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COMMUTATOR OF FRACTIONAL MAXIMAL FUNCTION ON LORENTZ SPACES

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ABSTRACT

In the paper we study the fractional maximal commutators $M_{b,\alpha}$ and the commutators of the fractional maximal operator $[b, M_\alpha]$ in the Lorentz spaces $L^{p,r}(R^n)$. The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators. The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces. Our main aim is to characterize the commutator functions b, involved in the boundedness on Lorentz spaces of the fractional maximal commutator $M_{b,\alpha}$ and the commutator of the fractional maximal operator $[b, M_\alpha]$. We give necessary and sufficient conditions for the boundedness of the operators $M_{b,\alpha}$ and $[b, M_\alpha]$ on Lorentz spaces $L^{p,r}(R^n)$ when b belongs to $BMO(R^n)$ spaces, whereby some new characterizations for certain subclasses of $BMO(R^n)$ spaces are obtained. We can apply this boundedness of fractional-maximal commutators in Lorentz spaces to study the regularity in Lorentz spaces of of the Navier-Stokes equations. Solutions to the Navier–Stokes equations are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. In particular, solutions of the Navier–Stokes equations often include turbulence, which remains one of the greatest unsolved problems in physics, despite its immense importance in science and engineering.

Keywords: Lorentz space; fractional maximal operator; commutator; space.

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1. Introduction

We give necessary and sufficient conditions for the boundedness of the fractional maximal commutators $M_{b,\alpha}$ and the commutators of the fractional maximal operator $[b, M_\alpha]$ on the Lorentz spaces $L^{p,r}(R^n)$. We obtain some new characterizations for certain subclasses of $BMO(R^n)$.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [1]). For $f \in L^1_{loc}(R^n)$, the fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{n}} \int_{B(x,r)} |f(y)| dy, \; 0 \le \alpha < n,$$

where $B(x,r)$ is the ball of radius r centered at $x \in R^n$, ${}^{C}B(x,r)$ is its complement and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$. If $\alpha=0$, then $M=M_0$ is the well-known Hardy-Littlewood maximal operator.

The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [1, 2]).

The fractional maximal commutator generated by $b \in L^1_{loc}(R^n)$ and the operator M_α is defined by

$$M_{b,\alpha} f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{n}} \int_{B(x,r)} |b(x)-b(y)||f(y)| dy,$$
$$0 \le \alpha < n.$$

The commutator generated by a suitable function b and M_α the operator is defined by

$$[b, M_\alpha](f)(x) = b(x) M_\alpha(f)(x) - M_\alpha(bf)(x).$$

The mapping properties of $M_{b,\alpha}$ and $[b, M_\alpha]$ have been studied extensively by many authors. See, for instance, [3-8]. The operator $M_b := M_{b,0}$ plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [6, 9-11]). The operators M_α, $[b, M_\alpha]$ and $M_{b,\alpha}$ play an important role in real and harmonic analysis

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and applications (see, for example [1, 8, 12]). The nonlinear commutator of Hardy-Littlewood maximal function $[b, M]$ can be used in studying the product of a function in H_1 and a function in *BMO* (see [13] for instance).

Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [6]. In [14] by Bastero et al. studied the necessary and sufficient condition for the boundedness of $[b, M]$ on L^p spaces. In [15] by Zhang and Wu considered the same problem for $[b, M_\alpha]$, see also [8].

Our main aim is to characterize the commutator functions b, involved in the boundedness on Lorentz spaces of the fractional maximal commutator $M_{b,\alpha}$ (Theorem 3.1) and the commutator of the fractional maximal operator $[b, M_\alpha]$ (Theorem 4.1).

The structure of the paper is as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\alpha}$ on $L^{p,r}(R^n)$ spaces. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{p,r}(R^n)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and some basic properties

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L_p spaces, also they are appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on R^n, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}$$

where

$$d_f(s) := \left|\{x \in R^n : |f(x)| > s\}\right|, \quad \forall s > 0 .$$

Definition 2.1. [16] The Lorentz space $L_{pq} \equiv L_{pq}(R^n)$, $0<p$, $q \le \infty$ is the collection of all measurable functions f on R^n such the quantity

$$\|f\|_{L_{p,q}} := \left\|t^{1/p-1/q} f^*(t)\right\|_{L_q(0,\infty)}$$

is finite. Clearly $L_{p,p} \equiv L_p$ and $L_{p,\infty} \equiv WL_p$. The functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if either $1 \le q \le p$ or $p=q=\infty$.

