




# Generalized fractional maximal and integral operators on Orlicz and generalized Orlicz–Morrey spaces of the third kind

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## Abstract

In the present paper, we will characterize the boundedness of the generalized fractional integral operators  $I_\rho$  and the generalized fractional maximal operators  $M_\rho$  on Orlicz spaces, respectively. Moreover, we will give a characterization for the Spanne-type boundedness and the Adams-type boundedness of the operators  $M_\rho$  and  $I_\rho$  on generalized Orlicz–Morrey spaces, respectively. Also we give criteria for the weak versions of the Spanne-type boundedness and the Adams-type boundedness of the operators  $M_\rho$  and  $I_\rho$  on generalized Orlicz–Morrey spaces.

**Keywords** Generalized fractional maximal function · Generalized fractional integral · Orlicz spaces · Generalized Orlicz–Morrey spaces

**Mathematical Subject Classification** 42B20 · 42B25 · 42B35 · 46E30

## 1 Introduction

The aim of this paper is to obtain the necessary conditions and the sufficient conditions for the generalized fractional maximal operator  $M_\rho$  and the generalized fractional integral operator  $I_\rho$  to be bounded on Orlicz spaces. Our results can be extended to generalized Orlicz–Morrey spaces of the third kind which will be defined later in this paper.

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Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space. For a function  $\rho : (0, \infty) \rightarrow (0, \infty)$ , let  $I_\rho$  be the generalized fractional integral operator:

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} f(y) dy.$$

Here  $f$  is a suitable measurable function. Note that this type of generalization goes back to [25]. If  $\rho(r) = r^\alpha$ ,  $0 < \alpha < n$ , then  $I_\rho$  is the fractional integral operator or the Riesz potential and denoted by  $I_\alpha$ . Hereafter, we assume that

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty \tag{1}$$

so that the fractional integrals  $I_\rho f$  are well defined, at least for characteristic functions of balls. The operator  $I_\rho$  was introduced in [19] and some partial results were announced in [18]. We refer to [16] for the boundedness of  $I_\rho$  on Orlicz space  $L^\Phi(\Omega)$  with bounded domain  $\Omega \subset \mathbb{R}^n$ . See also [20–23] for the boundedness of  $I_\rho$  on various spaces. In these papers we assumed that  $\rho$  satisfies the doubling condition:

$$\frac{1}{C_1} \leq \frac{\rho(r)}{\rho(s)} \leq C_1, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2, \tag{2}$$

and that  $r \mapsto \rho(r)/r^n$  is almost decreasing:

$$\frac{\rho(s)}{s^n} \leq C_2 \frac{\rho(r)}{r^n}, \quad \text{if } r < s, \tag{3}$$

where  $C_1$  and  $C_2$  are positive constants independent of  $r, s \in (0, \infty)$ . Under these conditions we proved the boundedness of  $I_\rho$  on Orlicz spaces in [18,19].

In this paper, instead of these conditions, we assume that there exist positive constants  $C, k_1$  and  $k_2$  with  $k_1 < k_2$  such that, for all  $r > 0$ ,

$$\sup_{r/2 \leq t \leq r} \rho(t) \leq C \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t} dt =: \tilde{\rho}(r). \tag{4}$$

The condition (4) was considered in [26] and also used in [31]. If  $\rho$  satisfies (2) or (3), then  $\rho$  satisfies (4). Let

$$\rho(r) = \begin{cases} r^n (\log(e/r))^{-1/2}, & 0 < r < 1, \\ e^{-(r-1)}, & 1 \leq r < \infty. \end{cases} \tag{5}$$

Then  $\rho$  satisfies (1) and (4), but fails (2) and (3). Therefore, the results in this paper improve ones in [19]. Moreover, we give necessary and sufficient conditions for the boundedness of  $I_\rho$  not only on Orlicz spaces but also on Orlicz–Morrey spaces of the third kind.

Next, we define the generalized fractional maximal operator  $M_\rho$ . For a function  $\rho : (0, \infty) \rightarrow (0, \infty)$ , let

$$M_\rho f(x) = \sup_{r>0} \frac{\rho(r)}{|B(x, r)|} \int_{B(x,r)} |f(y)|dy, \tag{6}$$

where  $|G|$  is the Lebesgue measure of a measurable set  $G \subset \mathbb{R}^n$ . We do not assume (1) on the function  $\rho$  in (6). Instead we suppose that  $\rho$  is an increasing function such that  $r \in (0, \infty) \mapsto r^{-n} \rho(r) \in (0, \infty)$  is decreasing.

If  $\rho \equiv 1$ , then  $M_\rho$  is the Hardy-Littlewood maximal operator denoted by  $M$ . If  $\rho(r) = r^\alpha$ , then  $M_\rho$  is the usual fractional maximal operator denoted by  $M_\alpha$ . We give some necessary conditions and some sufficient conditions for the boundedness of  $M_\rho$  on Orlicz and Orlicz–Morrey spaces.

The structure of the remaining part of the present paper is as follows: First we recall Young functions and Orlicz spaces in Sect. 2. In Sect. 3, we investigate the boundedness of generalized fractional integrals on Orlicz spaces. We will give a necessary and sufficient condition for the boundedness of the generalized fractional maximal operators on Orlicz spaces in Sect. 4. In Sect. 5 we discuss some properties of generalized Orlicz–Morrey spaces of the third kind. Moreover, we will give necessary and sufficient conditions for the Spanne and Adams-type boundedness of the generalized fractional integral operators on generalized Orlicz–Morrey spaces of the third kind in Sect. 6. Finally, in Sect. 7 we give criteria for the boundedness of the generalized fractional maximal operators on generalized Orlicz–Morrey spaces of the third kind.

## 2 Young functions and Orlicz spaces

We recall the definition of Young functions.

**Definition 1** A function  $\Phi : [0, \infty] \rightarrow [0, \infty]$  is called a Young function if  $\Phi$  is convex, left-continuous,  $\lim_{r \rightarrow +0} \Phi(r) = \Phi(0) = 0$  and  $\lim_{r \rightarrow \infty} \Phi(r) = \Phi(\infty) = \infty$ .

From the non-negativity, convexity and  $\Phi(0) = 0$  it follows that any Young function is increasing. We denote by  $\mathcal{Y}$  the set of all Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty.$$

If  $\Phi \in \mathcal{Y}$ , then  $\Phi$  is absolutely continuous on every compact interval in  $[0, \infty)$  and bijective from  $[0, \infty)$  to itself.

Next we recall the generalized inverse of Young function  $\Phi$  in the sense of O’Neil [24, Definition 1.2]. For a Young function  $\Phi$  and  $0 \leq s \leq \infty$ , let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}, \quad \text{where} \quad \inf \emptyset = \infty.$$

Note that if  $s < \infty$ , then so is  $\Phi^{-1}(s)$ . As in [24, p. 301, Remarks], we always have  $\Phi^{-1}(\infty) = \infty$ . An important inequality we use is

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)).$$

See [24, Property 1.3]. Then  $\Phi^{-1}(s)$  is finite for all  $s \in [0, \infty)$ , continuous on  $(0, \infty)$  and right continuous at  $s = 0$ . Observe that  $\Phi^{-1}(\Phi(r)) = r$  if  $0 < \Phi(r) < \infty$  and that  $\Phi(\Phi^{-1}(s)) = s$  if  $s \in [0, \Phi(\inf\{r > 0 : \Phi(r) = \infty\})]$ . Furthermore, if  $\Phi \in \mathcal{Y}$ , then  $\Phi^{-1}$  is the usual inverse function of  $\Phi$ .

**Remark 1** For a Young function  $\Phi$ , its inverse function  $\Phi^{-1}$  is increasing and concave. Hence, we have the following properties:

$$\begin{cases} \Phi^{-1}(t) \geq \Phi^{-1}(\alpha t) \geq \alpha\Phi^{-1}(t), & \text{if } 0 < \alpha < 1 \\ \Phi^{-1}(t) \leq \Phi^{-1}(\alpha t) \leq \alpha\Phi^{-1}(t), & \text{if } \alpha > 1. \end{cases}$$

Since  $\Phi^{-1}$  is increasing, the left inequality is clear. In particular,  $\Phi$  satisfies the doubling condition:  $\Phi^{-1}(2s) \leq 2\Phi^{-1}(s)$  for all  $s \geq 0$ .

In fact for  $0 < \alpha < 1$ ,

$$\Phi^{-1}(\alpha t) = \inf\{s \geq 0 : \Phi(s) > \alpha t\}.$$

Since  $\frac{1}{\alpha}\Phi(s) \leq \Phi\left(\frac{s}{\alpha}\right)$ , we have

$$\Phi^{-1}(\alpha t) \geq \inf\left\{s \geq 0 : \Phi\left(\frac{s}{\alpha}\right) > \alpha t\right\} = \alpha \inf\{s \geq 0 : \Phi(s) > t\} = \alpha\Phi^{-1}(t).$$

The right inequality for  $\alpha > 1$  is a consequence of the one for  $0 < \alpha < 1$ .

