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Global Regularity in Generalized Morrey Spaces of Solutions to Nondivergence Elliptic Equations with VMO Coefficients

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Abstract We show continuity in generalized Morrey spaces of sublinear integral operators generated by Calderón-Zygmund operator and their commutators with BMO functions. The obtained estimates are used to study global regularity of the solution of the Dirichlet problem for linear uniformly elliptic operators.

Keywords Generalized Morrey spaces · Sublinear integrals · Calderón-Zygmund integrals and commutators · BMO · VMO · Elliptic equations · Dirichlet problem

Mathematics Subject Classifications (2010) 35J25 · 35B40 · 42B20 · 42B35

1 Introduction

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced in [18] in order to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and the Hölder spaces permits to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems.

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Recall that for a bounded domain $\Omega \subset \mathbb{R}^n$ satisfying the cone property, the space $L_{p,\lambda}$ with $1 \le p < \infty$ consists of all functions $f \in L_p(\Omega)$ such that

$$\|f\|_{L_{p,\lambda}(\Omega)} = \left(\sup_{\mathcal{B}_r} \frac{1}{r^\lambda} \int_{\mathcal{B}_r \cap \Omega} |f(y)|^p \, dy \right)^{1/p} < \infty$$

where $\mathcal{B}_r$ ranges over all balls in $\mathbb{R}^n$ centered in some point $x \in \Omega$ and of radius $r > 0$. For the properties and applications of the classical Morrey spaces, we refer the readers to [3, 18, 21, 23] and the references there. In [5] Chiarenza and Frasca show boundedness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity of fractional and classical Calderón-Zygmund operators in these spaces. Recall that integral operators of that kind appear in the representation formulae of the solutions of elliptic/parabolic equations and systems. Thus the continuity of the Calderón-Zygmund integrals implies regularity of the solutions in the corresponding spaces. In [17] Mizuhara gives a generalization of these spaces considering a weight function $\omega(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. He studies also a continuity in $L_{p,\omega}$ of some classical integral operators. Later Nakai extends the results of Chiarenza and Frasca in $L_{p,\omega}$ imposing certain integral and doubling conditions on ω (see [19]). Taking a weight $\omega = \varphi^p r^n$ the conditions of Mizuhara-Nakai become

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C \varphi(x,r)^p, \qquad C^{-1} \le \frac{\varphi(x,t)}{\varphi(x,r)} \le C \quad \forall\, r \le t \le 2r$$

where the constants do not depend on t, r and $x \in \mathbb{R}^n$.

In series of works, the first author studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators as Calderón-Zygmund, Riesz and others (see [11, 12, 14]). The following theorem obtained in [11] extends the results of Nakai in Morrey-type spaces with weight $\omega = \varphi r^n$ (for the definition of the spaces see Section 2)

Theorem 1 *Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_t^\infty \varphi_1(x,r) \frac{dr}{r} \le C \varphi_2(x,t), \tag{1}$$

where C *does not depend on* x *and* t. *Then the maximal operator* M *and the Calderón-Zygmund integral operators* K *are bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to the weak space* WM_{1,φ_2}.

Later this result is extended on spaces with weaker condition on the weight pair (φ_1, φ_2) (see [14]). For more recent results on boundedness and continuity of singular integral operators in generalized Morrey and new functional spaces and their application in the differential equations theory see [2, 13, 20, 22, 25, 26] and the references there.

Throughout this paper the following notations will be used:

$D_i u = \partial u/\partial x_i,\ Du = (D_1 u, \ldots, D_n u)$ means the gradient of u,

$D_{ij} u = \partial^2 u/\partial x_i \partial x_j,\ D^2 u = \{D_{ij} u\}_{ij=1}^n$ is the Hessian matrix of u,

$\mathcal{B}_r = \mathcal{B}(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\},\ \mathcal{B}_r^c = \mathbb{R}^n \setminus \mathcal{B}_r,\ 2\mathcal{B}_r = \mathcal{B}(x_0, 2r)$,

$\mathbb{S}^{n-1}$ is a unit sphere in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a domain and $\Omega_r = \Omega \cap \mathcal{B}_r(x), x \in \Omega$,

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x = (x', x_n), x' \in \mathbb{R}^{n-1}, x_n > 0\}$,

$\mathcal{B}_r^+ \equiv \mathcal{B}^+(x^0, r) = \mathcal{B}(x^0, r) \cap \{x_n > 0\},\ 2\mathcal{B}_r^+ = \mathcal{B}^+(x^0, 2r)$ where $x^0 = (x', 0)$.

The standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various positive constants and may change from one occurrence to another.

2 Definitions and Statement of the Problem

In the present section we give the definitions of the functional spaces to which the coefficients and the data of the problem belong. The domain $\Omega \subset \mathbb{R}^n$ supposed to be bounded with $\partial\Omega \in C^{1,1}$.

Definition 2 Let $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $1 \le p < \infty$. The generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ consists of all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \left(r^{-n} \int_{\mathcal{B}(x,r)} |f(y)|^p dy \right)^{1/p} < \infty.$$

For any bounded domain Ω we define $M_{p,\varphi}(\Omega)$ taking $f \in L_p(\Omega)$ and Ω_r instead of $\mathcal{B}(x, r)$ in the norm above.

The generalized Sobolev-Morrey space $W_{2,p,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W_{2,p}(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega)$, endowed with the norm

$$\|u\|_{W_{2,p,\varphi}(\Omega)} = \sum_{0 \le |s| \le 2} \|D^s f\|_{M_{p,\varphi}(\Omega)}.$$

The space $W_{2,p,\varphi}(\Omega) \cap W^0_{1,p}(\Omega)$ consists of all functions $u \in W_{2,p}(\Omega) \cap W^0_{1,p}(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega)$, and is endowed by the same norm. Recall that $W^0_{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W_{1,p}$.

Definition 3 Let $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function, the generalized weak Morrey space $WM_{1,\varphi}(\mathbb{R}^n)$ consists of all measurable functions such that

$$\|f\|_{WM_{1,\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1}\, r^{-n}\, \|f\|_{WL_1(\mathcal{B}(x,r))}$$

where WL_1 denotes the weak L_1-space. For a bounded domain Ω we define the space $WM_{1,\varphi}(\Omega)$ taking $f \in WL_1(\Omega)$.

Definition 4 Let $a \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $a_{\mathcal{B}_r} = \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} a(y)dy$ is the mean integral of a. We say that

- $a \in BMO$ (*bounded mean oscillation*, [15]) if

$$\|a\|_* = \sup_{R>0}\sup_{\mathcal{B}_r, r\le R} \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} |a(y) - a_{\mathcal{B}_r}|dy < +\infty.$$

  The quantity $\|a\|_*$ is a norm in BMO modulo constant function under which BMO is a Banach space;
- $a \in VMO$ (*vanishing mean oscillation*, [24]) if $a \in BMO$ and

$$\lim_{R\to 0}\gamma_a(R) = \lim_{R\to 0}\sup_{\mathcal{B}_r, r\le R}\frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r}|a(y) - a_{\mathcal{B}_r}|dy = 0.$$

  The quantity $\gamma_a(R)$ is called VMO-modulus of a.

For any bounded domain $\Omega \subset \mathbb{R}^n$ we define $BMO(\Omega)$ and $VMO(\Omega)$ taking $a \in L_1(\Omega)$ and Ω_r instead of $\mathcal{B}_r$ in the definition above.

