(\vartheta_3)}\sum_{k=0}^{\infty}(\vartheta_1+1)_k\frac{{}^{K}\widehat{B}(\vartheta_2+1+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2+1,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}\\ &= \frac{(\vartheta_1)(\vartheta_2)}{(\vartheta_3)}\,{}^{K}\widehat{F}\left(\vartheta_1+1,\vartheta_2+1;\vartheta_3+1;z\right).\end{aligned}$$

Using the method of induction, we obtain the more general form as follows

$$\frac{d^r}{dz^r}\left\{{}^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)\right\} = \frac{(\vartheta_1)_r(\vartheta_2)_r}{(\vartheta_3)_r}\,{}^{K}\widehat{F}\left(\vartheta_1+r,\vartheta_2+r;\vartheta_3+r;z\right),$$

which is Eq. (43). Then we perform similar calculations for Eq. (16) and obtain Eq. (44).

Theorem 4.8. We have the following transformation formulas

$$ {}^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right) = (1-z)^{-\vartheta_1}\,{}^{K}\widehat{F}\left(\vartheta_1,\vartheta_3-\vartheta_2;\vartheta_3;\frac{z}{z-1}\right), \tag{45}$$

$$ {}^{K}\widehat{\Phi}\left(\vartheta_2;\vartheta_3;z\right) = \exp(z)\,{}^{K}\widehat{\Phi}\left(\vartheta_3-\vartheta_2;\vartheta_3;-z\right). \tag{46}$$

Proof. Using equation

$$\left(1-z(1-\omega)\right)^{-\vartheta_1} = (1-z)^{-\vartheta_1}\left(1+\frac{z\omega}{1-z}\right)^{-\vartheta_1},$$

and writing $\omega \to 1-\omega$ in Eq. (30), we obtain

$$\begin{aligned}{}^{K}\widehat{F}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right) &= \frac{(1-z)^{-\vartheta_1}}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1 \omega^{\vartheta_3-\vartheta_2-1}(1-\omega)^{\vartheta_2-1}\left(1-\frac{z\omega}{z-1}\right)^{-\vartheta_1}K\left(1-\omega,\mathbf{X}\right)d\omega\\ &= (1-z)^{-\vartheta_1}\,{}^{K}\widehat{F}\left(\vartheta_1,\vartheta_3-\vartheta_2;\vartheta_3;\frac{z}{z-1}\right),\end{aligned}$$

which is Eq. (45). Then we obtain Eq. (46) from Eq. (37).

Theorem 4.9. We have the following double Laplace transforms

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{\Gamma}(\sigma)\right](s_1,s_2)=\int\limits_0^\infty \omega^{\sigma-1}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega, \tag{47}$$

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{B}(\sigma,\tau)\right](s_1,s_2)=\int\limits_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega, \tag{48}$$

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{F}(\vartheta_1,\vartheta_2;\vartheta_3;z)\right](s_1,s_2)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int\limits_0^1 \omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega, \tag{49}$$

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{\Phi}(\vartheta_2;\vartheta_3;z)\right](s_1,s_2)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int\limits_0^1 \omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}\exp(z\omega)\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega, \tag{50}$$

where

$$T_1\left(\omega,\widehat{\mathbf{X}}\right):=\mathfrak{L}_q\mathfrak{L}_p\Big[K(\omega,\mathbf{X})\Big](s_1,s_2)\quad\text{and}\quad \widehat{\mathbf{X}}=\widehat{\mathbf{X}}(s_1,s_2,\kappa,\mu).$$

Proof. Using Eq. (10), we get

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{\Gamma}(\sigma)\right](s_1,s_2)=\int\limits_0^\infty \omega^{\sigma-1}\,\mathfrak{L}_q\mathfrak{L}_p\Big[K(\omega,\mathbf{X})\Big](s_1,s_2)d\omega.$$

Let the following equation be

$$\mathfrak{L}_q\mathfrak{L}_p\Big[K(\omega,\mathbf{X})\Big](s_1,s_2)=T_1\left(\omega,\widehat{\mathbf{X}}\right).$$

Then, we have

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{\Gamma}(\sigma)\right](s_1,s_2)=\int\limits_0^\infty \omega^{\sigma-1}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega,$$

which is Eq. (47). Also we apply Eq. (10) to Eqs. (14), (15) and (16), we obtain Eqs. (48), (49) and (50).