As in [24, Property 1.6], we have

$$r \leq \Phi^{-1}(r)\tilde{\Phi}^{-1}(r) \leq 2r \quad \text{for } r \geq 0, \tag{7}$$

where  $\tilde{\Phi}(r)$  is the complementary function of  $\Phi$  defined by

$$\tilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty. \end{cases}$$

Then  $\tilde{\Phi}$  is also a Young function and  $\tilde{\tilde{\Phi}} = \Phi$ .

A Young function  $\Phi$  is said to satisfy the  $\Delta_2$ -condition, denoted also by  $\Phi \in \Delta_2$ , if

$$\Phi(2r) \leq C\Phi(r), \quad r > 0$$

for some  $C \geq 1$ . If  $\Phi \in \Delta_2$ , then  $\Phi \in \mathcal{Y}$ . A Young function  $\Phi$  is said to satisfy the  $\nabla_2$ -condition, denoted also by  $\Phi \in \nabla_2$ , if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \quad r \geq 0$$

for some  $C > 1$ .

We denote by  $\chi_G$  the characteristic function of the set  $G \subset \mathbb{R}^n$ .

**Definition 2** (*Orlicz space*) For a Young function  $\Phi$ , the Orlicz space  $L^\Phi(\mathbb{R}^n)$  is defined by:

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}.$$

The space  $L^\Phi_{\text{loc}}(\mathbb{R}^n)$  is defined as the set of all measurable functions  $f$  such that  $f\chi_B \in L^\Phi(\mathbb{R}^n)$  for all balls  $B \subset \mathbb{R}^n$ .

If  $\Phi$  is a Young function, then  $L^\Phi(\mathbb{R}^n)$  is a Banach space under the Luxemburg–Nakano norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \leq 1 \right\}.$$

For example, if  $\Phi(r) = r^p$ ,  $1 \leq p < \infty$ , then  $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ . If  $\Phi(r) = 0$ , ( $0 \leq r \leq 1$ ) and  $\Phi(r) = \infty$ , ( $r > 1$ ), then  $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ .

For a measurable set  $\Omega \subset \mathbb{R}^n$ , a measurable function  $f$  and  $t > 0$ , let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case  $\Omega = \mathbb{R}^n$ , we abbreviate it to  $m(f, t)$ .

$L^0(\mathbb{R}^n)$  is the set of all measurable functions.

**Definition 3** For a Young function  $\Phi$ , the weak Orlicz space

$$WL^\Phi(\mathbb{R}^n) = \{f \in L^0(\mathbb{R}^n) : \|f\|_{WL^\Phi} < \infty\}$$

is defined by the quasi-norm

$$\|f\|_{WL^\Phi} = \sup_{\lambda > 0} \|\lambda \chi_{(\lambda, \infty)}(|f|)\|_{L^\Phi}.$$

For  $\Omega \subset \mathbb{R}^n$ , let

$$\|f\|_{L^\Phi(\Omega)} := \|f\chi_\Omega\|_{L^\Phi} \quad \text{and} \quad \|f\|_{WL^\Phi(\Omega)} := \|f\chi_\Omega\|_{WL^\Phi}.$$

A tacit understanding is that  $f$  is defined to be zero outside  $\Omega$ .

We note that  $\|f\|_{WL^\Phi(\Omega)} \leq \|f\|_{L^\Phi(\Omega)}$ , that

$$\begin{aligned} \|f\|_{WL^\Phi} &= \sup_{t>0} \Phi(t)m(\Omega, f, t) \\ &= \sup_{t>0} t m(\Omega, f, \Phi^{-1}(t)) \\ &= \sup_{t>0} t m(\Omega, \Phi(|f|), t) \end{aligned}$$

and that

$$\int_{\Omega} \Phi\left(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\Omega)}}\right) dx \leq 1, \quad \sup_{t>0} \Phi(t)m\left(\Omega, \frac{f}{\|f\|_{WL^{\Phi}(\Omega)}}, t\right) \leq 1 \tag{8}$$

according to [13, Proposition 4.2].

The following analogue of the Hölder inequality is well known; see [32] as well as the paper [24, § II] and the textbooks [14,29].

**Theorem 1** *Let  $\Omega \subset \mathbb{R}^n$  be a measurable set and, let  $f$  and  $g$  be measurable functions on  $\Omega$ . For a Young function  $\Phi$  and its complementary function  $\tilde{\Phi}$ , the following inequality is valid:*

$$\int_{\Omega} |f(x)g(x)| dx \leq 2\|f\|_{L^{\Phi}(\Omega)}\|g\|_{L^{\tilde{\Phi}}(\Omega)}.$$

By elementary calculations we have the following property:

**Lemma 1** *Let  $\Phi$  be a Young function and let  $B$  be a set in  $\mathbb{R}^n$  with finite Lebesgue measure. Then*

$$\|X_B\|_{L^{\Phi}} = \|X_B\|_{WL^{\Phi}} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

By Theorem 1, Lemma 1 and (7) we get the following estimate:

**Lemma 2** *For a Young function  $\Phi$  and  $B = B(x, r)$ , the following inequality is valid:*

$$\int_B |f(y)| dy \leq 2|B|\Phi^{-1}(|B|^{-1})\|f\|_{L^{\Phi}(B)}.$$

We recall the boundedness property of  $M$  on Orlicz spaces since we use it later.

**Theorem 2** *Let  $\Phi$  be a Young function.*

- [2, Theorem 1] *The maximal operator  $M$  is bounded from  $L^{\Phi}(\mathbb{R}^n)$  to  $WL^{\Phi}(\mathbb{R}^n)$ , and the inequality*

$$\|Mf\|_{WL^{\Phi}} \leq C_0\|f\|_{L^{\Phi}} \tag{9}$$

*holds with constant  $C_0$  independent of  $f$ .*

- [2, Theorem 1], [11, Corollary 3.3] *The maximal operator  $M$  is bounded on  $L^{\Phi}(\mathbb{R}^n)$ , and the inequality*

$$\|Mf\|_{L^{\Phi}} \leq C_0\|f\|_{L^{\Phi}} \tag{10}$$

*holds with constant  $C_0$  independent of  $f$  if and only if  $\Phi \in \nabla_2$ .*

See the textbooks [14,15,27,29] for more about Orlicz spaces.

### 3 Generalized fractional integrals on Orlicz spaces

The following theorem is one of our main results and gives necessary and sufficient conditions for the boundedness of the operator  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$  and from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$ .

**Theorem 3** *Let  $\Phi, \Psi$  be Young functions.*

1. *Let  $\rho$  satisfy the conditions (1) and (4). Then the condition*

$$\Phi^{-1}(r^{-n}) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \rho(t) \Phi^{-1}(t^{-n}) \frac{dt}{t} \leq C\Psi^{-1}(r^{-n}) \tag{11}$$

*for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , is sufficient for the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (11) is also sufficient for the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$ .*

2. *The condition*

$$\Phi^{-1}(r^{-n}) \int_0^r \frac{\rho(t)}{t} dt \leq C\Psi^{-1}(r^{-n}) \tag{12}$$

*for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , is necessary for the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$  and from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$ .*

3. *Let  $\rho$  satisfy the conditions (1) and (4). Assume the condition*

$$\int_r^\infty \rho(t) \Phi^{-1}(t^{-n}) \frac{dt}{t} \leq C\Psi^{-1}(r^{-n}) \tag{13}$$

*holds for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ . Then condition (12) is necessary and sufficient for the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (12) is necessary and sufficient for the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$ .*

**Remark 2** We cannot replace  $\int_0^r \frac{\rho(t)}{t} dt$  by  $\rho(r)$  in (11), see [23, Section 5].

We need a couple of auxiliary estimates. The following lemma was proved in [6, Lemma 2.1]:

**Lemma 3** *There exist a constant  $C > 0$  such that for all  $x \in B(0, r/2)$  and  $r > 0$ ,*

$$\int_0^{r/2} \frac{\rho(t)}{t} dt \leq CI_\rho \chi_{B(0,r)}(x)$$

*holds.*

**Proposition 1** *Let  $\rho$  satisfy (4). Define*

$$\tilde{\rho}(r) = \int_{k_1 r}^{k_2 r} \rho(s) \frac{ds}{s} \quad (r > 0). \tag{14}$$

Let  $\tau : (0, \infty) \rightarrow (0, \infty)$  be a doubling function in the sense that  $\tau(r) \sim \tau(s)$  if  $0 < s \leq r \leq 2s$ . Then, for each  $r > 0$ ,

$$\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) \lesssim \int_0^{k_2 r} \frac{\rho(s)}{s} ds, \tag{15}$$

$$\sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \tau((2^j r)^{-n}) \lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s} \tau(s^{-n}) ds. \tag{16}$$

**Proof** We invoke the overlapping property in [31] and by Remark 1 we have

$$\begin{aligned} \sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) &= \sum_{j=-\infty}^{-1} \int_{2^j k_1 r}^{2^j k_2 r} \rho(s) \frac{ds}{s} \\ &\leq \int_0^{k_2 r} \left( \sum_{j=-\infty}^{-1} \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} ds \\ &\lesssim \int_0^{k_2 r} \frac{\rho(s)}{s} ds \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \tau((2^j r)^{-n}) &= \int_{k_1 r}^{\infty} \left( \sum_{j=0}^{\infty} \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \frac{\rho(s)}{s} \tau((2^j r)^{-n}) \right) ds \\ &\lesssim \int_{k_1 r}^{\infty} \left( \sum_{j=0}^{\infty} \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} \tau(s^{-n}) ds \\ &\lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s} \tau(s^{-n}) ds. \end{aligned}$$

To prove Theorem 3, we need the following estimate of Hedberg-type [12]:

**Proposition 2** Under the assumption of Theorem 3, for any positive constant  $C_0$ , there exists a positive constant  $C_1$  such that, for all nonnegative functions  $f \in L^\Phi(\mathbb{R}^n)$  with  $f \neq 0$ ,

$$I_\rho f(x) \leq C_1 \|f\|_{L^\Phi} \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \quad (x \in \mathbb{R}^n). \tag{17}$$

**Proof** The idea of the proof comes from [6]. First note that

$$0 < \Phi^{-1}(0) \int_0^\infty \frac{\rho(t)}{t} dt \lesssim \Psi^{-1}(0)$$

as long as  $\Phi^{-1}(0) > 0$ .