According to [1, 16], having a function $a \in BMO(\Omega)$ or $VMO(\Omega)$ it is possible to extend it in the whole $\mathbb{R}^n$ preserving its BMO-norm or VMO-modulus, respectively. In the following we use this property without explicit references. Any bounded uniformly continuous function $f \in BUC$ with modulus of continuity $\omega_f(r)$ is also VMO and $\gamma_f(r) \equiv \omega_f(r)$. Besides that, BMO and VMO contain also discontinuous functions and the following example shows the inclusion $W_{1,n}(\mathbb{R}^n) \subset VMO \subset BMO$.

Example 5 $f_\alpha(x) = |\log|x||^\alpha \in VMO$ for any $\alpha \in (0,1)$;
$f_\alpha \in W_{1,n}(\mathbb{R}^n)$ for $\alpha \in (0, 1-1/n)$, $f_\alpha \notin W_{1,n}(\mathbb{R}^n)$ for $\alpha \in [1-1/n, 1)$;
$f(x) = |\log|x|| \in BMO \setminus VMO$; $\sin f_\alpha(x) \in VMO \cap L_\infty(\mathbb{R}^n)$.

In the Sections 3, 4 and 5 we study continuity in the spaces $M_{p,\varphi}$ of certain sublinear integrals and their commutators with BMO functions. These results unified withe known estimates in $L_p(\mathbb{R}^n)$ permit to obtain continuity of the Calderón-Zygmund operators in $M_{p,\varphi}(\mathbb{R}^n)$ that is shown in Section 6. The last section is dedicated to the Dirichlet problem for a linear uniformly elliptic operator with VMO coefficients. This problem is firstly studied by Chiarenza, Frasca and Longo. In their pioneer works [6, 7] they prove unique strong solvability of

$$\begin{cases} \mathcal{L}u \equiv a^{ij}(x)D_{ij}u = f(x) & \text{a.a. } x \in \Omega, \\ u \in W_{2,p}(\Omega) \cap W^0_{1,p}(\Omega), & p \in (1,\infty) \end{cases} \tag{2}$$

providing such way the classical theory on operators with continuous coefficients to those with discontinuous ones. Later their results are extended in the Sobolev-Morrey spaces $W_{2,p,\lambda}(\Omega) \cap W^0_{1,p}(\Omega)$, $\lambda \in (1,n)$ (see [8, 9]). In the present work we show that $\mathcal{L}u \in M_{p,\varphi}(\Omega)$ implies the same regularity of the second order derivatives $D_{ij}u$. The weight $\varphi(x,r)$ satisfies an integral condition weaker than Eq. 1.

3 Sublinear Operators and Commutators Generated by Singular Integrals in the Space $M_{p,\varphi}(\mathbb{R}^n)$

In this section we present results obtained by the first author in [14] concerning continuity of sublinear operators generated by singular integrals as Calderón-Zygmund. Let T be a sublinear operator such that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$ holds

$$|Tf(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\, dy \tag{3}$$

where C is independent of f.

Theorem 6 *Let $1 \le p < \infty$, $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions such that for any $x \in \mathbb{R}^n$ and for any $t > 0$*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\varphi_2(x,r) \tag{4}$$

and T be sublinear operator satisfying Eq. 3.

(i) *If $p > 1$ and T bounded on $L_p(\mathbb{R}^n)$ then T is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ and*

$$\|Tf\|_{M_{p,\varphi_2}(\mathbb{R}^n)} \le C\|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)}.$$

(ii) *If $p = 1$ and T bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ then it is bounded from $M_{1,\varphi_1}(\mathbb{R}^n)$ to $WM_{1,\varphi_2}(\mathbb{R}^n)$ and*

$$\|Tf\|_{WM_{1,\varphi_2}(\mathbb{R}^n)} \le C\|f\|_{M_{1,\varphi_1}(\mathbb{R}^n)}$$

with constants independent of f.

Note that condition 4 is weaker than the one in Theorem 1. Indeed, if condition 1 holds then

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le \int_r^\infty \varphi_1(x,t)\, \frac{dt}{t}$$

that implies Eq. 4. We give also two examples of admissible pairs of functions.

Example 7 For $\beta \in (0, \frac{n}{2p})$ consider the weight functions

$$\varphi_1(r) = r^{\beta - \frac{n}{p}} \left|\sin\left(\max\left\{1, \frac{\pi}{r}\right\}\right)\right|, \qquad \varphi_2(r) = r^{2\beta - \frac{n}{p}}.$$

If $r \in (0,1)$ then $\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(s) s^{\frac{n}{p}} = 0$ and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt = \begin{cases} 0 & r \in (0,1) \\ r^{\beta - \frac{n}{p}} & r \in (1,\infty) \end{cases} \le C\varphi_2(r).$$

Hence the pair (φ_1, φ_2) satisfies Eq. 4 but not Eq. 1.

Example 8 For $\beta \in (0, \frac{n}{p})$ consider the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p}-\beta}}, \qquad \varphi_2(r) = r^{-\frac{n}{p}}\left(1 + r^{\beta}\right).$$

They satisfy condition 4 but not Eq. 1.

Consider now the commutator $T_a f = T[a, f] = aTf - T(af)$ such that for any $f \in L_1(\mathbb{R}^n)$ with a compact support and $x \notin \operatorname{supp} f$ holds

$$|T_a f(x)| \le C \int_{\mathbb{R}^n} |a(x) - a(y)| \, \frac{|f(y)|}{|x - y|^n} dy, \tag{5}$$

where C is independent of f and x. Suppose in addition that T_a is bounded in $L_p(\mathbb{R}^n)$ satisfying the estimate $\|T_a f\|_{L_p(\mathbb{R}^n)} \le C\|a\|_* \|f\|_{L^p(\mathbb{R}^n)}$. Then the following result holds (see [14]).

Theorem 9 *Let* $1 < p < \infty$, $a \in BMO$ *and* (φ_1, φ_2) *satisfy*

$$\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C \varphi_2(x, r) \tag{6}$$

where C does not depend on x and r. Suppose T_a *be a sublinear operator satisfying Eq. 5 and bounded on* $L_p(\mathbb{R}^n)$. *Then the operator* T_a *is bounded from* M_{p,φ_1} *to* M_{p,φ_2}

$$\|T_a f\|_{M_{p,\varphi_2}(\mathbb{R}^n)} \le C\|a\|_* \, \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)}.$$

4 Sublinear Operators Generated by Nonsingular Integral Operators in the Space $M_{p,\varphi}(\mathbb{R}^n_+)$

We start with a known result concerning the Hardy operator

$$Hg(r) := \frac{1}{r} \int_0^r g(t) dt \qquad 0 < r < \infty.$$

Theorem 10 [4] *The inequality*

$$\operatorname*{ess\,sup}_{r>0} w(r) Hg(r) \le A \operatorname*{ess\,sup}_{r>0} v(r) g(r) \tag{7}$$

holds for all non-negative and non-increasing g on $(0, \infty)$ *iff*

$$A = C \sup_{r>0} \frac{w(r)}{r} \int_0^r \frac{dt}{\operatorname*{ess\,sup}_{0<s<t} v(s)} < \infty. \tag{8}$$

For any $x \in \mathbb{R}^n_+$ define $\tilde{x} = (x', -x_n)$ and recall that $x^0 = (x', 0)$. Let $\widetilde{T}$ be a sublinear operator such that for any $f \in L_1(\mathbb{R}^n_+)$ with a compact support holds

$$|\widetilde{T} f(x)| \le C \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n} dy. \tag{9}$$

Lemma 11 *Let* $f \in L_p^{\mathrm{loc}}(\mathbb{R}_+^n)$, $1 \le p < \infty$ *be such that*

$$\int_1^\infty t^{-\frac{n}{p}-1} \|f\|_{L_p(\mathcal{B}^+(x^0,t))} dt < \infty \tag{10}$$

and $\widetilde{T}$ *be a sublinear operator satisfying Eq.* 9.