5 Applications to fractional partial differential equations

In this section, we obtain the solutions of fractional partial differential equations involving special functions with general kernels via the double Laplace transform and present graphs for some specific values.

Application 5.1. Let $1<\Re(\varepsilon_1),\Re(\varepsilon_2)\leq 2$. We consider the fractional partial differential equation

$${}_q^cD_{0^+}^{\varepsilon_2}\,{}_p^cD_{0^+}^{\varepsilon_1}y(p,q)={}^K\widehat{\Gamma}(\varepsilon_1\varepsilon_2\sigma),$$

with the initial conditions

$$y(0,0)=\frac{\partial y(0,0)}{\partial p}=\frac{\partial y(0,0)}{\partial q}=\frac{\partial^2 y(0,0)}{\partial q\partial p}=0$$

and

$$y(p,0)=\frac{\partial y(p,0)}{\partial q}=0,\quad y(0,q)=\frac{\partial y(0,q)}{\partial p}=0.$$

Application of Eq. (10) and considering Eqs. (12) and (47) gives

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}_q^cD_{0^+}^{\varepsilon_2}\ {}_p^cD_{0^+}^{\varepsilon_1}y(p,q)\right](s_1,s_2)=\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{\Gamma}(\varepsilon_1\varepsilon_2\sigma)\right](s_1,s_2),$$

then

$$s_1^{\varepsilon_1}s_2^{\varepsilon_2}\left(\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2)-s_1^{-1}\mathfrak{L}_q\left[y(0,q)\right](s_2)-s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2)-s_2^{-1}\mathfrak{L}_p\left[y(p,0)\right](s_1)\right.$$
$$\left.-s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1)+s_1^{-1}s_2^{-1}y(0,0)+s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p}+s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q}+s_1^{-2}s_2^{-2}\frac{\partial^2y(0,0)}{\partial q\partial p}\right)$$
$$=\int_0^\infty\omega^{\varepsilon_1\varepsilon_2\sigma-1}\ T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.$$

Using the initial conditions, we get

$$\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2)=\int_0^\infty\omega^{\varepsilon_1\varepsilon_2\sigma-1}\ s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\ T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.$$

Application of Eq. (11) gives

$$y(p,q)=\int_0^\infty\omega^{\varepsilon_1\varepsilon_2\sigma-1}\ T_2\left(\omega,\mathbf{X}\right)d\omega,$$

where

$$T_2\left(\omega,\mathbf{X}\right):=\mathfrak{L}_q^{-1}\mathfrak{L}_p^{-1}\left[s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\ T_1\left(\omega,\widehat{\mathbf{X}}\right)\right](p,q)\quad\text{and}\quad\widehat{\mathbf{X}}=\widehat{\mathbf{X}}(s_1,s_2,\kappa,\mu).$$

Application 5.2. Let $1<\Re(\varepsilon_1),\Re(\varepsilon_2)\leq 2$. We consider the fractional partial differential equation

$${}_q^cD_{0^+}^{\varepsilon_2}\ {}_p^cD_{0^+}^{\varepsilon_1}y(p,q)={}^K\widehat{B}(\varepsilon_1\sigma,\varepsilon_2\tau),$$

with the initial conditions

$$y(0,0)=\frac{\partial y(0,0)}{\partial p}=\frac{\partial y(0,0)}{\partial q}=\frac{\partial^2y(0,0)}{\partial q\partial p}=0$$

and

$$y(p,0)=\frac{\partial y(p,0)}{\partial q}=0,\quad y(0,q)=\frac{\partial y(0,q)}{\partial p}=0.$$