Let  $x \in \mathbb{R}^n$ . Keeping in mind that  $Mf(x) > 0$ , we may assume

$$0 < \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} < \infty, \quad 0 \leq \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) < \infty;$$

otherwise there is nothing to prove. If

$$\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) = 0,$$

then

$$\frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \leq \sup\{u \geq 0 : \Phi(u) = 0\} = \Phi^{-1}(0)$$

and hence

$$\begin{aligned} I_\rho f(x) &\leq C \sum_{j=-\infty}^{\infty} \frac{\tilde{\rho}(2^j)}{2^{jn}} \int_{|x-y|<2^j} |f(y)| dy \\ &\leq C \left( \int_0^\infty \frac{\rho(s)}{s} ds \right) Mf(x) \\ &\leq C \frac{\Psi^{-1}(0)}{\Phi^{-1}(0)} Mf(x) \\ &\leq C \frac{1}{\Phi^{-1}(0)} \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) \right) Mf(x) \\ &\leq C \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) \right) \|f\|_{L^\phi}. \end{aligned}$$

So, this case the result is valid.

If

$$\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) > 0,$$

choose  $r \in (0, \infty)$  so that

$$r^{-n} = \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right).$$

We have

$$I_\rho f(x) \leq C \left[ \sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\tilde{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y|<2^j r} f(y) dy \right] = C(I + II)$$

for given  $x \in \mathbb{R}^n$  and  $r > 0$ .

Then from Proposition 1

$$\begin{aligned} \text{I} &\leq C \sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) Mf(x) \leq C \left( \int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) Mf(x) \\ \text{II} &\leq C \sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \Phi^{-1}((2^j r)^{-n}) \|f\|_{L\Phi(B(x, 2^j r))} \\ &\leq C \|f\|_{L\Phi} \int_{k_1 r}^{\infty} \Phi^{-1}(s^{-n}) \frac{\rho(s)}{s} ds. \end{aligned}$$

Consequently, we have

$$I_\rho f(x) \lesssim \left( \int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) Mf(x) + \|f\|_{L\Phi} \int_{k_1 r}^{\infty} \Phi^{-1}(s^{-n}) \frac{\rho(s)}{s} ds.$$

Thus, by the doubling property of  $\Phi^{-1}$  and  $\Psi^{-1}$ , (11) and Remark 1 we obtain

$$\begin{aligned} I_\rho f(x) &\lesssim Mf(x) \frac{\Psi^{-1}((k_2 r)^{-n})}{\Phi^{-1}((k_2 r)^{-n})} + \|f\|_{L\Phi} \Psi^{-1}((k_1 r)^{-n}) \\ &\lesssim Mf(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \|f\|_{L\Phi} \Psi^{-1}(r^{-n}). \end{aligned}$$

Recall that  $\Phi^{-1}(\Phi(r)) = r$  if  $0 < \Phi(r) < \infty$ . Thus  $\Phi^{-1}(r^{-n}) = \frac{Mf(x)}{C_0 \|f\|_{L\Phi}}$  and

$$I_\rho f(x) \lesssim \|f\|_{L\Phi} \Psi^{-1}(r^{-n}) = \|f\|_{L\Phi} \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L\Phi}} \right) \right).$$

Therefore, we get (17).

Now we move on to the proof of Theorem 3. The third statement is a consequence of the remaining statements. So we concentrate on the first and the second ones.

- Let  $C_0$  be as in (9). Let  $f$  be a non-negative measurable function. Then by (9) and (17),

$$\begin{aligned} \sup_{r>0} \Psi(r) m \left( \frac{I_\rho f(x)}{C_1 \|f\|_{L\Phi}}, r \right) &= \sup_{r>0} r m \left( \Psi \left( \frac{I_\rho f(x)}{C_1 \|f\|_{L\Phi}} \right), r \right) \\ &\leq \sup_{r>0} r m \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L\Phi}} \right), r \right) \leq \sup_{r>0} \Phi(r) m \left( \frac{Mf(x)}{\|Mf\|_{W_L\Phi}}, r \right) \leq 1, \end{aligned}$$

i.e.

$$\|I_\rho f\|_{WL^\Psi} \lesssim \|f\|_{L^\Phi}.$$

– Assume in addition that  $\Phi \in \nabla_2$ , so that we have (10). By (10), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Psi \left( \frac{I_\rho f(x)}{C_1 \|f\|_{L^\Phi}} \right) dx &\leq \int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) dx \\ &\leq \int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{\|Mf\|_{L^\Phi}} \right) dx \leq 1, \end{aligned}$$

i.e.

$$\|I_\rho f\|_{L^\Psi} \lesssim \|f\|_{L^\Phi}.$$

– We can and do concentrate on the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ , since the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$  is stronger than the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ . With this in mind, assume that  $I_\rho$  is bounded from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ .

Then we have by Lemma 3

$$\int_0^{r/2} \frac{\rho(s)}{s} ds \|\chi_{B(0,r/2)}\|_{WL^\Psi(B(0,r/2))} \lesssim \|I_\rho \chi_{B(0,r)}\|_{WL^\Psi(B(0,r/2))}.$$

Therefore, by the doubling property of  $\Phi^{-1}$  and Lemma 1, we have

$$\begin{aligned} \int_0^{r/2} \frac{\rho(s)}{s} ds &\lesssim \Psi^{-1}(r^{-n}) \|I_\rho \chi_{B(0,r)}\|_{WL^\Psi(B(0,r/2))} \\ &\lesssim \Psi^{-1}(r^{-n}) \|I_\rho \chi_{B(0,r)}\|_{WL^\Psi} \\ &\lesssim \Psi^{-1}(r^{-n}) \|\chi_{B(0,r)}\|_{L^\Phi} \\ &\lesssim \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})}. \end{aligned}$$

**Remark 3** In [19, Corollary 3.2] the third author found the sufficient conditions which ensures the boundedness of the operator  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$ , including its weak version. Theorem 3 improves the third author’s result in that Theorem 3 also covers the necessity by imposing a weaker condition on  $\rho$ .

**Remark 4** In the case  $\Phi(t) = t^p$ , Theorem 3 was proved in [6, Corollary 1.5].

**Example 1** Let  $\rho$  be as in (5) and

$$\Phi(t) = \begin{cases} t^{3/2}, & 0 \leq t \leq 1, \\ t(\log(et))^{1/2}, & t > 1, \end{cases} \quad \Psi(t) = \begin{cases} \frac{2e}{3} t^{3/2}, & 0 \leq t \leq 1, \\ \frac{2e}{3} \frac{\exp \exp(t)}{\exp \exp(1)}, & t > 1. \end{cases}$$

Then the pair  $(\rho, \Phi, \Psi)$  satisfies (11). In fact, we have

$$\begin{aligned}\Phi^{-1}(u) &\sim \begin{cases} u^{2/3}, & 0 \leq u \leq 1, \\ u(\log(eu))^{-1/2}, & u > 1, \end{cases} \\ \Psi^{-1}(u) &\sim \begin{cases} u^{2/3}, & 0 \leq u \leq 1, \\ \log(\log(e^e u)), & u > 1, \end{cases}\end{aligned}$$

and for all  $r > 0$

$$\begin{aligned}\int_0^r \frac{\rho(t)}{t} dt \Phi^{-1}(1/r^n) &\lesssim \min(1, r^{-2n/3}), \\ \int_r^\infty \frac{\rho(t) \Phi^{-1}(1/t^n)}{t} dt &\lesssim \min(\log \log(e^e/r), r^{-2n/3}), \\ \Psi^{-1}(1/r^n) &\sim \min(\log \log(e^e/r), r^{-2n/3}).\end{aligned}$$

See [20] for other examples.

#### 4 Generalized fractional maximal operators on Orlicz spaces

We recall that, for a function  $\rho : (0, \infty) \rightarrow (0, \infty)$ ,  $M_\rho$  is defined by (6). Here we suppose that  $\rho$  is an increasing function such that  $r \in (0, \infty) \mapsto r^{-n} \rho(r) \in (0, \infty)$  is decreasing.