(i) *If* $p > 1$ *and* $\widetilde{T}$ *bounded on* $L_p(\mathbb{R}_+^n)$ *then*

$$\|\widetilde{T} f\|_{L_p(\mathcal{B}^+(x^0,r))} \le C r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}-1} \|f\|_{L_p(\mathcal{B}^+(x^0,t))} dt. \tag{11}$$

(ii) *If* $p = 1$ *and* $\widetilde{T}$ *bounded from* $L_1(\mathbb{R}_+^n)$ *on* $WL_1(\mathbb{R}_+^n)$ *then*

$$\|\widetilde{T} f\|_{WL_1(\mathcal{B}^+(x^0,r))} \le C r^n \int_{2r}^\infty t^{-n-1} \|f\|_{L_1(\mathcal{B}^+(x^0,t))} dt, \tag{12}$$

where the constants are independent of x^0, r *and* f.

Proof

(i) Denote by $\mathcal{B}_r^+ = \mathcal{B}^+(x^0, r)$, $\mathcal{B}_t^+ = \mathcal{B}^+(x^0, t)$ and for any $f \in L_p^{\mathrm{loc}}(\mathbb{R}_+^n)$, $p \in (1, \infty)$ write $f = f_1 + f_2$ with $f_1 = f\chi_{2\mathcal{B}_r^+}$ and $f_2 = f\chi_{(2\mathcal{B}_r^+)^c}$. Because of the (p, p)-boundedness of the operator $\widetilde{T}$ and $f_1 \in L_p(\mathbb{R}_+^n)$ we have

$$\|\widetilde{T} f_1\|_{L_p(\mathcal{B}_r^+)} \le \|\widetilde{T} f_1\|_{L_p(\mathbb{R}_+^n)} \le C\|f_1\|_{L_p(\mathbb{R}_+^n)} = C\|f\|_{L_p(2\mathcal{B}_r^+)}.$$

It is easy to see that for arbitrary points $x \in \mathcal{B}_r^+$ and $y \in (2\mathcal{B}_r^+)^c$ it holds

$$\frac{1}{2}|x^0 - y| \le |\tilde{x} - y| \le \frac{3}{2}|x^0 - y|. \tag{13}$$

Applying Eq. 9 and the Fubuni theorem to $\widetilde{T} f_2$ we get

$$\begin{aligned} |\widetilde{T} f_2(x)| &\le C \int_{\mathbb{R}_+^n} \frac{|f_2(y)|}{|\tilde{x} - y|^n} dy \\ &\le C \int_{(2\mathcal{B}_r^+)^c} \frac{|f(y)|}{|x^0 - y|^n} dy \le C \int_{(2\mathcal{B}_r^+)^c} |f(y)| \int_{|x^0-y|}^\infty \frac{dt}{t^{n+1}} \\ &\le C \int_{2r}^\infty \left(\int_{2r \le |x^0-y| < t} |f(y)| dy \right) \frac{dt}{t^{n+1}} \le C \int_{2r}^\infty \left(\int_{\mathcal{B}_t^+} |f(y)| dy \right) \frac{dt}{t^{n+1}}. \end{aligned}$$

Direct calculations give

$$\|\widetilde{T} f_2\|_{L_p(\mathcal{B}_r^+)} \le C r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}} \tag{14}$$

and the last estimate holds for all $f \in L_p(\mathbb{R}_+^n)$, $1 \le p < \infty$ satisfying Eq. 10. Thus

$$\|\widetilde{T} f\|_{L_p(\mathcal{B}_r^+)} \le C \left(\|f\|_{L_p(2\mathcal{B}_r^+)} + r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}} \right). \tag{15}$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2\mathcal{B}_r)} &= Cr^{\frac{n}{p}}\|f\|_{L_p(2\mathcal{B}_r)}\int_{2r}^{\infty}\frac{dt}{t^{\frac{n}{p}+1}}\\ &\leq Cr^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L_p(\mathcal{B}_t)}\frac{dt}{t^{\frac{n}{p}+1}} \qquad (16)\end{aligned}$$

which unified with Eq. 15 gives Eq. 11.

(ii) Let now $f \in L_1(\mathbb{R}^n_+)$, the weak (1,1)-boundedness of $\widetilde{T}$ implies

$$\|\widetilde{T}f_1\|_{WL_1(\mathcal{B}^+_r)} \leq \|\widetilde{T}f_1\|_{WL_1(\mathbb{R}^n_+)} \leq C\|f_1\|_{L_1(\mathbb{R}^n_+)} = C\|f\|_{L_1(2\mathcal{B}^+_r)}.$$

The estimate Eq. 12 follows by Eq. 14. □

Theorem 12 *Let $1 \leq p < \infty$, $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying Eq. 4 and $\widetilde{T}$ be a sublinear operator satisfying Eq. 9.*

(i) *If $p > 1$ and $\widetilde{T}$ bounded in $L_p(\mathbb{R}^n_+)$ then it is bounded from $M_{p,\varphi_1}(\mathbb{R}^n_+)$ in $M_{p,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}f\|_{M_{p,\varphi_2}(\mathbb{R}^n_+)} \leq C\|f\|_{M_{p,\varphi_1}(\mathbb{R}^n_+)}. \qquad (17)$$

(ii) *If $p = 1$ and $\widetilde{T}$ bounded from $L_1(\mathbb{R}^n_+)$ to $WL_1(\mathbb{R}^n_+)$ then it is bounded from $M_{1,\varphi_1}(\mathbb{R}^n_+)$ to $WM_{1,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}f\|_{M_{1,\varphi_2}(\mathbb{R}^n_+)} \leq C\|f\|_{WM_{1,\varphi_1}(\mathbb{R}^n_+)}$$

with constants independent of f.

Proof Let $p > 1$ than by Lemma 11 we have

$$\begin{aligned}\|\widetilde{T}f\|_{M_{p,\varphi_2}(\mathbb{R}^n_+)} &\leq C\sup_{x^0,\, r>0}\varphi_2(x^0,r)^{-1}\int_r^{\infty}\|f\|_{L_p(\mathcal{B}^+(x^0,t))}\frac{dt}{t^{\frac{n}{p}+1}}\\ &= C\sup_{x^0,\, r>0}\varphi_2(x^0,r)^{-1}\int_0^{r^{-\frac{n}{p}}}\|f\|_{L_p(\mathcal{B}^+(x^0,t^{-\frac{p}{n}}))}dt\\ &= C\sup_{x^0,\, r>0}\varphi_2(x^0,r^{-\frac{p}{n}})^{-1}\int_0^{r}\|f\|_{L_p(\mathcal{B}^+(x^0,t^{-\frac{p}{n}}))}dt.\end{aligned}$$

Applying the Theorem 10 to the above integral with

$$w(x^0,r) = \varphi_2(x^0,r^{-\frac{p}{n}})^{-1}r, \qquad v(x^0,r) = \varphi_1(x^0,r^{-\frac{p}{n}})^{-1}r,$$

$$g(x^0,r) = \|f\|_{L_p(\mathcal{B}^+(x^0,t^{-\frac{p}{n}}))}, \qquad Hg(x^0,r) = r^{-1}\int_0^r\|f\|_{L_p(\mathcal{B}^+(x^0,t^{-\frac{p}{n}}))}dt$$

where the condition 8 is equivalent to Eq. 4 we get

$$\|\widetilde{T}f\|_{M_{p,\varphi_2}(\mathbb{R}^n_+)} \leq C\sup_{x^0,r>0}\varphi_1(x^0,r^{-\frac{p}{n}})^{-1}r\|f\|_{L_p(\mathcal{B}^+(x^0,r^{-\frac{p}{n}}))} = C\|f\|_{M_{p,\varphi_1}(\mathbb{R}^n_+)}.$$