Application of Eq. (10) and considering Eqs. (12) and (48) gives

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}_q^cD_{0^+}^{\varepsilon_2}\ {}_p^cD_{0^+}^{\varepsilon_1}y(p,q)\right](s_1,s_2)=\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{B}(\varepsilon_1\sigma,\varepsilon_2\tau)\right](s_1,s_2),$$

then

$$s_1^{\varepsilon_1}s_2^{\varepsilon_2}\left(\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2)-s_1^{-1}\mathfrak{L}_q\left[y(0,q)\right](s_2)-s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2)-s_2^{-1}\mathfrak{L}_p\left[y(p,0)\right](s_1)\right.$$
$$\left.-s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1)+s_1^{-1}s_2^{-1}y(0,0)+s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p}+s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q}+s_1^{-2}s_2^{-2}\frac{\partial^2y(0,0)}{\partial q\partial p}\right)$$
$$=\int_0^1\omega^{\varepsilon_1\sigma-1}(1-\omega)^{\varepsilon_2\tau-1}\ T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.$$

Using the initial conditions, we get

$$\mathfrak{L}_q\mathfrak{L}_p\Big[y(p,q)\Big](s_1,s_2)=\int\limits_0^1 \omega^{\varepsilon_1\sigma-1}(1-\omega)^{\varepsilon_2\tau-1}\, s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\, T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.$$

Application of Eq. (11) gives

$$y(p,q)=\int\limits_0^1 \omega^{\varepsilon_1\sigma-1}(1-\omega)^{\varepsilon_2\tau-1}\, T_2\left(\omega,\mathbf{X}\right)d\omega,$$

where

$$T_2\left(\omega,\mathbf{X}\right):=\mathfrak{L}_q^{-1}\mathfrak{L}_p^{-1}\left[s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\, T_1\left(\omega,\widehat{\mathbf{X}}\right)\right](p,q)\quad\text{and}\quad \widehat{\mathbf{X}}=\widehat{\mathbf{X}}(s_1,s_2,\kappa,\mu).$$

Application 5.3. Let $1<\Re(\varepsilon_1),\Re(\varepsilon_2)\leq 2$. We consider the fractional partial differential equation

$${}^{c}_{q}D^{\varepsilon_2}_{0^+}\,{}^{c}_{p}D^{\varepsilon_1}_{0^+}y(p,q)={}^{K}\widehat{F}(\vartheta_1,\varepsilon_1\vartheta_2;\varepsilon_2\vartheta_3;z),$$

with the initial conditions

$$y(0,0)=\frac{\partial y(0,0)}{\partial p}=\frac{\partial y(0,0)}{\partial q}=\frac{\partial^2 y(0,0)}{\partial q\partial p}=0$$

and

$$y(p,0)=\frac{\partial y(p,0)}{\partial q}=0,\quad y(0,q)=\frac{\partial y(0,q)}{\partial p}=0.$$

Application of Eq. (10) and considering Eqs. (12) and (49) gives

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^{c}_{q}D^{\varepsilon_2}_{0^+}\,{}^{c}_{p}D^{\varepsilon_1}_{0^+}y(p,q)\right](s_1,s_2)=\mathfrak{L}_q\mathfrak{L}_p\left[{}^{K}\widehat{F}(\vartheta_1,\varepsilon_1\vartheta_2;\varepsilon_2\vartheta_3;z)\right](s_1,s_2),$$

then

$$\begin{aligned}
&s_1^{\varepsilon_1}s_2^{\varepsilon_2}\Bigg(\mathfrak{L}_q\mathfrak{L}_p\Big[y(p,q)\Big](s_1,s_2)-s_1^{-1}\mathfrak{L}_q\Big[y(0,q)\Big](s_2)-s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2)-s_2^{-1}\mathfrak{L}_p\Big[y(p,0)\Big](s_1)\\
&-s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1)+s_1^{-1}s_2^{-1}y(0,0)+s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p}+s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q}+s_1^{-2}s_2^{-2}\frac{\partial^2 y(0,0)}{\partial q\partial p}\Bigg)\\
&=\frac{1}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\int\limits_0^1 \omega^{\varepsilon_1\vartheta_2-1}(1-\omega)^{\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.
\end{aligned}$$