Lemma 2.1. [16] Let $1 < p,\, p',\, r,\, r' < \infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$. Suppose that $f \in L_{p,r}(R^n)$ and $f \in L_{p',r'}(R^n)$. Then

$$\|f\,g\|_{L_1(R^n)} \le 2\|f\|_{L_{p,r}(R^n)}\|g\|_{L_{p',r'}(R^n)} .$$

The following result completely characterizes the boundedness of M on Lorentz spaces.

Lemma 2. 2. [16] Let $1<p,\, r \le \infty$.

(*I*) If $1<p\le\infty$, then the operator M is bounded on the Lorentz spaces $L^{p,r}(R^n)$.

(*II*) If $p=1$, then the operator M is bounded on the Lorentz spaces $L^{1,r}(R^n)$ to $WL^1(R^n)$.

The following result completely characterizes the boundedness of M_α on Lorentz spaces.

Lemma 2.3. [17] Let $0 \le \alpha < n$, $1 \le p < \frac{n}{\alpha}$ and $p \le q < \infty$.

(*I*) If $1 < p < \frac{n}{\alpha}$, $1 \le r \le s \le \infty$, then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ is necessary and sufficient for the boundedness of the operator M_α from the Lorentz spaces $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

(*II*) If $p=1$, $1 \le r \le \infty$, then the condition $1-\frac{1}{q}=\frac{\alpha}{n}$ is necessary and sufficient for the boundedness of the operator M_α from the Lorentz spaces $L^{1,r}(R^n)$ to $WL^q(R^n)$.

3. ($L^{p,r}$, $L^{q,s}$) - boundedness of the fractional maximal commutator operator $M_{b,\alpha}$

In this section we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\alpha}$ from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

Definition 3.1. We define the space $BMO(R^n)$ as the set of all locally integrable functions f with finite norm

$$\|f\|_* = \sup_{x\in R^n,\, t>0} |B(x,t)|^{-1} \int_{B(x,t)} \left|f(y) - f_{B(x,t)}\right| dy < \infty$$

where

$$f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y)\,dy.$$

Lemma 3.1. ([9, Corollary 1.11]) If $b \in BMO(R^n)$, then there exists a positive constant C such that

$$M_b f(x) \le C\|b\|_* M^2 f(x)$$

for almost every $x \in R^n$ and for all $f \in L^1_{loc}(R^n)$.

Lemma 3.2. [18] Let $0<\alpha<n$ and $b \in BMO(R^n)$. Then there exists a positive constant C such that

$$M_{b,\alpha} f(x) \le C\|b\|_* \left(M(M_\alpha f)(x) + M_\alpha(Mf)(x)\right) \tag{1}$$

holds for almost every $x \in R^n$ and for all functions from $f \in L^1_{loc}(R^n)$.

Theorem 3.1. Let $0 \le \alpha < n$, $b \in L^1_{loc}(R^n)$, $1 < p < \frac{n}{\alpha}$, $1 \le r \le s \le \infty$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$. The following assertions are equivalent:

(*I*) $b \in BMO(R^n)$.

(*II*) The operator $M_{b,\alpha}$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

(*III*) There exist *a* constant C>0 such that

$$\sup_B \frac{\left\|(b(\cdot)-b_B)\chi_B\right\|_{L^{q,s}(R^n)}}{\|\chi_B\|_{L^{q,s}(R^n)}} \le C .$$

(*IV*) There exist *a* constant C>0 such that

$$\sup_B \frac{\left\|(b(\cdot)-b_B)\chi_B\right\|_{L^1(R^n)}}{|B|} \le C .$$

Proof. (*I*)⇒(*II*). Suppose that $b \in BMO(R^n)$. Combining Lemmas 2.2 and 3.1, we get

$$\begin{aligned}\|M_{b,\alpha} f\|_{L^{q,s}(R^n)} &\lesssim \|b\|_* \left\|M(M_\alpha f) + M_\alpha(Mf)\right\|_{L^{q,s}(R^n)} \\ &\lesssim \|b\|_* \left(\|M_\alpha f\|_{L^{q,s}(R^n)} + \|M f\|_{L^{p,r}(R^n)}\right) \\ &\lesssim \|b\|_* \|f\|_{L^{p,r}(R^n)}.\end{aligned}$$