Under this assumption, we have the following localized estimate:

**Lemma 4** *There exists a positive constant  $C$  such that, for all balls  $B = B(x, r)$  and all measurable functions  $f$  supported on  $B$ ,*

$$M_\rho f(x) \leq C \rho(r) Mf(x). \quad (18)$$

**Proof** Let  $B(R) = B(x, R)$  with  $x = 0$  for  $R > 0$ . By the definition of  $M_\rho$ , we have

$$\begin{aligned}M_\rho f(x) &= \max \left\{ \sup_{0 < R < 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| dy, \sup_{R \geq 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| dy \right\}.\end{aligned}$$

For the first term, we use the fact that  $\rho$  is increasing and doubling to have

$$\begin{aligned}\sup_{0 < R < 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| dy &\lesssim \sup_{0 < R < 3r} \frac{\rho(r)}{|B(x, R)|} \int_{B(x, R)} |f(y)| dy \\ &\leq \rho(r) Mf(x).\end{aligned}$$

For the second term, since  $r \in (0, \infty) \mapsto r^{-n}\rho(r) \in (0, \infty)$  is decreasing and  $f$  is supported on  $B(x, r)$

$$\begin{aligned} \sup_{R \geq 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| dy &\leq \sup_{R \geq 3r} \frac{\rho(3r)}{|B(x, 3r)|} \int_{B(x, R)} |f(y)| dy \\ &\leq \rho(r) Mf(x). \end{aligned}$$

Thus, combining these estimates, we obtain the desired result.

The Hedberg inequality for  $M_\rho$  and  $L^\Phi$  can be stated as follows:

**Lemma 5** *Let  $\Phi, \Psi$  be Young functions. Assume that there exists a positive constant  $C$  such that, for all  $r > 0$ ,*

$$\rho(r) \leq C \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})}. \tag{19}$$

*Then, for any positive constant  $C_0$ , there exists a positive constant  $C_1$  such that, for all  $f \in L^\Phi(\mathbb{R}^n)$  with  $f \neq 0$ ,*

$$M_\rho f(x) \leq C_1 \|f\|_{L^\Phi} (\Psi^{-1} \circ \Phi) \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \quad (x \in \mathbb{R}^n). \tag{20}$$

**Proof** First note that

$$\lim_{r \rightarrow \infty} \rho(r) \lesssim \frac{\Psi^{-1}(0)}{\Phi^{-1}(0)} \tag{21}$$

if  $\Phi^{-1}(0) > 0$ . Let  $x \in \mathbb{R}^n$  be an arbitrary point. We may assume that  $0 < Mf(x) < \infty$  keeping in mind that  $f$  does not vanish on a set of positive measure. Furthermore, we can assume that

$$\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) < \infty;$$

otherwise there is nothing to do since  $\Psi^{-1}(\infty) = \infty$ . If

$$\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) = 0,$$

then

$$\Phi^{-1}(0) \geq \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} > 0$$

according to the definition of  $\Phi^{-1}$ . Thus, thanks to (21)

$$\begin{aligned} \lim_{r \rightarrow \infty} \rho(r) &\lesssim \frac{\Psi^{-1}(0)}{\Phi^{-1}(0)} \\ &= \frac{1}{\Phi^{-1}(0)} \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \\ &\leq \frac{C_0 \|f\|_{L^\Phi}}{Mf(x)} \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right). \end{aligned}$$

Thus by (18) we have

$$M_\rho f(x) \lesssim \lim_{r \rightarrow \infty} \rho(r) Mf(x) \lesssim \|f\|_{L^\Phi} \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right).$$

It thus remains to handle the case where

$$0 < \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) < \infty.$$

In the case we can choose  $r > 0$  such that

$$r^{-n} = \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right).$$

Let  $B = B(x, r)$  and represent  $f$  as

$$f = f_1 + f_2, \quad f_1 = f \chi_B, \quad f_2 = f \chi_{\mathbb{R}^n \setminus B}$$

so that  $M_\rho f(x) \leq M_\rho f_1(x) + M_\rho f_2(x)$ .

We have (18) for  $f_1$ . Meanwhile by Lemma 2,

$$\begin{aligned} M_\rho f_2(x) &= \sup_{t>0} \frac{\rho(t)}{|B(x, t)|} \int_{B(x, t) \cap \mathbb{R}^n \setminus B(x, r)} |f(z)| dz \\ &= \sup_{r < t < \infty} \frac{\rho(t)}{|B(x, t)|} \int_{B(x, t)} |f(z)| dz \\ &\lesssim \sup_{r < t < \infty} \rho(t) \Phi^{-1}(|B(x, t)|^{-1}) \|f\|_{L^\Phi(B(x, t))} \\ &\lesssim \|f\|_{L^\Phi} \sup_{r < t < \infty} \rho(t) \Phi^{-1}(t^{-n}). \end{aligned}$$

Consequently we have by Lemma 4

$$M_\rho f(x) \lesssim \rho(r) Mf(x) + \|f\|_{L^\Phi} \sup_{r < t < \infty} \rho(t) \Phi^{-1}(t^{-n}).$$

Thus, by (19) and the monotonicity of  $\Psi^{-1}$  we obtain

$$M_\rho f(x) \lesssim Mf(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}).$$

Since  $\Phi^{-1}(r^{-n}) = \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}$ , we have

$$M_\rho f(x) \lesssim \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}) = \|f\|_{L^\Phi} \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \right).$$

Therefore, we get (20).

In [10] we obtain a counterpart to generalized Orlicz–Morrey spaces of the second kind defined in [7]. However, as is written in [7] generalized Orlicz–Morrey spaces of the second kind do not cover  $L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ . So, the following theorem can be viewed as a different theorem from [7]:

**Theorem 4** *Let  $\Phi, \Psi$  be Young functions. Assume that  $\rho$  is increasing and that  $r \mapsto r^{-n} \rho(r)$  is decreasing. Then the condition (19) is necessary and sufficient for the boundedness of  $M_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (19) is necessary and sufficient for the boundedness of  $M_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$ .*

**Proof** We start with the necessity. For the necessity, we can concentrate on the boundedness of  $M_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ , since the boundedness of  $M_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $L^\Psi(\mathbb{R}^n)$  is stronger than the boundedness of  $M_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ . With this in mind, assume that  $M_\rho$  is bounded from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$ . We utilize a trivial pointwise estimate

$$\rho(r) \chi_{B(0,r)} \leq M_\rho \chi_{B(0,2r)}. \tag{22}$$

Therefore, by the doubling property of  $\Phi^{-1}$  and Lemma 1, we have

$$\begin{aligned} \rho(r) &\lesssim \Psi^{-1}(r^{-n}) \|M_\rho \chi_{B(0,2r)}\|_{WL^\Psi(B(0,r))} \\ &\lesssim \Psi^{-1}(r^{-n}) \|M_\rho \chi_{B(0,2r)}\|_{WL^\Psi} \\ &\lesssim \Psi^{-1}(r^{-n}) \|\chi_{B(0,2r)}\|_{L^\Phi} \\ &\lesssim \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})}. \end{aligned}$$

We move on to the sufficiency. Here and below we let  $f$  be a nonzero measurable function.

– Let  $C_0$  be as in (9). Then by (9) and (20), we have

$$\begin{aligned} \sup_{r>0} \Psi(r) m\left(\frac{M_\rho f(y)}{C_1 \|f\|_{L^\Phi}}, r\right) &= \sup_{r>0} r m\left(\Psi\left(\frac{M_\rho f(y)}{C_1 \|f\|_{L^\Phi}}\right), r\right) \\ &\leq \sup_{r>0} r m\left(\Phi\left(\frac{Mf(y)}{C_0 \|f\|_{L^\Phi}}\right), r\right) \leq \sup_{r>0} \Phi(r) m\left(\frac{Mf(y)}{\|Mf\|_{WL^\Phi}}, r\right) \leq 1, \end{aligned}$$

i.e.

$$\|M_\rho f\|_{WL^\Psi} \lesssim \|f\|_{L^\Phi}. \tag{23}$$

– Assume in addition that  $\Phi \in \nabla_2$ . Let  $C_0$  be as in (10). By (10) and (20), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Psi\left(\frac{M_\rho f(y)}{C_1 \|f\|_{L^\Phi}}\right) dy &\leq \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(y)}{C_0 \|f\|_{L^\Phi}}\right) dy \\ &\leq \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(y)}{\|Mf\|_{L^\Phi}}\right) dy \leq 1, \end{aligned}$$

i.e.

$$\|M_\rho f\|_{L^\Psi} \lesssim \|f\|_{L^\Phi}. \tag{24}$$

### 5 Generalized Orlicz–Morrey spaces of the third kind

In [3], the generalized Orlicz–Morrey space  $M^{\Phi, \varphi}(\mathbb{R}^n)$  was introduced to unify Orlicz spaces and generalized Morrey spaces. Other definitions of generalized Orlicz–Morrey spaces can be found in [22,31]. In words of [8], our generalized Orlicz–Morrey space is the third kind and the ones in [22,31] are the first kind and the second kind, respectively. Notice that the definition of the space of the third kind relies only on the fact that  $L^\Phi(\mathbb{R}^n)$  is a normed linear space, which is independent of the condition that it is generated by modulars.