Using the initial conditions, we get

$$\begin{aligned}
\mathfrak{L}_q\mathfrak{L}_p\Big[y(p,q)\Big](s_1,s_2)=\frac{1}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\int\limits_0^1 \omega^{\varepsilon_1\vartheta_2-1}(1-\omega)^{\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\\
\times s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.
\end{aligned}$$

Application of Eq. (11) gives

$$y(p,q)=\frac{1}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\int\limits_0^1 \omega^{\varepsilon_1\vartheta_2-1}(1-\omega)^{\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\, T_2\,(\omega,\mathbf{X})\,d\omega,$$

where

$$T_2\,(\omega,\mathbf{X}):=\mathfrak{L}_q^{-1}\mathfrak{L}_p^{-1}\left[s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)\right](p,q)\quad\text{and}\quad \widehat{\mathbf{X}}=\widehat{\mathbf{X}}(s_1,s_2,\kappa,\mu).$$

Application 5.4. Let $1<\Re(\varepsilon_1),\Re(\varepsilon_2)\leq 2$. We consider the fractional partial differential equation

$${}_q^cD_{0^+}^{\varepsilon_2}\;{}_p^cD_{0^+}^{\varepsilon_1}y(p,q)={}^K\widehat{\Phi}(\varepsilon_1\vartheta_2;\varepsilon_2\vartheta_3;z),$$

with the initial conditions

$$y(0,0)=\frac{\partial y(0,0)}{\partial p}=\frac{\partial y(0,0)}{\partial q}=\frac{\partial^2 y(0,0)}{\partial q\partial p}=0$$

and

$$y(p,0)=\frac{\partial y(p,0)}{\partial q}=0,\quad y(0,q)=\frac{\partial y(0,q)}{\partial p}=0.$$

Application of Eq. (10) and considering Eqs. (12) and (50) gives

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}_q^cD_{0^+}^{\varepsilon_2}\;{}_p^cD_{0^+}^{\varepsilon_1}y(p,q)\right](s_1,s_2)=\mathfrak{L}_q\mathfrak{L}_p\left[{}^K\widehat{\Phi}(\varepsilon_1\vartheta_2;\varepsilon_2\vartheta_3;z)\right](s_1,s_2),$$

then

$$\begin{aligned}&s_1^{\varepsilon_1}s_2^{\varepsilon_2}\Bigg(\mathfrak{L}_q\mathfrak{L}_p\Big[y(p,q)\Big](s_1,s_2)-s_1^{-1}\mathfrak{L}_q\Big[y(0,q)\Big](s_2)-s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2)-s_2^{-1}\mathfrak{L}_p\Big[y(p,0)\Big](s_1)\\&-s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1)+s_1^{-1}s_2^{-1}y(0,0)+s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p}+s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q}+s_1^{-2}s_2^{-2}\frac{\partial^2 y(0,0)}{\partial q\partial p}\Bigg)\\&=\frac{1}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\int\limits_0^1\omega^{\varepsilon_1\vartheta_2-1}(1-\omega)^{\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-1}\exp(z\omega)\;T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.\end{aligned}$$

Using the initial conditions, we get

$$\begin{aligned}\mathfrak{L}_q\mathfrak{L}_p\Big[y(p,q)\Big](s_1,s_2)=\frac{1}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}&\int\limits_0^1\omega^{\varepsilon_1\vartheta_2-1}(1-\omega)^{\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-1}\exp(z\omega)\\&\times s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\;T_1\left(\omega,\widehat{\mathbf{X}}\right)d\omega.\end{aligned}$$