The case $p = 1$ is treated in the same manner making use of Eqs. 12 and 7

$$\|\widetilde{T} f\|_{WM_{1,\varphi_2}(\mathbb{R}^n_+)} \le C \sup_{x^0,\, r>0} \varphi_2(x^0, r)^{-1} \int_r^{\infty} \|f\|_{L_1(\mathcal{B}^+(x^0,t))} \frac{dt}{t^{n+1}}$$

$$= C \sup_{x^0,\, r>0} \varphi_2(x^0, r)^{-1} \int_0^{r^{-n}} \|f\|_{L_1(\mathcal{B}^+(x^0,t^{-\frac{1}{n}}))} dt$$

$$= C \sup_{x^0,\, r>0} \varphi_2(x, r^{-\frac{1}{n}})^{-1} \int_0^{r} \|f\|_{L_1(\mathcal{B}^+(x^0,t^{-\frac{1}{n}}))} dt$$

$$\le C \sup_{x^0,\, r>0} \varphi_1(x^0, r^{-\frac{1}{n}})^{-1}\, r\, \|f\|_{L_1(\mathcal{B}^+(x^0,r^{-\frac{1}{n}}))} = C\|f\|_{M_{1,\varphi_1}(\mathbb{R}^n_+)}.$$

□

5 Commutators of Sublinear Operators Generated by Nonsingular Integrals in the Space $M_{p,\varphi}(\mathbb{R}^n_+)$

For a function $a \in BMO$ and sublinear operator $\widetilde{T}$ satisfying Eq. 9 define the commutator $\widetilde{T}_a = \widetilde{T}[a, f] = a\widetilde{T} f - \widetilde{T}(af)$. Suppose that for any $f \in L_1(\mathbb{R}^n_+)$ with compact support and $x \notin supp f$ holds

$$|\widetilde{T}_a f(x)| \le C \int_{\mathbb{R}^n_+} |a(x) - a(y)| \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy, \tag{18}$$

with a constant independent of f and x. Suppose in addition that $\widetilde{T}_a$ is bounded in $L_p(\mathbb{R}^n_+)$, $p \in (1, \infty)$ satisfying $\|\widetilde{T}_a f\|_{L^p(\mathbb{R}^n_+)} \le C\|a\|_* \|f\|_{L_p(\mathbb{R}^n_+)}$. Our aim is to show boundedness of $\widetilde{T}_a$ in $M_{p,\varphi}(\mathbb{R}^n_+)$. For this goal we recall some well known properties of the BMO functions.

Lemma 13 (John-Nirenberg lemma, [15]) *Let $a \in BMO$ and $p \in (1, \infty)$. Then for any ball $\mathcal{B}$ there holds*

$$\left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |a(y) - a_{\mathcal{B}}|^p dy\right)^{\frac{1}{p}} \le C(p)\|a\|_*.$$

As an immediate consequence of Lemma 13 we get the following property.

Corollary 14 *Let $a \in BMO$ then for all $0 < 2r < t$ holds*

$$\left|a_{\mathcal{B}_r} - a_{\mathcal{B}_t}\right| \le C\|a\|_* \ln \frac{t}{r}. \tag{19}$$

To estimate the commutator we shall employ the same idea which we used in the proof of Lemma 11.

Lemma 15 *Let* $1 < p < \infty$, $a \in BMO$ *and* $\widetilde{T}_a$ *be a bounded operator in* $L_p(\mathbb{R}^n_+)$ *satisfying Eq.* 18. *Suppose that for all* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n_+)$ *and* $r > 0$ *holds*

$$\int_1^\infty \Big(1 + \ln \frac{t}{r}\Big) t^{-\frac{n}{p}-1} \|f\|_{L_p(\mathcal{B}_t^+(x^0,t))} dt < \infty. \tag{20}$$

Then

$$\|\widetilde{T}_a f\|_{L_p(\mathcal{B}_r^+)} \le C\|a\|_* r^{\frac{n}{p}} \int_{2r}^\infty \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_p(\mathcal{B}^+(x^0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Proof Decompose f as $f = f\chi_{2\mathcal{B}_r^+} + f\chi_{(2\mathcal{B}_r^+)^c} = f_1 + f_2$. From the boundedness of $\widetilde{T}_a$ in $L_p(\mathbb{R}^n_+)$ it follows that

$$\|\widetilde{T}_a f_1\|_{L_p(\mathcal{B}_r^+)} \le \|\widetilde{T}_a f_1\|_{L_p(\mathbb{R}^n_+)} \le C\|a\|_* \, \|f_1\|_{L_p(\mathbb{R}^n_+)} = C\|a\|_* \, \|f\|_{L_p(2\mathcal{B}_r^+)}.$$

On the other hand, because of Eq. 13 we can write

$$\begin{aligned}
\|\widetilde{T}_a f_2\|_{L_p(\mathcal{B}_r^+)} &\le C\left(\int_{\mathcal{B}_r^+}\left(\int_{(2\mathcal{B}_r^+)^c} \frac{|a(x)-a(y)||f(y)|}{|x^0-y|^n} dy\right)^p dx\right)^{\frac{1}{p}} \\
&\le C\left(\int_{\mathcal{B}_r^+}\left(\int_{(2\mathcal{B}_r^+)^c} \frac{|a(y)-a_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n} dy\right)^p dx\right)^{\frac{1}{p}} \\
&\quad + C\left(\int_{\mathcal{B}_r^+}\left(\int_{(2\mathcal{B}_r^+)^c} \frac{|a(x)-a_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n} dy\right)^p dx\right)^{\frac{1}{p}} = I_1 + I_2.
\end{aligned}$$

We estimate I_1 as follows

$$\begin{aligned}
I_1 &\le Cr^{\frac{n}{p}} \int_{(2\mathcal{B}_r^+)^c} \frac{|a(y)-a_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n} dy \\
&= Cr^{\frac{n}{p}} \int_{(2\mathcal{B}_r^+)^c} |a(y)-a_{\mathcal{B}_r^+}||f(y)| \int_{|x^0-y|}^\infty \frac{dt}{t^{n+1}} dy \\
&= Cr^{\frac{n}{p}} \int_{2r}^\infty \int_{2r\le|x^0-y|\le t} |a(y)-a_{\mathcal{B}_r^+}||f(y)| dy \frac{dt}{t^{n+1}} \\
&\le Cr^{\frac{n}{p}} \int_{2r}^\infty \int_{\mathcal{B}_t^+} |a(y)-a_{\mathcal{B}_r^+}||f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality, Lemma 13 and Eq. 19, we get

$$\begin{aligned}
I_1 &\le C\Big(r^{\frac{n}{p}} \int_{2r}^{\infty} \int_{\mathcal{B}_t^+} |a(y) - a_{\mathcal{B}_t^+}||f(y)|dy \,\frac{dt}{t^{n+1}} \\
&\qquad + r^{\frac{n}{p}} \int_{2r}^{\infty} |a_{\mathcal{B}_r^+} - a_{\mathcal{B}_t^+}| \int_{\mathcal{B}_t^+} |f(y)|dy \,\frac{dt}{t^{n+1}} \Big) \\
&\le C\Big(r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(\int_{\mathcal{B}_t^+} |a(y) - a_{\mathcal{B}_t^+}|^{\frac{p}{p-1}} dy \Big)^{\frac{p-1}{p}} \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{n+1}} \\
&\qquad + r^{\frac{n}{p}} \int_{2r}^{\infty} |a_{\mathcal{B}_r^+} - a_{\mathcal{B}_t^+}| \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}} \Big) \\
&\le C\|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}}.
\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \Big(\int_{\mathcal{B}_r^+} |a(x) - a_{\mathcal{B}_r^+}|^p dx \Big)^{\frac{1}{p}} \int_{(2\mathcal{B}_r^+)^c} \frac{|f(y)|}{|x^0 - y|^n} dy.$$