The definition of generalized Orlicz–Morrey spaces of the third kind is as follows:

**Definition 4** Let  $\varphi$  be a positive measurable function on  $(0, \infty)$  and  $\Phi$  any Young function. We denote by  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  the generalized Orlicz–Morrey space of the third kind, the space of all functions  $f \in L_{loc}^\Phi(\mathbb{R}^n)$  with finite norm

$$\|f\|_{\mathcal{M}^{\Phi, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x,r))}.$$

Also by  $WM^{\Phi, \varphi}(\mathbb{R}^n)$  we denote the weak generalized Orlicz–Morrey space of the third kind of all measurable functions  $f \in WL_{loc}^\Phi(\mathbb{R}^n)$  for which

$$\|f\|_{WM^{\Phi, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{WL^\Phi(B(x,r))} < \infty.$$

A function  $\varphi : (0, \infty) \rightarrow (0, \infty)$  is said to be almost increasing (resp. almost decreasing) if there exists a constant  $C > 0$  such that

$$\varphi(r) \leq C\varphi(s) \quad (\text{resp. } \varphi(r) \geq C\varphi(s)) \quad \text{for } r \leq s.$$

For a Young function  $\Phi$ , we denote by  $\mathcal{G}_\Phi$  the set of all  $\varphi : (0, \infty) \rightarrow (0, \infty)$  functions such that  $t \in (0, \infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-n})}$  is almost increasing and  $t \in (0, \infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-n})t^n}$  is almost decreasing. Note that  $\varphi \in \mathcal{G}_\Phi$  implies doubling condition of  $\varphi$ .

We investigate the structure of  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$ . We denote by  $\Theta$  the set of all measurable functions equivalent to 0 on  $\mathbb{R}^n$ . To exclude some trivial cases, we can use the following lemma was proved in [4]:

**Lemma 6** *Let  $\Phi$  be a Young function and  $\varphi$  be a positive measurable function on  $(0, \infty)$ .*

(i) *If*

$$\sup_{t < r < \infty} \frac{\Phi^{-1}(r^{-n})}{\varphi(r)} = \infty \quad \text{for some } t > 0 \tag{25}$$

*then  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) = \Theta$ .*

(ii) *If*

$$\sup_{0 < r < \tau} \varphi(r)^{-1} = \infty \quad \text{for some } \tau > 0 \tag{26}$$

*then  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) = \Theta$ .*

**Remark 5** *If*

$$\sup_{0 < r \leq t} \frac{\Phi^{-1}(r^{-n})r^n}{\varphi(r)} = \infty \quad \text{for some } t > 0,$$

then  $\mathcal{M}^{\Phi, \varphi} = \Theta$ . Actually, by Remark 1 one has

$$\Phi^{-1}(r^{-n})r^n \leq \Phi^{-1}(t^{-n})t^n < \infty$$

and then

$$\sup_{0 < r < t} \varphi(r)^{-1} = \infty.$$

**Remark 6** Based on Lemma 6 and Remark 5 and an observation similar to the one made by Nakai [17, p. 446] it can be assumed that  $\varphi \in \mathcal{G}_\Phi$  in the definition of  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$ . More explicitly, we have the following observation:

(i) By Lemma 6 we may assume that  $\inf_{r \leq t < \infty} \frac{\varphi(t)}{\Phi^{-1}(t^{-n})} > 0$  for every  $r > 0$ . Let

$$\psi(r) = \Phi^{-1}(r^{-n}) \inf_{r \leq t < \infty} \frac{\varphi(t)}{\Phi^{-1}(t^{-n})}, \quad r > 0.$$

Then  $r \in (0, \infty) \mapsto \frac{\psi(r)}{\Phi^{-1}(r^{-n})}$  is increasing and  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) = \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n)$  with equivalent norms. Indeed, it is clear that  $\psi(r) \leq \varphi(r)$  by the definition of  $\psi$ . Hence  $\mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \subset \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  and  $\|f\|_{\mathcal{M}^{\Phi, \varphi}} \leq \|f\|_{\mathcal{M}^{\Phi, \psi}}$ . On the other hand,

$$\begin{aligned} \sup_{r > 0} \psi(r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x, r))} &= \sup_{r > 0} \frac{1}{\inf_{r \leq t < \infty} \frac{\varphi(t)}{\Phi^{-1}(t^{-n})}} \|f\|_{L^\Phi(B(x, r))} \\ &= \sup_{r > 0} \left( \sup_{r \leq t < \infty} \frac{\Phi^{-1}(t^{-n})}{\varphi(t)} \right) \|f\|_{L^\Phi(B(x, r))} \\ &\leq \sup_{t > 0} \varphi(t)^{-1} \Phi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x, t))}. \end{aligned}$$

Hence  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \subset \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n)$  and  $\|f\|_{\mathcal{M}^{\Phi, \psi}} \leq \|f\|_{\mathcal{M}^{\Phi, \varphi}}$ .

(ii) By Remark 5 we may assume that  $\inf_{0 < t \leq r} \frac{\varphi(t)}{\Phi^{-1}(t^{-n})t^n} > 0$  for every  $r > 0$ . Define  $\psi(r)$  by the formula:

$$\sup_{t \in (0, r]} \frac{\Phi^{-1}(t^{-n})t^n}{\varphi(t)} = \frac{\Phi^{-1}(r^{-n})r^n}{\psi(r)}.$$

It is easy to see that  $\psi(r) \leq \varphi(r)$  for any  $r > 0$ . Thus,  $\mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \subset \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  and  $\|f\|_{\mathcal{M}^{\Phi, \varphi}} \leq \|f\|_{\mathcal{M}^{\Phi, \psi}}$ . Conversely, let  $f \in \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$ . For any  $r \in (0, \infty)$ , choose  $t \in (0, r]$  so that

$$\frac{\Phi^{-1}(r^{-n})r^n}{\psi(r)} \leq 2 \frac{\Phi^{-1}(t^{-n})t^n}{\varphi(t)},$$

and cover  $B(x, r)$  with a family of  $N$  balls  $\{B(x_j, t)\}_{j=1}^N$ , where  $N \lesssim r^{-n}t^n$ . Let  $j_0$  be such that

$$\begin{aligned} \|f\|_{L^\Phi(B(x, r))} &\leq N \|f\|_{L^\Phi(B(x_{j_0}, t))} \\ &= N \max_{j=1, 2, \dots, N} \|f\|_{L^\Phi(B(x_j, t))} \\ &\lesssim t^{-n}r^n \|f\|_{L^\Phi(B(x_{j_0}, r))}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\Phi^{-1}(r^{-n})}{\psi(r)} \|f\|_{L^\Phi(B(x,r))} &\lesssim \frac{r^n \Phi^{-1}(r^{-n})}{\psi(r)t^n} \|f\|_{L^\Phi(B(x_{j_0},t))} \\ &\leq 2 \frac{\Phi^{-1}(t^{-n})}{\varphi(t)} \|f\|_{L^\Phi(B(x_{j_0},t))} \\ &\leq 2 \|f\|_{\mathcal{M}^{\Phi,\varphi}}, \end{aligned}$$

implying  $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  and  $\|f\|_{\mathcal{M}^{\Phi,\psi}} \leq \|f\|_{\mathcal{M}^{\Phi,\varphi}}$ . Thus,  $\mathcal{M}^{\Phi,\psi}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ .

As the following lemma shows,  $\mathcal{G}_\Phi$  is useful:

**Lemma 7** ([5]) *Let  $B_0 := B(x_0, r_0)$ . If  $\varphi \in \mathcal{G}_\Phi$  is almost decreasing, then there exist  $C > 0$  such that*

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \leq \frac{C}{\varphi(r_0)}.$$

## 6 Generalized fractional integrals on generalized Orlicz–Morrey spaces

We remark that there are two types of the boundedness of the fractional integral operators. One is the Spanne-type boundedness obtained in [28]. Another boundedness is of Adams-type obtained by Adams [1]. In the classical case due to the fact that Morrey spaces are nested we can say that the Adams-type boundedness is stronger than the Spanne-type boundedness. However, we need to depend on the pointwise estimate of Hedberg-type [12], so the Adams-type boundedness is unavailable for local Morrey spaces. In this section we give a characterization for the Spanne-type boundedness and the Adams-type boundedness of the operator  $I_\rho$  on generalized Orlicz–Morrey spaces, respectively.