(*II*)⇒(*III*). Assume that the operator $M_{b,\alpha}$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$. Let $B=B(x,r)$ be a fixed ball. We consider $f=\chi_B$. It is easy to compute that

$$\|\chi_B\|_{L^{p,r}(R^n)} \approx r^{\frac{n}{p}}.$$

On the other hand, for all $x \in B$ we have

$$\left|b(x)-b_B\right| \le \frac{1}{|B|}\int_B \left|b(x)-b(y)\right|dy$$

$$= \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B \left|b(x)-b(y)\right|\chi_B(y)dy$$

$$\le |B|^{-\frac{\alpha}{n}} M_{b,\alpha}(\chi_B)(x).$$

Since $M_{b,\alpha}$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$, then by (1) we obtain

$$\frac{\left\|(b-b_B)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \le \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\left\|M_{b,\alpha}(\chi_B)\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}}$$

$$\lesssim \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\left\|\chi_B\right\|_{L^{p,r}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} = \frac{\left\|\chi_B\right\|_{L^{p,r}(R^n)}}{\left\|\chi_B\right\|_{L^{p,r}(R^n)}} = 1$$

which implies that (3.1) holds since the ball $B\subset R^n$ is arbitrary.

(III)$\Rightarrow$(IV). Assume that (3.1) holds, we will prove (3.2). For any fixed ball B, by Lemma 2.1 and (3.1), (1), it is easy to see

$$\frac{1}{|B|}\int_B \left|b(x)-b(y)\right|dy \lesssim \frac{1}{|B|}\left\|(b-b_B)\chi_B\right\|_{L^{q,s}(R^n)}\left\|\chi_B\right\|_{L^{q',s'}(R^n)}$$

$$\lesssim \frac{\left\|(b-b_B)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}}$$

$$\lesssim 1 .$$

(IV)$\Rightarrow$(I). For any fixed ball B, we have

$$\frac{1}{|B|}\int_B \left|b(x)-b_B\right|dy \lesssim \frac{\left\|(b-b_B)\chi_B\right\|_{L^1}}{|B|}$$

$$\lesssim \frac{\left\|(b-b_B)\chi_B\right\|_{L^1}}{|B|}$$

$$\lesssim 1$$

which implies that $b\in BMO(R^n)$.

The proof of Theorem 3.1 is finished.

4. ($L^{p,r}$, $L^{q,s}$) - boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$

In this section we find necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$ from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

For a function b defined on R^n, we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x)\ge 0 \\ |b(x)|, & \text{if } b(x)<0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid.

Let b be any non-negative locally integrable function. Then for all $f\in L^1_{loc}(R^n)$ and $x\in R^n$ the following inequality is valid

$$\left|\left[b,M_\alpha\right]f(x)\right| = \left|b(x)M_\alpha f(x) - M_\alpha(bf)(x)\right| =$$

$$= \left|M_\alpha\left(b(x)f\right)(x) - M_\alpha(bf)(x)\right| \le M_\alpha\left(\left|b(x)-b\right|f\right)(x) =$$

$$= M_{b,\alpha}f(x).$$

If b is any locally integrable function on R^n, then

$$\left|\left[b,M_\alpha\right]f(x)\right| \le M_{b,\alpha}f(x) + 2b^-(x)M_\alpha f(x), \quad x\in R^n$$

holds for all $f\in L^1_{loc}(R^n)$ (see, for example [8, 12]).

Obviously, the $M_{b,\alpha}$ and $[b, M_\alpha]$ operators are essentially different from each other because $M_{b,\alpha}$ is positive and sublinear and $[b, M_\alpha]$ is neither positive nor sublinear.

Let $B=B(x, r)$ be a fixed ball. Denote by $M_{\alpha,B}f$ the local fractional maximal function of f:

$$M_{\alpha,B}f(x) := \sup_{B'\ni x:\, B'\subset B}\frac{1}{|B'|^{1-\frac{\alpha}{n}}}\int_{B'}\left|f(y)\right|dy, \quad x\in R^n.$$

Applying Theorem 3.1, we obtain the following result

Theorem 4.1. Let $0\le\alpha<n$, $b\in L^1_{loc}(R^n)$, $1<p<\frac{n}{\alpha}$, $1\le r\le s\le\infty$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$. The following assertions are equivalent:

(I) $b\in BMO(R^n)$ and $b^-\in L^\infty(R^n)$.