By Lemma 13 and Eq. 14 we get

$$I_2 \le C\|a\|_* \, r^{\frac{n}{p}} \int_{(2\mathcal{B}_r^+)^c} \frac{|f(y)|}{|x^0 - y|^n} dy \le C\|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Summing up I_1 and I_2 we get that for all $p \in (1, \infty)$

$$\|\widetilde{T}_a f_2\|_{L_p(\mathcal{B}_r^+)} \le C\|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Finally,

$$\|\widetilde{T}_a f\|_{L_p(\mathcal{B}_r^+)} \le \|a\|_* \, \|f\|_{L_p(2\mathcal{B}_r^+)} + \|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L_p(\mathcal{B}_t^+)} \frac{dt}{t^{\frac{n}{p}+1}}$$

and the statement follows by Eq. 16. □

Theorem 16 *Let $1 < p < \infty$, $a \in BMO$ and $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying Eq. 6. Suppose $\widetilde{T}_a$ be a sublinear operator bounded on $L_p(\mathbb{R}^n_+)$ and satisfying Eq. 18. Then $\widetilde{T}_a$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n_+)$ to $M_{p,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}_a f\|_{M_{p,\varphi_2}(\mathbb{R}^n_+)} \le C\|a\|_* \, \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n_+)} \tag{21}$$

with a constant independent of f.

The statement of the theorem follows by Lemma 15 and Theorem 10 in the same manner as the proof of Theorem 12.

6 Singular and Nonsingular Integral Operators in the Spaces $M_{p,\varphi}$

In the present section we deal with Calderón-Zygmund type integrals and their commutators with BMO functions. We start with the definition of the corresponding kernel.

Definition 17 A measurable function $\mathcal{K}(x,\xi):\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ is called a variable Calderón-Zygmund kernel if:

(i) $\mathcal{K}(x,\cdot)$ is a Calderón-Zygmund kernel for almost all $x\in\mathbb{R}^n$:

  (a) $\mathcal{K}(x,\cdot)\in C^\infty(\mathbb{R}^n\setminus\{0\})$,
  (b) $\mathcal{K}(x,\mu\xi)=\mu^{-n}\mathcal{K}(x,\xi)\quad\forall\mu>0$,
  (c) $\displaystyle\int_{\mathbb{S}^{n-1}}\mathcal{K}(x,\xi)d\sigma_\xi=0\quad\int_{\mathbb{S}^{n-1}}|\mathcal{K}(x,\xi)|d\sigma_\xi<+\infty$,

(ii) $\displaystyle\max_{|\beta|\le 2n}\left\|D_\xi^\beta\mathcal{K}(x,\xi)\right\|_{L_\infty(\mathbb{R}^n\times\mathbb{S}^{n-1})}=M<\infty$ independently of x.

The singular integrals

$$\mathfrak{K}f(x)=P.V.\int_{\mathbb{R}^n}\mathcal{K}(x,x-y)f(y)dy,$$

$$\mathfrak{C}[a,f](x)=P.V.\int_{\mathbb{R}^n}\mathcal{K}(x,x-y)f(y)[a(x)-a(y)]dy=a(x)\mathfrak{K}f(x)-\mathfrak{K}(af)(x)$$

are bounded in $L_p(\mathbb{R}^n)$ (see [6]), moreover

$$|\mathcal{K}(x,\xi)|\le|\xi|^{-n}\Big|\mathcal{K}\big(x,\frac{\xi}{|\xi|}\big)\Big|\le M|\xi|^{-n}$$

which implies

$$|\mathfrak{K}f(x)|\le C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy,\quad|\mathfrak{C}[a,f](x)|\le C\int_{\mathbb{R}^n}\frac{|a(x)-a(y)||f(y)|}{|x-y|^n}dy$$

and hence the validity of all results from Section 3. Let us note that any measurable function $\varphi:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ satisfying the condition 6 satisfies also Eq. 4 with $\varphi_1\equiv\varphi_2\equiv\varphi$. Hence the following results hold as a simple application of the estimates from Section 3.

Theorem 18 *Let* $1<p<\infty$ *and* $\varphi:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ *be measurable function such that for all* $x\in\mathbb{R}^n$ *and* $r>0$

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt\le C\varphi(x,r).\tag{22}$$

Then for any $f \in M_{p,\varphi}(\mathbb{R}^n)$ *and* $a \in BMO$ *there exist constants depending on* n, p, φ *and the kernel such that*

$$\|\mathfrak{K} f\|_{M_{p,\varphi}(\mathbb{R}^n)} \le C\|f\|_{M_{p,\varphi}(\mathbb{R}^n)}, \ \ \|\mathfrak{C}[a, f]\|_{M_{p,\varphi}(\mathbb{R}^n)} \le C\|a\|_*\|f\|_{M_{p,\varphi}(\mathbb{R}^n)}.$$

The assertion follows by Eqs. 17 and 21.

Example 19 The weight $\varphi(r) = r^{\beta-\frac{n}{p}}$, $0 < \beta < \frac{n}{p}$ satisfies condition 22. Let us note that in this case $M_{p,\varphi}$ coincides with $L_{p,\beta p}$.

Example 20 The weight $\varphi(r) = r^{\beta-\frac{n}{p}} \ln^m(e+r)$, $m \ge 1$, $0 < \beta < \frac{n}{p}$ satisfies condition 22 and the space $M_{p,\varphi}$ does not coincide with any Morrey space.

Since we aim at studying regularity properties of the solution of the Dirichlet problem 2 we need of some additional local results.

Corollary 21 *Let* $\Omega \subset \mathbb{R}^n$, $\partial\Omega \in C^{1,1}$, $a \in BMO(\Omega)$ *and* $f \in M_{p,\varphi}(\Omega)$ *with* p *and* φ *as in Theorem 18. Then*

$$\|\mathfrak{K} f\|_{M_{p,\varphi}(\Omega)} \le C\|f\|_{M_{p,\varphi}(\Omega)} \quad \|\mathfrak{C}[a, f]\|_{M_{p,\varphi}(\Omega)} \le C\|a\|_*\|f\|_{M_{p,\varphi}(\Omega)} \tag{23}$$

with $C = C(n, p, \varphi, \Omega, \mathcal{K})$.

Corollary 22 *Let* p *and* φ *be as in Theorem 18 and* $a \in VMO$ *with* VMO*-modulus* γ_a*. Then for any* $\varepsilon > 0$ *there exists a positive number* $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ *such that for any ball* $\mathcal{B}_r$ *with a radius* $r \in (0, \rho_0)$ *and all* $f \in M_{p,\varphi}(\mathcal{B}_r)$ *holds*

$$\|\mathfrak{C}[a, f]\|_{M_{p,\varphi}(\mathcal{B}_r^+)} \le C\varepsilon\|f\|_{M_{p,\varphi}(\mathcal{B}_r^+)}, \tag{24}$$

with $C = C(n, p, \varphi, \Omega, \mathcal{K})$.

To obtain the above estimates it is sufficient to extend $\mathcal{K}(x, \cdot)$ and $f(\cdot)$ as zero outside Ω (see [6, Theorem 2.11] for details). Recall that the extension of a keeps its BMO norm or VMO-modulus according to [1, 16].