Application of Eq. (11) gives

$$y(p,q)=\frac{1}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\int\limits_0^1\omega^{\varepsilon_1\vartheta_2-1}(1-\omega)^{\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-1}\exp(z\omega)\;T_2\,(\omega,\mathbf{X})\,d\omega,$$

where

$$T_2\,(\omega,\mathbf{X}):=\mathfrak{L}_q^{-1}\mathfrak{L}_p^{-1}\left[s_1^{-\varepsilon_1}s_2^{-\varepsilon_2}\,T_1\left(\omega,\widehat{\mathbf{X}}\right)\right](p,q)\quad\text{and}\quad \widehat{\mathbf{X}}=\widehat{\mathbf{X}}(s_1,s_2,\kappa,\mu).$$

Now, with the same initial conditions, we give an illustrative application for the fractional partial differential equations of Applications (5.2), (5.3) and (5.4) using Eqs. (5), (8) and (9) defined by Şahin et al. [14]. We also present graphs of the solution functions for some specific values in Figures 1, 2 and 3.

Application 5.5. We let the general kernel as follows

$$K(\omega,\mathbf{X}):=\exp\left(-\frac{p}{\omega^{\kappa}}-\frac{q}{(1-\omega)^{\mu}}\right).$$

So the fractional partial differential equation for Application (5.2) is

$${}^{c}_{q}D^{\varepsilon_2}_{0^+}\,{}^{c}_{p}D^{\varepsilon_1}_{0^+}y(p,q)=B^{(\kappa,\mu)}_{p,q}(\varepsilon_1\sigma,\varepsilon_2\tau),$$

the fractional partial differential equation for Application (5.3) is

$${}^{c}_{q}D^{\varepsilon_2}_{0^+}\,{}^{c}_{p}D^{\varepsilon_1}_{0^+}y(p,q)=F^{(\kappa,\mu)}_{p,q}(\vartheta_1,\varepsilon_1\vartheta_2;\varepsilon_2\vartheta_3;z),$$

the fractional partial differential equation for Application (5.4) is

$${}^{c}_{q}D^{\varepsilon_2}_{0^+}\,{}^{c}_{p}D^{\varepsilon_1}_{0^+}y(p,q)=\Phi^{(\kappa,\mu)}_{p,q}(\varepsilon_1\vartheta_2;\varepsilon_2\vartheta_3;z).$$

Then the solution of the first fractional partial differential equation is

$$y(p,q)=p^{\varepsilon_1}q^{\varepsilon_2}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-p)^{u}}{\Gamma(1+\varepsilon_1+u)}\frac{(-q)^{v}}{\Gamma(1+\varepsilon_2+v)}B(\varepsilon_1\sigma-\kappa u,\varepsilon_2\tau-\mu v), \tag{51}$$

the solution of the second fractional partial differential equation is

$$\begin{aligned}y(p,q)=&\frac{p^{\varepsilon_1}q^{\varepsilon_2}}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)^{u}}{\Gamma(1+\varepsilon_1+u)}\frac{(-q)^{v}}{\Gamma(1+\varepsilon_2+v)}\frac{(\vartheta_1)_k\,z^k}{k!}\\&\times B(\varepsilon_1\vartheta_2+k-\kappa u,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-\mu v),\end{aligned} \tag{52}$$

the solution of the third fractional partial differential equation is

$$\begin{aligned}y(p,q)=&\frac{p^{\varepsilon_1}q^{\varepsilon_2}}{B(\varepsilon_1\vartheta_2,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2)}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)^{u}}{\Gamma(1+\varepsilon_1+u)}\frac{(-q)^{v}}{\Gamma(1+\varepsilon_2+v)}\frac{z^k}{k!}\\&\times B(\varepsilon_1\vartheta_2+k-\kappa u,\varepsilon_2\vartheta_3-\varepsilon_1\vartheta_2-\mu v).\end{aligned} \tag{53}$$