(II) The operator $[b, M_\alpha]$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

(III) There exist a constant C>0 such that

$$\sup_B \frac{\left\|\left(b(\cdot)-M_B(b)(\cdot)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \le C$$

(IV) There exist a constant C>0 such that

$$\sup_B \frac{\left\|\left(b(\cdot)-M_B(b)(\cdot)\right)\chi_B\right\|_{L^1(R^n)}}{|B|} \le C$$

Proof. (I)$\Rightarrow$(II). Suppose that $b\in BMO(R^n)$ and $b^-\in L^\infty(R^n)$ Combining Lemma 2.2 and 3.1, and inequality (4.8), we get

$$\left\|\left[b,M_\alpha\right]f\right\|_{L^{q,s}(R^n)} \le \left\|M_{b,\alpha}f + 2b^-M_\alpha f\right\|_{L^{q,s}(R^n)}$$

$$\le \left\|M_{b,\alpha}f\right\|_{L^{q,s}(R^n)} + 2\left\|b^-\right\|_{L^\infty}\left\|M_\alpha f\right\|_{L^{q,s}(R^n)}$$

$$\lesssim \left(\|b\|_* + \left\|b^-\right\|_{L^\infty}\right)\|f\|_{L^{p,r}(R^n)}.$$

Thus, we obtain that $[b, M_\alpha]$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

(II)$\Rightarrow$(III). Assume that $[b, M_\alpha]$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$.

We divide the proof into two cases according to the range of α.

Case 1. Assume α=0. For any fixed ball B and $x\in B$, we have

$$b(x) - M_B(b)(x) = b(x)M(\chi_B)(x) - M(b\chi_B)(x) =$$

$$= [b,M](\chi_B)(x).$$

Assume that $[b, M]$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$, then by Lemma 2.3, we get

$$\frac{\left\|\left(b-M_B(b)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \le \frac{\left\|[b,M](\chi_B)\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}}$$

$$\lesssim \frac{\left\|\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} = 1$$

which implies that (4.9) holds since the ball $B\subset R^n$ is arbitrary.

Case 2. Assume $0<\alpha<n$. For any fixed ball B and $x\in B$, we have, since

$$M_\alpha(b\chi_B)\chi_B = M_{\alpha,B}(b)$$

and

$$M_\alpha(\chi_B)\chi_B = M_{\alpha,B}(\chi_B)\chi_B = |B|^{\frac{\alpha}{n}}\chi_B$$

we obtain

$$\frac{\left\|\left(b-M_B(b)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \le \frac{\left\|\left(b-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} + \frac{\left\|\left(|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)-M_B(b)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} := I + II.$$

For I. For any $x\in B$,

$$\begin{aligned} b(x)-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)(x) &= |B|^{-\frac{\alpha}{n}}\left(b(x)|B|^{\frac{\alpha}{n}}-M_{\alpha,B}(b)(x)\right) \\ &= |B|^{-\frac{\alpha}{n}}\left(b(x)M_\alpha(\chi_B)(x)-M_\alpha(b\chi_B)(x)\right) \\ &= |B|^{-\frac{\alpha}{n}}\left[b,M_\alpha\right](\chi_B)(x). \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$, then by Lemma 2.3, we get

$$\begin{aligned} I &= \frac{\left\|\left(b-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \\ &= \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\left\|\left[b,M_\alpha\right](\chi_B)\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \\ &= \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\left\|\chi_B\right\|_{L^{p,r}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \\ &\lesssim \frac{\left\|\chi_B\right\|_{L^{p,r}(R^n)}}{\left\|\chi_B\right\|_{L^{p,r}(R^n)}} = 1. \end{aligned}$$

Next, we estimate I_2. For any $x\in B$, $M_B(\chi_B)(x)=(\chi_B)(x)$ (see, for example, [7]) and then $M(\chi_B)(x)=(\chi_B)(x)$ and $M(b\chi_B)(x)= =M_B(b)(x)$ for any $x\in B$. Then