For any $x, y \in \mathbb{R}^n_+$, $\tilde{x} = (x', -x_n)$ define the *generalized reflection* $\mathcal{T}(x; y)$ as

$$\mathcal{T}(x; y) = x - 2x_n \frac{\mathbf{a}^n(y)}{a^{nn}(y)} \qquad \mathcal{T}(x) = \mathcal{T}(x; x) : \mathbb{R}^n_+ \to \mathbb{R}^n_-$$

where $\mathbf{a}^n$ is the last row of the coefficients matrix $\mathbf{a}$. Then there exist positive constants C_1, C_2 depending on n and Λ, such that

$$C_1|\tilde{x} - y| \le |\mathcal{T}(x) - y| \le C_2|\tilde{x} - y| \qquad \forall\, x, y \in \mathbb{R}^n_+.$$

For any $f \in M_{p,\varphi}(\mathbb{R}^n_+)$ and $a \in BMO$ consider the nonsingular integral operators

$$\widetilde{\mathfrak{K}} f(x) = \int_{\mathbb{R}^n_+} \mathcal{K}(x, \mathcal{T}(x) - y) f(y)dy, \quad \widetilde{\mathfrak{C}}[a, f](x) = a(x)\mathfrak{K} f(x) - \mathfrak{K}(af)(x).$$

The kernel $\mathcal{K}(x, \mathcal{T}(x) - y) : \mathbb{R}^n \times \mathbb{R}^n_+ \to \mathbb{R}$ is not singular and verifies the conditions (b) and (ii) from the Definition 17. Moreover

$$|\mathcal{K}(x, \mathcal{T}(x) - y)| \le M|\mathcal{T}(x) - y|^{-n} \le C|\tilde{x} - y|^{-n}$$

that implies

$$|\widetilde{\mathfrak{K}} f(x)| \le C \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy, \quad |\widetilde{\mathfrak{C}}[a, f](x)| \le C \int_{\mathbb{R}^n_+} |a(x) - a(y)| \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy.$$

The following estimates are simple consequence of the results in Sections 4 and 5.

Theorem 23 *Let* $a \in BMO(\mathbb{R}^n_+)$, $p \in (1, \infty)$ *and* φ *be measurable function satisfying Eq. 22. Then the operators* $\widetilde{\mathfrak{K}} f$ *and* $\widetilde{\mathfrak{C}}[a, f]$ *are continuous in* $M_{p,\varphi}$ *and for all* $f \in M_{p,\varphi}(\mathbb{R}^n_+)$ *holds*

$$\|\widetilde{\mathfrak{K}} f\|_{M_{p,\varphi}(\mathbb{R}^n_+)} \le C \|f\|_{M_{p,\varphi}(\mathbb{R}^n_+)}$$

$$\|\widetilde{\mathfrak{C}}[a, f]\|_{M_{p,\varphi}(\mathbb{R}^n_+)} \le C \|a\|_* \, \|f\|_{M_{p,\varphi}(\mathbb{R}^n_+)} \tag{25}$$

with a constant dependent on known quantities only.

Corollary 24 *Let* p *and* φ *be as in Theorem 23 and* $a \in VMO$ *with a* VMO*-modulus* γ_a. *Then for any* $\varepsilon > 0$ *there exists a positive number* $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ *such that for any ball* $\mathcal{B}^+_r$ *with a radius* $r \in (0, \rho_0)$ *and all* $f \in M_{p,\varphi}(\mathcal{B}^+_r)$ *holds*

$$\|\widetilde{\mathfrak{C}}[a, f]\|_{M_{p,\varphi}(\mathcal{B}^+_r)} \le C\varepsilon \|f\|_{M_{p,\varphi}(\mathcal{B}^+_r)}, \tag{26}$$

where C *is independent of* ε, f *and* r.

The proof is as [6, Theorem 2.13].

7 The Dirichlet Problem

We consider the Dirichlet problem for second order linear equations

$$\begin{cases} \mathcal{L}u := a^{ij}(x) D_{ij} u = f(x) & \text{a.a. } x \in \Omega, \\ u \in W_{2,p,\varphi}(\Omega) \cap W^0_{1,p}(\Omega), & p \in (1, \infty) \end{cases} \tag{27}$$

subject to the following conditions:

(H_1) *Uniform ellipticity of* $\mathcal{L}$: there exists a constant $\Lambda > 0$, such that

$$\begin{cases} \Lambda^{-1} |\xi|^2 \le a^{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 & \text{a.a. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n \\ a^{ij}(x) = a^{ji}(x) & 1 \le i, j \le n. \end{cases}$$

The last assumption implies immediately essential boundedness of the coefficients $a^{ij} \in L_\infty(\Omega)$.

(H_2) *Regularity of the data:* $a^{ij} \in VMO(\Omega)$ and $f \in M_{p,\varphi}(\Omega)$ with $1 < p < \infty$ and $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable.

Theorem 25 (Interior Estimate) *Let $u \in W^{\mathrm{loc}}_{2,p}(\Omega)$ and $\mathcal{L}$ be a linear uniformly elliptic operator with VMO coefficients such that $\mathcal{L}u \in M^{\mathrm{loc}}_{p,\varphi}(\Omega)$ with $p \in (1,\infty)$ and φ satisfying Eq. 22. Then $D_{ij}u \in M_{p,\varphi}(\Omega')$ for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and*

$$\|D^2u\|_{M_{p,\varphi}(\Omega')} \le C\big(\|u\|_{M_{p,\varphi}(\Omega'')} + \|\mathcal{L}u\|_{M_{p,\varphi}(\Omega'')}\big) \tag{28}$$

where the constant depends on known quantities and dist $(\Omega', \partial\Omega'')$.

Proof Take an arbitrary point $x \in \operatorname{supp} u$ and a ball $\mathcal{B}_r(x) \subset \Omega'$, choose a point $x_0 \in \mathcal{B}_r(x)$ and fix the coefficients of $\mathcal{L}$ in x_0. Consider the constant coefficients operator $\mathcal{L}_0 = a^{ij}(x_0)D_{ij}$. From the classical theory we know that a solution $v \in C_0^\infty(\mathcal{B}_r(x))$ of $\mathcal{L}_0 v = (\mathcal{L}_0 - \mathcal{L})v + \mathcal{L}v$ can be presented as Newtonian type potential

$$v(x) = \int_{\mathcal{B}_r} \Gamma^0(x-y)[(\mathcal{L}_0 - \mathcal{L})v(y) + \mathcal{L}v(y)]dy$$

where $\Gamma^0(x-y) = \Gamma(x_0, x-y)$ is the fundamental solution of $\mathcal{L}_0$. Taking $D_{ij}v$ and unfreezing the coefficients we get for all $i, j = 1, \dots, n$ (cf. [6])

$$\begin{aligned} D_{ij}v(x) &= P.V.\int_{\mathcal{B}_r} \Gamma_{ij}(x, x-y)\left[\mathcal{L}v(y) + \big(a^{hk}(x) - a^{hk}(y)\big)D_{hk}v(y)\right]dy \\ &\quad + \mathcal{L}v(x)\int_{\mathbb{S}^n} \Gamma_j(x,y)y_i d\sigma_y \\ &= \mathfrak{K}_{ij}\mathcal{L}v(x) + \mathfrak{C}_{ij}[a^{hk}, D_{hk}v](x) + \mathcal{L}v(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x;y)y_i d\sigma_y. \end{aligned} \tag{29}$$