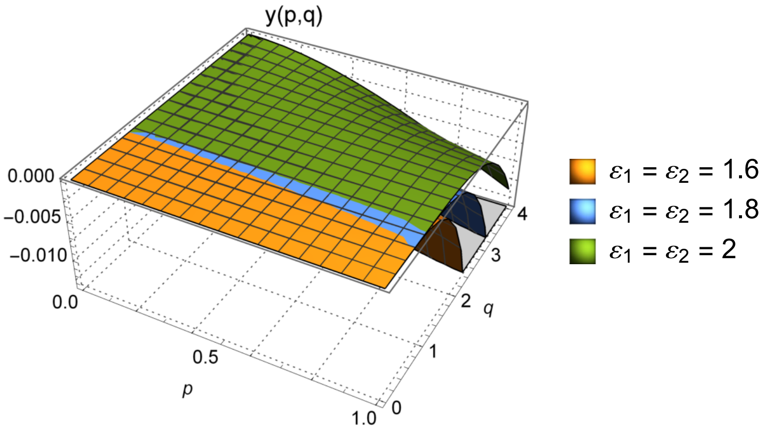

Fig. 1 The approximate graphs of Eq. (51) for the values $u = v = 0, 1, 2, 3$, $\kappa = \mu = 1$, $\sigma = \tau = 3$, $0 < p < 1$, $0 < q < 4$, $\varepsilon_1 = \varepsilon_2 = 1.6$ (yellow), $\varepsilon_1 = \varepsilon_2 = 1.8$ (blue) and $\varepsilon_1 = \varepsilon_2 = 2$ (green).

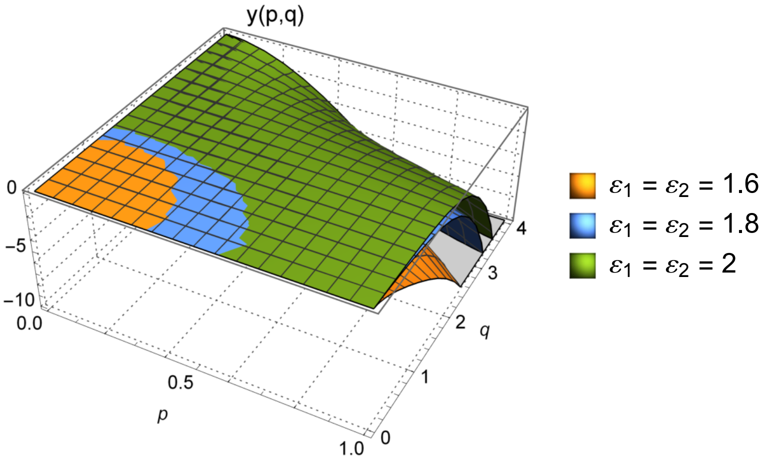

Fig. 2 The approximate graphs of Eq. (52) for the values $u = v = k = 0, 1, 2, 3$, $\kappa = \mu = \vartheta_1 = 1$, $\vartheta_2 = 2$, $\vartheta_3 = 5$, $z = 0.5$, $0 < p < 1$, $0 < q < 4$, $\varepsilon_1 = \varepsilon_2 = 1.6$ (yellow), $\varepsilon_1 = \varepsilon_2 = 1.8$ (blue) and $\varepsilon_1 = \varepsilon_2 = 2$ (green).