$$\begin{aligned} &\left||B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)(x)-M_B(b)(x)\right| = \\ &= |B|^{-\frac{\alpha}{n}}\left|M_{\alpha,B}(b)(x)-|B|^{\frac{\alpha}{n}}M_B(b)(x)\right| = \\ &= |B|^{-\frac{\alpha}{n}}\left|M_\alpha(b\chi_B)(x)-M_\alpha(\chi_B)(x)M(b\chi_B)(x)\right| \\ &\le |B|^{-\frac{\alpha}{n}}\left|M_\alpha(b\chi_B)(x)-|b(x)|M_\alpha(\chi_B)(x)\right| \\ &+ |B|^{-\frac{\alpha}{n}}\left||b(x)|M_\alpha(\chi_B)(x)-M_\alpha(\chi_B)(x)M(b\chi_B)(x)\right| \\ &= |B|^{-\frac{\alpha}{n}}\left|M_\alpha(|b|\chi_B)(x)-|b(x)|M_\alpha(\chi_B)(x)\right| \\ &+ |B|^{-\frac{\alpha}{n}}M_\alpha(\chi_B)(x)\left||b(x)|M(\chi_B)(x)-M(b\chi_B)(x)\right| \\ &= |B|^{-\frac{\alpha}{n}}\left|\left[|b|,M_\alpha\right](\chi_B)(x)\right| + \left|\left[|b|,M\right](\chi_B)(x)\right|. \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $L^{p,r}(R^n)$ to $L^{q,s}(R^n)$, then by Lemma 2.3, we get

$$\begin{aligned} II &= \frac{\left\|\left(|B|^{-\frac{\alpha}{n}}\left|\left[|b|,M_\alpha\right](\chi_B)\right| + \left|\left[|b|,M\right](\chi_B)\right|\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \\ &\le \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\left\|\left[|b|,M_\alpha\right](\chi_B)\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} + \frac{\left\|\left[|b|,M\right](\chi_B)\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\left\|\chi_B\right\|_{L^{p,r}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} + \frac{\left\|\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \approx 1. \end{aligned}$$

This gives the desired estimate

$$\frac{\left\|\left(b-M_B(b)\right)\chi_B\right\|_{L^{q,s}(R^n)}}{\left\|\chi_B\right\|_{L^{q,s}(R^n)}} \lesssim 1$$

which deduces that (III).

$(III)\Rightarrow(IV)$. Assume that (4.9) holds, then for any fixed ball B, by Lemma 2.1, we conclude that

$$\begin{aligned} \frac{1}{|B|}\int_B\left|b(x)-M_B(b)(x)\right|dx &\lesssim \frac{1}{|B|}\left\|\left(b-M_B(b)\right)\chi_B\right\|_{L^{p,r}(R^n)}\left\|\chi_B\right\|_{L^{p',r'}} \\ &\lesssim \frac{\left\|\left(b-M_B(b)\right)\chi_B\right\|_{L^{p,r}(R^n)}}{\left\|\chi_B\right\|_{L^{p,r}(R^n)}} \\ &\lesssim 1. \end{aligned}$$

$(IV)\Rightarrow(I)$ Assume that (4.10) holds, we will prove $b\in BMO(R^n)$ and $b^-\in L^\infty(R^n)$

Denote by

$$E:=\left\{x\in B: b(x)\le b_B\right\},\ \ F:=\left\{x\in B: b(x)>b_B\right\}.$$

Since

$$\int_E\left|b(t)-b_B\right|dt = \int_F\left|b(t)-b_B\right|dt$$

in view of the inequality $b(x)\le b_B\le M_B(b)$, $x\in E$, we get

$$\begin{aligned} \frac{1}{|B|}\int_B\left|b-b_B\right|dt &= \frac{2}{|B|}\int_E\left|b-b_B\right|dt \\ &\le \frac{2}{|B|}\int_E\left|b-M_B(b)\right|dt \\ &\le \frac{2}{|B|}\int_B\left|b-M_B(b)\right|dt \le c. \end{aligned}$$

Consequently, $b\in BMO(R^n)$.

In order to show that $b^-\in L^\infty(R^n)$, note that $M_B(b)\ge b$. Hence

$$0\le b^- = |b|-b^+ \le M_B(b)-b^+ + b^- = M_B(b)-b.$$

Thus

$$\left(b^-\right)_B \le c$$

and by the Lebesgue Differentiation theorem we get that

$$0\le b^-(x) = \lim_{|B|\to 0}\frac{1}{|B|}\int_B b^-(y)dy \le c \quad \text{for } a.e.\ x\in R^n.$$

Remark 1. Note that in the case of $\alpha=0$ from Theorem 3.1 we get [19, Theorem 3.1] and Theorem 4.1 we get [19, Theorem 4.1], see also [5].

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