### 6.1 Spanne-type result

We need the following lemma is valid:

**Lemma 8** *Let  $\Phi, \Psi$  be Young functions, and let  $\rho$  satisfy (1) and (4). Assume that the condition (11) is fulfilled. Then there exists a positive constant  $C$  such that, for all  $f \in L^\Phi_{loc}(\mathbb{R}^n)$  and  $B = B(x, r)$ ,*

$$\begin{aligned} &\|I_\rho f\|_{\mathcal{WL}^\Psi(B)} \\ &\leq C \|f\|_{L^\Phi(2B)} + \frac{C}{\Psi^{-1}(r^{-n})} \int_{2k_1r}^\infty \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}. \end{aligned} \tag{27}$$

Moreover if we assume  $\Phi \in \nabla_2$ , the following inequality is also valid:

$$\begin{aligned} & \|I_\rho f\|_{L^\Psi(B)} \\ & \leq C \|f\|_{L^\Phi(2B)} + \frac{C}{\Psi^{-1}(r^{-n})} \int_{2k_1r}^\infty \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}. \end{aligned} \tag{28}$$

**Proof** We represent  $f$  as

$$f = f_1 + f_2, \quad f_1 = f \chi_{2B}, \quad f_2 = f - f_1, \quad r > 0.$$

Then we have

$$\|I_\rho f\|_{WL^\Psi(B)} \leq 2(\|I_\rho f_1\|_{WL^\Psi(B)} + \|I_\rho f_2\|_{WL^\Psi(B)}).$$

From the boundedness of  $I_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$  (see Theorem 3) it follows that:

$$\|I_\rho f_1\|_{WL^\Psi(B)} \lesssim \|I_\rho f_1\|_{WL^\Psi(\mathbb{R}^n)} \leq C \|f_1\|_{L^\Phi(\mathbb{R}^n)} = C \|f\|_{L^\Phi(2B)}, \tag{29}$$

where constant  $C > 0$  is independent of  $f$ .

For  $f_2$  we have

$$|I_\rho f_2(y)| \leq \int_{\mathbb{C}(2B)} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)| dz = \sum_{j=1}^\infty \int_{2^{j+1}B \setminus 2^jB} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)| dz.$$

A geometric observation shows that  $y \in B, z \in \mathbb{R}^n \setminus (2B)$  implies

$$\frac{1}{2}|x-z| \leq |y-z| \leq \frac{3}{2}|x-z|.$$

Using (4) and Lemma 2, we have

$$\begin{aligned} & \int_{2^{j+1}B \setminus 2^jB} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)| dz \\ & \lesssim \left( \sup_{2^{j-1}r \leq t \leq 3 \cdot 2^j r} \rho(t) \right) \frac{1}{|2^jB|} \int_{2^{j+1}B \setminus 2^jB} |f(z)| dz \\ & \lesssim \int_{2^j k_1 r}^{3 \cdot 2^j k_2 r} \frac{\rho(t)}{t} dt \|f\|_{L^\Phi(B(x, 2^{j+1}r))} \Phi^{-1}(|2^{j+1}B|^{-1}) \\ & \lesssim \int_{2^j k_1 r}^{2^j \max(3k_2, 4)r} \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \frac{\rho(t)}{t} dt. \end{aligned}$$

Then

$$|I_\rho f_2(y)| \lesssim \int_{2k_1r}^\infty \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \frac{\rho(t)}{t} dt. \tag{30}$$

Thus by Lemma 1 we have

$$\|I_\rho f_2\|_{WL^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})} \int_{2k_1r}^\infty \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n})\rho(t) \frac{dt}{t}. \tag{31}$$

Therefore we obtain (27) by (29) and (31).

If  $\Phi \in \nabla_2$ , then we can use strong type inequality instead of (29) and obtain (28) by using the same argument.

**Remark 7** In the case  $\Phi(t) = t^p$  ( $1 \leq p < \infty$ ) Lemma 8 was proved in [9].

The following theorem gives necessary and sufficient conditions for Spanne-type boundedness of the operator  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ .

**Theorem 5** (Spanne-type result) *Let  $\Phi, \Psi$  be Young functions, and let  $\varphi_1 \in \mathcal{G}_\Phi$  and  $\varphi_2 \in \mathcal{G}_\Psi$ .*

1. *Let  $\rho$  satisfy (1) and (4). Assume that (11) is fulfilled. Then the conditions*

$$\frac{\varphi_1(r)}{\Phi^{-1}(r^{-n})} \leq C \frac{\varphi_2(r)}{\Psi^{-1}(r^{-n})}, \tag{32}$$

$$\int_r^\infty \varphi_1(t)\rho(t) \frac{dt}{t} \leq C \varphi_2(r), \tag{33}$$

for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , are sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{WM}^{1,\varphi_2}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the conditions (32) and (33) are sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ .

2. *Let  $\varphi_1$  be almost decreasing. Then the condition*

$$\varphi_1(r) \int_0^r \frac{\rho(t)}{t} dt \leq C \varphi_2(r), \tag{34}$$

for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , is necessary for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{WM}^{1,\varphi_2}(\mathbb{R}^n)$  and hence  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ .

3. *Let  $\rho$  satisfy (1) and (4). Assume that (11) is fulfilled, that  $\varphi_1$  is almost decreasing and that  $\varphi_1$  and  $\varphi_2$  satisfy (32). Assume also that  $\varphi_1$  and  $\rho$  satisfy the condition*

$$\int_r^\infty \varphi_1(t)\rho(t) \frac{dt}{t} \leq C \varphi_1(r)\rho(r), \tag{35}$$

for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ . Then the condition (34) is necessary and sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{WM}^{\Psi,\varphi_2}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (34) is necessary and sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$ .

**Proof 1.** By (27), (32) and (33) we have

$$\begin{aligned} \|I_\rho f\|_{\mathbb{W}\mathcal{M}^{\psi,\varphi_2}} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \Psi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x,2r))} \\ &\quad + \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \int_{2k_1r}^\infty \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t} \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x,r))} \\ &\quad + \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \int_{2k_1r}^\infty \varphi_1(t)^{-1} \rho(t) \frac{dt}{t} \|f\|_{\mathcal{M}^{\Phi,\varphi_1}} \\ &\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}}. \end{aligned}$$

Simply replace  $\mathbb{W}L^\psi(B)$  with  $L^\psi(B)$  and  $\mathbb{W}\mathcal{M}^{\psi,\eta}(\mathbb{R}^n)$  with  $\mathcal{M}^{\psi,\eta}(\mathbb{R}^n)$  and use (28), (32) and (33) for the strong estimate.

2. We will now prove the second part. Let  $B_R = B(0, R)$  and  $x \in B_{R/2}$ . By Lemma 3 we have

$$\rho^*(R/2) := \int_0^{R/2} \frac{\rho(t)}{t} dt \leq C I_\rho \chi_{B_R}(x).$$

Therefore, by Lemma 7 and the doubling property of  $\varphi_1$ ,

$$\begin{aligned} \rho^*(R/2) &\lesssim |B_{R/2}|^{-1} \|I_\rho \chi_{B_R}\|_{\mathbb{W}L^1(B_{R/2})} \lesssim \varphi_2(R/2) \|I_\rho \chi_{B_R}\|_{\mathbb{W}\mathcal{M}^{1,\varphi_2}} \\ &\lesssim \varphi_2(R/2) \|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(R/2)}{\varphi_1(R)} \lesssim \frac{\varphi_2(R/2)}{\varphi_1(R/2)}. \end{aligned}$$

Since this is true for every  $R > 0$ , we are done.

3. The third statement of the theorem follows from the first and second parts of the theorem.

### 6.2 Adams-type result

The following theorem was proved in [3, Theorem 4.6]:

**Theorem 6 1.** Let  $\varphi \in \mathcal{G}_\Phi$  be almost decreasing. Then the maximal operator  $M$  is bounded from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathbb{W}\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ .

2. Let  $\Phi \in \nabla_2$  and  $\varphi \in \mathcal{G}_\Phi$  be almost decreasing. Then the maximal operator  $M$  is bounded on  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ .

The following theorem gives necessary and sufficient conditions for Adams-type boundedness of the operator  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{M}^{\psi,\eta}(\mathbb{R}^n)$ :

**Theorem 7** (Adams-type result) Let  $\Phi$  be a Yougn function, and let  $\varphi \in \mathcal{G}_\Phi$  be almost decreasing. Let  $\beta \in (0, 1)$  and define  $\eta(t) \equiv \varphi(t)^\beta$  for  $t > 0$  and  $\Psi(t) \equiv \Phi(t^{1/\beta})$  for  $t > 0$ .

1. Let  $\rho$  satisfy (1) and (4). Then the condition

$$\varphi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \rho(t) \varphi(t) \frac{dt}{t} \leq C\eta(r), \tag{36}$$

for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , is sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{WM}^{\Psi,\eta}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (36) is sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ .

2. The condition

$$\varphi(r) \int_0^r \frac{\rho(t)}{t} dt \leq C\eta(r), \tag{37}$$

for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , is necessary for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{WM}^{1,\eta}(\mathbb{R}^n)$  and hence for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ .

3. Let  $\rho$  satisfy (1) and (4). Assume that  $\varphi$  satisfies the condition

$$\int_r^\infty \rho(t) \varphi(t) \frac{dt}{t} \leq C\rho(r)\varphi(r), \tag{38}$$

for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ . Then the condition (37) is necessary and sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{WM}^{\Psi,\eta}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (37) is necessary and sufficient for the boundedness of  $I_\rho$  from  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$ .