Here $\Gamma_{ij}(x,\xi)$ stand for the derivatives $D_{\xi_i\xi_j}\Gamma(x,\xi)$. The known properties of the fundamental solution imply that $\Gamma_{ij}(x,\xi)$ are variable Calderón-Zygmund kernels in the sense of Definition 17. The representation formula 29 still holds for any $v \in W_{2,p}(\mathcal{B}_r) \cap W^0_{1,p}(\mathcal{B}_r)$ because of the approximation properties of the Sobolev functions with C_0^∞ functions. In view of Eqs. 23, 24 and 29 for each $\varepsilon > 0$ there exists $r_0(\varepsilon)$ such that for any $r < r_0(\varepsilon)$ it holds

$$\|D^2v\|_{p,\varphi;r} \le C\left(\varepsilon\|D^2v\|_{p,\varphi;r} + \|\mathcal{L}v\|_{p,\varphi;r}\right) \qquad \|\cdot\|_{p,\varphi;r} := \|\cdot\|_{M_{p,\varphi}(\mathcal{B}_r^+)}.$$

Choosing ε (and hence also r!) small enough we can move the norm of D^2v on the left-hand side that gives

$$\|D^2v\|_{p,\varphi;r} \le C\|\mathcal{L}v\|_{p,\varphi;r}\,. \tag{30}$$

Define a cut-off function $\eta(x)$ such that for $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and $s = 0, 1, 2$ we have

$$\eta(x) = \begin{cases} 1 & x \in \mathcal{B}_{\theta r} \\ 0 & x \notin \mathcal{B}_{\theta' r} \end{cases} \qquad \eta(x) \in C_0^\infty(\mathcal{B}_r), \quad |D^s\eta| \le C[\theta(1-\theta)r]^{-s}.$$

Applying Eq. 30 to $v(x) = \eta(x)u(x) \in W_{2,p,\varphi}(\mathcal{B}_r) \cap W^0_{1,p}(\mathcal{B}_r)$ we get

$$\begin{aligned} \|D^2u\|_{p,\varphi;\theta r} &\le C\|\mathcal{L}v\|_{p,\varphi;\theta' r} \\ &\le C\left(\|\mathcal{L}u\|_{p,\varphi;\theta' r} + \frac{\|Du\|_{p,\varphi;\theta' r}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi;\theta' r}}{[\theta(1-\theta)r]^2}\right). \end{aligned}$$

Define the weighted semi-norm

$$\Theta_s = \sup_{0<\theta<1} \big[\theta(1-\theta)r\big]^s \|D^s u\|_{p,\varphi;\theta r} \qquad s = 0, 1, 2.$$

Because of the choice of θ' we have $\theta(1-\theta) \le 2\theta'(1-\theta')$. Thus, after standard transformations and taking the supremum with respect to $\theta \in (0,1)$ the last inequality rewrites as

$$\Theta_2 \le C\left(r^2\|\mathcal{L}u\|_{p,\varphi;r} + \Theta_1 + \Theta_0\right) . \tag{31}$$

Lemma 26 (Interpolation Inequality) *There exists a constant C independent of r such that*

$$\Theta_1 \le \varepsilon\Theta_2 + \frac{C}{\varepsilon}\Theta_0 \qquad \textit{for any } \varepsilon \in (0,2).$$

Proof By simple scaling arguments we get in $M_{p,\varphi}(\mathbb{R}^n)$ an interpolation inequality analogous to [10, Theorem 7.28]

$$\|Du\|_{p,\varphi;r} \le \delta\|D^2u\|_{p,\varphi;r} + \frac{C}{\delta}\|u\|_{p,\varphi;r} \qquad \delta \in (0,r)\,.$$

We can always find some $\theta_0 \in (0,1)$ such that

$$\begin{aligned}\Theta_1 &\le 2[\theta_0(1-\theta_0)r]\|Du\|_{p,\varphi;\theta_0 r}\\ &\le 2[\theta_0(1-\theta_0)r]\left(\delta\|D^2u\|_{p,\varphi;\theta_0 r} + \frac{C}{\delta}\|u\|_{p,\varphi;\theta_0 r}\right).\end{aligned}$$

The assertion follows choosing $\delta = \frac{\varepsilon}{2}[\theta_0(1-\theta_0)r] < \theta_0 r$ for any $\varepsilon \in (0,2)$. □

Interpolating Θ_1 in Eq. 31 we get

$$\frac{r^2}{4}\|D^2u\|_{p,\varphi;r/2} \le \Theta_2 \le C\left(r^2\|\mathcal{L}u\|_{p,\varphi;r} + \|u\|_{p,\varphi;r}\right)$$

and hence the Caccioppoli-type estimate

$$\|D^2u\|_{p,\varphi;r/2} \le C\left(\|\mathcal{L}u\|_{p,\varphi;r} + \frac{1}{r^2}\|u\|_{p,\varphi;r}\right). \tag{32}$$

Let $\mathbf{v} = \{v_{ij}\}_{ij=1}^n \in [M_{p,\omega}(\mathcal{B}_r)]^{n^2}$ be arbitrary function matrix. Define the operators

$$\mathcal{S}_{ijhk}(v_{hk})(x) = \mathfrak{C}_{ij}[a^{hk}, v_{hk}](x) \qquad i, j, h, k = 1, \dots, n.$$

Because of the VMO properties of a^{ij}'s we can choose r so small that

$$\sum_{i,j,h,k=1}^n \|\mathcal{S}_{ijhk}\| < 1. \tag{33}$$

Now for a given $u \in W_{2,p}(\mathcal{B}_r) \cap W^0_{1,p}(\mathcal{B}_r)$ with $\mathcal{L}u \in M_{p,\varphi}(\mathcal{B}_r)$ define

$$\mathcal{H}_{ij}(x) = \mathfrak{K}_{ij}\mathcal{L}u(x) + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x;\, y)y_i d\sigma_y$$

and Eq. 23 implies $\mathcal{H}_{ij} \in M_{p,\varphi}(\mathcal{B}_r)$. Define the operator $\mathcal{W}$ by the setting

$$\mathcal{W}\mathbf{v} = \left\{\sum_{h,k=1}^{n} \left(\mathcal{S}_{ijhk}v_{hk} + \mathcal{H}_{ij}(x)\right)\right\}_{ij=1}^{n} : \left[M_{p,\varphi}(\mathcal{B}_r)\right]^{n^2} \to \left[M_{p,\varphi}(\mathcal{B}_r)\right]^{n^2}.$$

By virtue of Eq. 33 the operator $\mathcal{W}$ is a contraction mapping and there exists a unique fixed point $\widetilde{\mathbf{v}} = \{\widetilde{v}_{ij}\}_{ij=1}^n \in [M_{p,\varphi}(\mathcal{B}_r)]^{n^2}$ of $\mathcal{W}$ such that $\mathcal{W}\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}$. On the other hand it follows from the representation formula 29 that also $D^2u = \{D_{ij}u\}_{ij=1}^n$ is a fixed point of $\mathcal{W}$. Hence $D^2u \equiv \widetilde{\mathbf{v}}$, that is $D_{ij}u \in M_{p,\varphi}(\mathcal{B}_r)$ and in addition Eq. 32 holds. The interior estimate Eq. 28 follows from Eq. 32 by a finite covering of Ω' with balls $\mathcal{B}_{r/2}$, $r < \operatorname{dist}(\Omega', \partial\Omega'')$. □

To prove a local boundary estimate for the norm of $D_{ij}u$ we define the space $W^{\gamma_0}_{2,p}(\mathcal{B}^+_r)$ as a closure of $C_{\gamma_0} = \{u \in C^\infty_0(\mathcal{B}(x^0, r)) :\, u(x) = 0 \text{ for } x_n \le 0\}$ with respect to the norm of $W_{2,p}$.