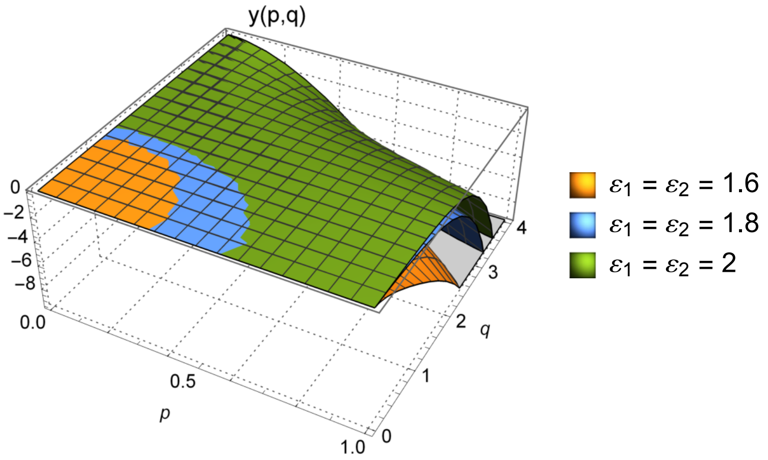

Fig. 3 The approximate graphs of Eq. (53) for the values $u = v = k = 0, 1, 2, 3$, $\kappa = \mu = 1$, $\vartheta_2 = 2$, $\vartheta_3 = 5$, $z = 0.5$, $0 < p < 1$, $0 < q < 4$, $\varepsilon_1 = \varepsilon_2 = 1.6$ (yellow), $\varepsilon_1 = \varepsilon_2 = 1.8$ (blue) and $\varepsilon_1 = \varepsilon_2 = 2$ (green).

6 Beta distribution with general kernel

One of the application areas of various generalized beta functions is statistics. The beta distribution is a continuous probability distribution that is widely used in Bayesian statistics and in modelling ratios and proportions. The beta function with general kernel is a general function that encompasses various beta functions that find applications in various fields such as physics, engineering and finance. Now we give the beta distribution with general kernel and describe the incomplete beta function with general kernel.

Application 6.1. We give the beta distribution with general kernel by

$$F(\omega)=\begin{cases}\frac{\omega^{\sigma-1}(1-\omega)^{\tau-1}\;K(\omega,\mathbf{X})}{{}^{K}\widehat{B}(\sigma,\tau)} & ,0<\omega<1\\ 0 & ,\text{otherwise.}\end{cases}$$

If $\lambda\in\mathbb{R}$, then for $-\infty<\sigma<\infty$, $-\infty<\tau<\infty$

$$E\left(X^{\lambda}\right)=\frac{{}^{K}\widehat{B}(\sigma+\lambda,\tau)}{{}^{K}\widehat{B}(\sigma,\tau)}.$$

The variance of the distribution is

$$E\left(X^{2}\right)-\{E(X)\}^{2}=\frac{{}^{K}\widehat{B}(\sigma,\tau)\;{}^{K}\widehat{B}(\sigma+2,\tau)-\left({}^{K}\widehat{B}(\sigma+1,\tau)\right)^{2}}{\left({}^{K}\widehat{B}(\sigma,\tau)\right)^{2}}.$$

The moment generation function of the distribution is

$$M(\omega)=\sum_{k=0}^{\infty}E\left(X^{k}\right)\frac{\omega^{k}}{k!}=\sum_{k=0}^{\infty}\frac{{}^{K}\widehat{B}(\sigma+k,\tau)}{{}^{K}\widehat{B}(\sigma,\tau)}\frac{\omega^{k}}{k!}.$$

The cummulative distribution of $F(\omega)$ can be written as

$$F(X)=\frac{{}^{K}\widehat{B}_{X}(\sigma,\tau)}{{}^{K}\widehat{B}(\sigma,\tau)},$$

where

$${}^{K}\widehat{B}_{X}(\sigma,\tau)=\int_{0}^{X}\omega^{\sigma-1}(1-\omega)^{\tau-1}\,K(\omega,\mathbf{X})\,d\omega$$

is incomplete beta function with general kernel.

7 Conclusions and remarks

In this paper, we introduced the gamma, beta, Gauss hypergeometric and confluent hypergeometric functions with general kernel. We also examined that special functions with general kernel generate other special functions in literature. Furthermore, we gave fundamental properties and presented some applications of special functions with general kernel. Finally, we obtained the incomplete beta function with general kernel by defining the beta distribution with general kernel. We conclude this paper by stating that in future works we will introduce their new structures with general kernel of various special functions such as Horn, Appell, Lauricella, Srivastava and also their new structures with general kernel of various fractional operators such as Riemann-Liouville, Caputo, Kober-Erdelyi and give their various potential properties and applications.

8 Declarations

8.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

8.2 Funding:

Not applicable.

8.3 Author's Contribution:

E.A.-Writing-Original Draft. İ.O.K.-Writing, Review and Editing.

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8.5 Data availability statement:

All data that support the findings of this study are included within the paper.

8.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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