We notice that the function  $\varphi$  and  $\eta$  come into play, unlike Spanne-type. Similar to Lemma 5, we have the following pointwise estimate:

**Lemma 9** Let  $\Phi$  be a Young function,  $\varphi \in \mathcal{G}_\Phi$ ,  $\beta \in (0, 1)$ ,  $\eta(t) \equiv \varphi(t)^\beta$  and  $\Psi(t) \equiv \Phi(t^{1/\beta})$ . If (36) holds, then there exists a positive constant  $C$  such that, for all non-negative measurable functions  $f$  and for every  $x \in \mathbb{R}^n$ ,

$$I_\rho f(x) \leq C(Mf(x))^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}. \tag{39}$$

**Proof** Let  $\tilde{\rho}$  be defined by (14). We have

$$I_\rho f(x) \leq C \left[ \sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\tilde{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y| < 2^j r} f(y) dy \right] = C(\text{I} + \text{II})$$

for given  $x \in \mathbb{R}^n$  and  $r > 0$ . Thus from (15) and (16) with  $\tau = \varphi$  we deduce

$$\begin{aligned} \text{I} &\leq C \sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) Mf(x) \leq C \left( \int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) Mf(x) \\ \text{II} &\leq C \sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \varphi(2^j r) \|f\|_{\mathcal{M}^{\Phi,\varphi}} \leq C \|f\|_{\mathcal{M}^{\Phi,\varphi}} \int_{k_1 r}^\infty \varphi(s) \frac{\rho(s)}{s} ds. \end{aligned}$$

Consequently we have

$$I_\rho f(x) \lesssim \left( \int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) Mf(x) + \|f\|_{\mathcal{M}^{\Phi,\varphi}} \int_{k_1 r}^\infty \varphi(s) \frac{\rho(s)}{s} ds.$$

Thus, the technique in [30, p. 6492] by (36) and the doubling property of  $\varphi$  we obtain

$$\begin{aligned} I_\rho f(x) &\lesssim \min \{ \varphi(r)^{\beta-1} Mf(x), \varphi(r)^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}} \} \\ &\lesssim \sup_{s>0} \min \{ s^{\beta-1} Mf(x), s^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}} \} \\ &= (Mf(x))^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}, \end{aligned}$$

where we have used that the supremum is achieved when the minimum parts are balanced. Hence we have  $I_\rho f(x) \lesssim (Mf(x))^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}$ .

We have the following scaling law:

**Lemma 10** *Let  $\beta > 0$ . Let  $\Psi$  and  $\Phi$  be Yougn functions, and let  $B$  be a ball. Then  $\| |f|^\beta \|_{L^\Psi(B)} = \|f\|_{L^\Phi(B)}^\beta$  and  $\| |f|^\beta \|_{WL^\Psi(B)} = \|f\|_{WL^\Phi(B)}^\beta$  for all measurable functions  $f$ .*

**Proof** Simply note that

$$\int_B \Psi \left( \frac{|f(x)|^\beta}{\|f\|_{L^\Phi(B)}^\beta} \right) dx = \int_B \Phi \left( \frac{|f(x)|}{\|f\|_{L^\Phi(B)}} \right) dx$$

for  $L^\Phi(B)$ . The equality for weak spaces can be proved similarly.

**Proof of Theorem 7 1.**

- We deal with the weak-type estimate. By using inequality (39) we have for an arbitrary ball  $B$

$$\|I_\rho f\|_{WL^\Psi(B)} \lesssim \|(Mf)^\beta\|_{WL^\Psi(B)} \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}.$$

Consequently by using this inequality and Lemma 10 we have

$$\|I_\rho f\|_{WL^\Psi(B)} \lesssim \|Mf\|_{WL^\Phi(B)}^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta}. \tag{40}$$

From Theorem 6 and (40), we get

$$\begin{aligned} \|I_\rho f\|_{W\mathcal{M}^{\Psi,\eta}} &= \sup_B \eta(r)^{-1} \Psi^{-1}(r^{-n}) \|I_\rho f\|_{WL^\Psi(B)} \\ &\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \sup_B \eta(r)^{-1} \Psi^{-1}(r^{-n}) \|Mf\|_{WL^\Phi(B)}^\beta \\ &= \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \left( \sup_B \varphi(r)^{-1} \Phi^{-1}(r^{-n}) \|Mf\|_{WL^\Phi(B)} \right)^\beta \\ &\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi}}. \end{aligned}$$

- Simply replace  $WL^\Psi(B)$  with  $L^\Psi(B)$  and  $WM^{\Psi,\eta}(\mathbb{R}^n)$  with  $M^{\Psi,\eta}(\mathbb{R}^n)$  for the strong estimate.

2. We will now prove the second part. Let  $B_R = B(0, R)$  and  $x \in B_{R/2}$ . By Lemmas 1, 3 and 7 and the doubling property of  $\varphi$ , we have

$$\begin{aligned} \rho^*(R/2) &\leq C|B_{R/2}|^{-1} \|I_\rho \chi_{B_R}\|_{WL^1(B_{R/2})} \leq C\eta(R/2) \|I_\rho \chi_{B_R}\|_{WM^{1,\eta}} \\ &\leq C\eta(R/2) \|\chi_{B_R}\|_{M^{\Phi,\varphi}} \leq C \frac{\eta(R/2)}{\varphi(R)} \leq C \frac{\eta(R/2)}{\varphi(R/2)} = C\varphi(R/2)^{\beta-1}. \end{aligned}$$

Since this is true for every  $R > 0$ , the proof is complete.

- 3. This part follows from the first and second parts.

### 7 Generalized fractional maximal operators on generalized Orlicz–Morrey spaces

In this section we give a characterization for the Spanne-type boundedness and the Adams-type boundedness of the operator  $M_\rho$  on generalized Orlicz–Morrey spaces, respectively.

#### 7.1 Spanne-type result

We use the following lemma:

**Lemma 11** *Let  $\Phi, \Psi$  be Young functions. Assume that  $\rho$  is increasing and that  $r \mapsto r^{-n}\rho(r)$  is decreasing. Assume also that the condition (19) is fulfilled. Then there exists a positive constant  $C$  such that, for all  $f \in L^\Phi_{loc}(\mathbb{R}^n)$  and  $B = B(x, r)$ ,*

$$\begin{aligned} &\|M_\rho f\|_{WL^\Psi(B)} \\ &\leq C\|f\|_{L^\Phi(2B)} + \frac{C}{\Psi^{-1}(r^{-n})} \sup_{r < t < \infty} \|f\|_{L^\Phi(B(x, 2t))} \Phi^{-1}(t^{-n})\rho(t). \end{aligned} \tag{41}$$

Moreover if we assume  $\Phi \in \nabla_2$ , the following inequality is also valid:

$$\begin{aligned} &\|M_\rho f\|_{L^\Psi(B)} \\ &\leq C\|f\|_{L^\Phi(2B)} + \frac{C}{\Psi^{-1}(r^{-n})} \sup_{r < t < \infty} \|f\|_{L^\Phi(B(x, 2t))} \Phi^{-1}(t^{-n})\rho(t). \end{aligned} \tag{42}$$

**Proof** We represent  $f$  as

$$f = f_1 + f_2, \quad f_1 = f\chi_{2B}, \quad f_2 = f - f_1, \quad r > 0.$$

Then we have

$$\|M_\rho f\|_{WL^\Psi(B)} \lesssim \|M_\rho f_1\|_{WL^\Psi(B)} + \|M_\rho f_2\|_{WL^\Psi(B)}.$$

From the boundedness of  $M_\rho$  from  $L^\Phi(\mathbb{R}^n)$  to  $WL^\Psi(\mathbb{R}^n)$  (see Theorem 4) it follows that:

$$\|M_\rho f_1\|_{WL^\Psi(B)} \leq \|M_\rho f_1\|_{WL^\Psi(\mathbb{R}^n)} \leq C \|f_1\|_{L^\Phi(\mathbb{R}^n)} = C \|f\|_{L^\Phi(2B)}, \tag{43}$$

where constant  $C > 0$  is independent of  $f$ .

If  $y \in B$  and  $r < t$ , then  $B(y, t) \subset B(x, 2t)$ . Then, using Lemma 2, we have

$$\begin{aligned} M_\rho f_2(y) &= \sup_{t>0} \frac{\rho(t)}{|B(y, t)|} \int_{B(y,t) \setminus (B(x,2r))} |f(z)| dz \\ &\leq \sup_{t>r} \frac{\rho(t)}{|B(x, t)|} \int_{B(x,2t)} |f(z)| dz \\ &\lesssim \sup_{r<t<\infty} \rho(t) \Phi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x,2t))} \quad \text{for } y \in B. \end{aligned} \tag{44}$$

Thus by Lemma 1 we have

$$\|M_\rho f_2\|_{WL^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})} \sup_{r<t<\infty} \rho(t) \Phi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x,2t))}. \tag{45}$$

Therefore we obtain (41) by (43) and (45).

If  $\Phi \in \nabla_2$ , then we can use strong type inequality instead of (43) and obtain (42) by using the same argument.

The following theorem gives a necessary and sufficient condition for Spanne-type boundedness of the operator  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi, \varphi_2}(\mathbb{R}^n)$ : We notice that the requirement is the same as the Orlicz spaces.