Theorem 27 (Boundary Estimate) *Let $u \in W^{\gamma_0}_{2,p}(\mathcal{B}^+_r)$ and suppose that $\mathcal{L}u \in M_{p,\varphi}(\mathcal{B}^+_r)$ with $p \in (1,\infty)$ and φ satisfying Eq. 22. Then $D_{ij}u \in M_{p,\varphi}(\mathcal{B}^+_r)$ and for each $\varepsilon > 0$ there exists $r_0(\varepsilon)$ such that*

$$\|D_{ij}u\|_{p,\varphi;\mathcal{B}^+_r} \le C\|\mathcal{L}u\|_{p,\varphi;\mathcal{B}^+_r} \quad \forall r \in (0, r_0). \tag{34}$$

Proof For $u \in W^{\gamma_0}_{2,p}(\mathcal{B}^+_r)$ the boundary representation formula holds (see [7])

$$\begin{aligned} D_{ij}u(x) = {} & P.V.\int_{\mathcal{B}^+_r} \Gamma_{ij}(x, x-y)\mathcal{L}u(y)dy \\ & + P.V.\int_{\mathcal{B}^+_r} \Gamma_{ij}(x, x-y)\big[a^{hk}(x) - a^{hk}(y)\big]D_{hk}u(y)dy \\ & + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x, y)y_i d\sigma_y + I_{ij}(x) \\ \forall\ & i, j = 1, \ldots, n, \end{aligned} \tag{35}$$

where we have set

$$
\begin{aligned}
I_{ij}(x) =& \int_{\mathcal{B}_r^+} \Gamma_{ij}(x, \mathcal{T}(x)-y)\mathcal{L}u(y)dy \\
&+ \int_{\mathcal{B}_r^+} \Gamma_{ij}(x, \mathcal{T}(x)-y)\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y)dy \\
&\qquad \forall\, i, j = 1, \dots, n-1, \\
I_{in}(x) =& I_{ni}(x) = \int_{\mathcal{B}_r^+} \Gamma_{il}(x, \mathcal{T}(x)-y)(D_n\mathcal{T}(x))^l \\
&\times \left\{\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y) + \mathcal{L}u(y)\right\} dy \\
&\qquad \forall\, i = 1, \dots, n-1, \\
I_{nn}(x) =& \int_{\mathcal{B}_r^+} \Gamma_{ls}(x, \mathcal{T}(x)-y)(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s \\
&\times \left\{\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y) + \mathcal{L}u(y)\right\} dy
\end{aligned}
$$

where $D_n\mathcal{T}(x) = \big((D_n\mathcal{T}(x))^1, \dots, (D_n\mathcal{T}(x))^n\big) = \mathcal{T}(e_n, x)$. Applying the estimates Eqs. 25 and 26, taking into account the VMO properties of the coefficients a^{ij}'s, it is possible to choose r_0 so small that

$$\|D_{ij}u\|_{p,\varphi;\mathcal{B}_r^+} \le C\|\mathcal{L}u\|_{p,\varphi;\mathcal{B}_r^+} \qquad \text{for each } r < r_0.$$

For arbitrary function matrix $\mathbf{w} = \{w_{ij}\}_{ij=1}^n \in [M_{p,\varphi}(\mathcal{B}_r^+)]^{n^2}$ define

$$
\begin{aligned}
\mathcal{S}_{ijhk}(w_{hk})(x) &= \mathfrak{C}_{ij}[a^{hk}, w_{hk}](x) \qquad i, j, h, l = 1, \dots, n, \\
\widetilde{\mathcal{S}}_{ijhk}(w_{hk})(x) &= \widetilde{\mathfrak{C}}_{ij}[a^{hk}, w_{hk}](x) \quad i, j = 1, \dots, n-1;\ h, k = 1, \dots, n, \\
\widetilde{\mathcal{S}}_{inhk}(w_{hk})(x) &= \widetilde{\mathfrak{C}}_{il}[a^{hk}, w_{hk}](D_n\mathcal{T}(x))^l, \quad i, h, k = 1, \dots, n, \\
\widetilde{\mathcal{S}}_{nnhk}(w_{hk})(x) &= \widetilde{\mathfrak{C}}_{ls}[a^{hk}, w_{hk}](x)(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s \quad h, k = 1, \dots, n.
\end{aligned}
$$

Because of Eqs. 24 and 26 we can take r so small that

$$\sum_{i,j,h,k=1}^{n} \|\mathcal{S}_{ijhk} + \widetilde{\mathcal{S}}_{ijhk}\| < 1. \tag{36}$$

Now, given $u \in W_{2,p}^{\gamma_0}(\mathcal{B}_r^+)$ with $\mathcal{L}u \in M_{p,\varphi}(\mathcal{B}_r^+)$ we set

$$
\begin{aligned}
\widetilde{\mathcal{H}}_{ij}(x) =& \mathfrak{K}_{ij}\mathcal{L}u(x) + \widetilde{\mathfrak{K}}_{ij}\mathcal{L}u(x) + \widetilde{\mathfrak{K}}_{il}\mathcal{L}u(x)(D_n\mathcal{T}(x))^l \\
&+ \widetilde{\mathfrak{K}}_{ls}\mathcal{L}u(x)(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x, y)y_i d\sigma_y
\end{aligned}
$$

and the Theorems 18 and 23 imply $\widetilde{\mathcal{H}}_{ij} \in M_{p,\varphi}(\mathcal{B}_r^+)$. Define the operator

$$\mathcal{U}\mathbf{w} = \left\{ \sum_{h,k=1}^{n} \big(\mathcal{S}_{ijhk}(w_{hk}) + \widetilde{\mathcal{S}}_{ijhk}(w_{hk})\big) + \widetilde{\mathcal{H}}_{ij}(x) \right\}_{ij=1}^{n}.$$

By virtue of Eq. 36 it is a contraction mapping in $\left[M_{p,\varphi}(\mathcal{B}_r^+)\right]^{n^2}$ and there is unique fixed point $\widetilde{\mathbf{w}} = \{\widetilde{w}_{ij}\}_{ij=1}^n$ such that $\mathcal{U}\widetilde{\mathbf{w}} = \widetilde{\mathbf{w}}$. On the other hand, it follows from the representation formula 35 that also $D^2u = \{D_{ij}u\}_{ij=1}^n$ is a fixed point of $\mathcal{U}$. Hence $D^2u \equiv \widetilde{\mathbf{w}}$, $D_{ij}u \in M_{p,\varphi}(\mathcal{B}_r^+)$ and the estimate Eq. 34 holds. □

Theorem 28 (Main Result) *Let $\mathcal{L}$ be uniformly elliptic operator satisfying conditions H_1) and H_2). Then for any function $f \in M_{p,\varphi}(\Omega)$ the unique solution of the problem 27 has second derivatives in $M_{p,\varphi}(\Omega)$. Moreover*

$$\|D^2u\|_{M_{p,\varphi}(\Omega)} \le C\big(\|u\|_{M_{p,\varphi}(\Omega)} + \|f\|_{M_{p,\varphi}(\Omega)}\big) \tag{37}$$

and the constant C depends on known quantities only.

Proof Since $M_{p,\varphi}(\Omega) \subset L_p(\Omega)$ the problem 27 is uniquely solvable in the Sobolev space $W_{2,p}(\Omega) \cap W^0_{1,p}(\Omega)$ according to [7]. By local flattering of the boundary, covering with semi-balls, taking a partition of unity subordinated to that covering and applying of estimate Eq. 34 we get a boundary a priori estimate that unified with Eq. 28 ensures validity of Eq. 37. □

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