**Theorem 8** (Spanne-type result) *Let  $\Phi, \Psi$  be Young functions, and let  $\varphi_1 \in \mathcal{G}_\Phi$  and  $\varphi_2 \in \mathcal{G}_\Psi$ .*

1. *Assume that  $\rho$  is increasing and that  $r \mapsto r^{-n} \rho(r)$  is decreasing. Assume also that the conditions (19) and (32) are satisfied. Then the condition*

$$\sup_{r<t<\infty} \varphi_1(t) \rho(t) \leq C \varphi_2(r), \tag{46}$$

*for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , are sufficient for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{WM}^{\Psi, \varphi_2}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (46) is sufficient for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi, \varphi_2}(\mathbb{R}^n)$ .*

2. *Let  $\varphi_1$  be almost decreasing. Then the condition*

$$\varphi_1(r) \rho(r) \leq C \varphi_2(r), \tag{47}$$

*for all  $r > 0$ , where  $C > 0$  does not depend on  $r$ , is necessary for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{WM}^{1, \varphi_2}(\mathbb{R}^n)$  and hence  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi, \varphi_2}(\mathbb{R}^n)$ .*

3. Assume that  $\rho$  is increasing and that  $r \mapsto r^{-n} \rho(r)$  is decreasing. Assume also that the conditions (19) and (32) are satisfied. Let  $\varphi_1$  and  $\varphi_2$  be almost decreasing. Then the condition (47) is necessary and sufficient for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{WM}^{\Psi, \varphi_2}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (47) is necessary and sufficient for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi, \varphi_2}(\mathbb{R}^n)$ .

**Proof** 1. By (32), (41), (46) and the doubling properties of  $\varphi_1$  and  $\Phi^{-1}$  we have

$$\begin{aligned} \|M_\rho f\|_{\mathcal{WM}^{\Psi, \varphi_2}} &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \Psi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x, 2r))} \\ &\quad + \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \sup_{r < t < \infty} \|f\|_{L^\Phi(B(x, 2t))} \Phi^{-1}(t^{-n}) \rho(t) \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x, r))} \\ &\quad + \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \sup_{r < t < \infty} \varphi_1(t) \rho(t) \|f\|_{\mathcal{M}^{\Phi, \varphi_1}} \\ &\lesssim \|f\|_{\mathcal{M}^{\Phi, \varphi_1}}. \end{aligned}$$

Simply replace  $\mathcal{WL}^\Psi(B)$  with  $L^\Psi(B)$  and  $\mathcal{WM}^{\Psi, \eta}(\mathbb{R}^n)$  with  $\mathcal{M}^{\Psi, \eta}(\mathbb{R}^n)$  for the strong estimate.

2. We will now prove the second part. We utilize (22). By Lemma 7, we have

$$\begin{aligned} \rho(r) &\lesssim |B(0, r)|^{-1} \|M_\rho \chi_{B(0, 2r)}\|_{WL^1(B(0, r))} \lesssim \varphi_2(r) \|M_\rho \chi_{B(0, 2r)}\|_{\mathcal{WM}^{\Phi, \varphi_2}} \\ &\lesssim \varphi_2(r) \|\chi_{B(0, 2r)}\|_{\mathcal{M}^{\Phi, \varphi_1}} \lesssim \frac{\varphi_2(r)}{\varphi_1(r)}. \end{aligned}$$

3. Since  $\varphi_2$  is almost decreasing, (46) and (47) are equivalent. Then the third statement of the theorem follows from the first and second parts of the theorem.

### 7.2 Adams-type result

The following theorem gives necessary and sufficient conditions for Adams-type boundedness of the operator  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi, \eta}(\mathbb{R}^n)$ .

Here we suppose that  $\rho$  is an increasing function such that  $r \in (0, \infty) \mapsto r^{-n} \rho(r) \in (0, \infty)$  is decreasing.

**Theorem 9** *Let  $\Phi$  be a Young function, and let  $\varphi \in \mathcal{G}_\Phi$  be almost decreasing. Assume that  $\rho$  is increasing and that  $r \mapsto r^{-n} \rho(r)$  is decreasing. Let  $\beta \in (0, 1)$ ,  $\eta(t) \equiv \varphi(t)^\beta$  and  $\Psi(t) \equiv \Phi(t^{1/\beta})$ . Then the condition*

$$\rho(t) \lesssim \varphi(t)^{\beta-1}, \tag{48}$$

*is necessary and sufficient for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  to  $\mathcal{WM}^{\Psi, \eta}(\mathbb{R}^n)$ . Moreover, if  $\Phi \in \nabla_2$ , then the condition (48) is necessary and sufficient for the boundedness of  $M_\rho$  from  $\mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Psi, \eta}(\mathbb{R}^n)$ .*

As before, we start with an auxiliary pointwise estimate.

**Lemma 12** *Under the assumption of Theorem 9 including (48), there exists a positive constant  $C$  such that, for all  $f \in \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n)$  and all  $x \in \mathbb{R}^n$ ,*

$$M_{\rho} f(x) \leq C(Mf(x))^{\beta} \|f\|_{\mathcal{M}^{\Phi, \varphi}}^{1-\beta}. \quad (49)$$

**Proof** For arbitrary ball  $B = B(x, r)$  we represent  $f$  as

$$f = f_1 + f_2, \quad f_1 = f\chi_B, \quad f_2 = f - f_1, \quad r > 0,$$

so that

$$M_{\rho} f(x) \leq M_{\rho} f_1(x) + M_{\rho} f_2(x).$$

Hence by Lemma 2,

$$\begin{aligned} M_{\rho} f_2(x) &= \sup_{t>0} \frac{\rho(t)}{|B(x, t)|} \int_{B(x, t) \setminus (B(x, r))} |f(z)| dz \\ &\leq \sup_{t>r} \frac{\rho(t)}{|B(x, t)|} \int_{B(x, t)} |f(z)| dz \\ &\lesssim \sup_{t>r} \rho(t) \Phi^{-1}(|B(x, t)|^{-1}) \|f\|_{L^{\Phi}(B(x, t))} \\ &\lesssim \|f\|_{\mathcal{M}^{\Phi, \varphi}} \sup_{t>r} \rho(t) \varphi(t). \end{aligned}$$

Consequently by Lemma 4 we have

$$M_{\rho} f(x) \lesssim \rho(r)Mf(x) + \|f\|_{\mathcal{M}^{\Phi, \varphi}} \sup_{t>r} \rho(t) \varphi(t).$$

Thus, using the technique in [30, p. 6492] as before and (48) we obtain

$$\begin{aligned} M_{\rho} f(x) &\lesssim \min\{\varphi(r)^{\beta-1}Mf(x), \varphi(r)^{\beta}\|f\|_{\mathcal{M}^{\Phi, \varphi}}\} \\ &\lesssim \sup_{s>0} \min\{s^{\beta-1}Mf(x), s^{\beta}\|f\|_{\mathcal{M}^{\Phi, \varphi}}\} \\ &= (Mf(x))^{\beta} \|f\|_{\mathcal{M}^{\Phi, \varphi}}^{1-\beta}, \end{aligned}$$

where we have used that the supremum is achieved when the minimum parts are balanced. This shows (49).

We prove Theorem 9.

**Proof of Theorem 9** By using inequality (49) and Lemma 10 we have, for all balls  $B$ ,

$$\|M_{\rho} f\|_{WL^{\Psi}(B)} \lesssim \|(Mf)^{\beta}\|_{WL^{\Psi}(B)} \|f\|_{\mathcal{M}^{\Phi, \varphi}}^{1-\beta} = \|Mf\|_{WL^{\Phi}(B)}^{\beta} \|f\|_{\mathcal{M}^{\Phi, \varphi}}^{1-\beta}.$$

Consequently, by using the boundedness of the maximal operator  $M$ , we get

$$\begin{aligned} \eta(r)^{-1} \Psi^{-1}(|B|^{-1}) \|M_\rho f\|_{WL^\Psi(B)} &\lesssim \eta(r)^{-1} \Psi^{-1}(|B|^{-1}) \|Mf\|_{WL^\Phi(B)}^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \\ &= \left( \varphi(r)^{-1} \Phi^{-1}(|B|^{-1}) \|Mf\|_{WL^\Phi(B)} \right)^\beta \|f\|_{\mathcal{M}^{\Phi,\varphi}}^{1-\beta} \lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi}}. \end{aligned}$$

By taking the supremum over all balls  $B$ , we get the desired result. Moreover, if  $\Phi \in \nabla_2$ , then we have the strong type estimate.

We will now prove the necessity. We utilize (22). By Lemmas 1 and 7, we have

$$\begin{aligned} \rho(r) &\lesssim \Psi^{-1}(r^{-n}) \|M_\rho \chi_{B(0,2r)}\|_{WL^\Psi(B(0,r))} \lesssim \eta(r) \|M_\rho \chi_{B(0,2r)}\|_{W\mathcal{M}^{\Psi,\eta}} \\ &\lesssim \eta(r) \|\chi_{B(0,2r)}\|_{\mathcal{M}^{\Phi,\varphi}} \lesssim \frac{\eta(r)}{\varphi(r)} \lesssim \varphi(r)^{\beta-1}. \end{aligned}$$

Then the proof is complete.

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