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POINTWISE HEMISLANT SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

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We study pointwise hemislant submersions as a generalization of pointwise slant submersions and hemislant submersions from cosymplectic manifolds onto Riemannian manifolds. We investigate the integrability of distributions and the geometry of totally geodesic foliations arising from the definition of these submersions. Moreover, we study the ϕ-pluriharmonicity of these maps and obtain some inequalities connecting the Ricci curvature with the scalar curvature, depending on whether ξ is vertical or horizontal, for pointwise hemislant submersions from cosymplectic space forms onto Riemannian manifolds.

1. Introduction

The theory of submersions especially the theory of Riemannian submersions is one of the important topics in Riemannian geometry. Riemannian submersions between Riemannian manifolds were introduced by O'Neill [27] and Gray [19]. Later, such submersions were considered by Watson [34] under the name of almost Hermitian submersions between almost Hermitian manifolds by proving that the base manifold and each fiber have the same kind of structure as the total space in most cases. Since then, many works considering different types of Riemannian submersion have been done (see, [1, 15, 22, 30, 31, 33]).

B. Şahin studied slant submersions from almost Hermitian manifolds to Riemannian manifolds [31] and examined geometric properties of such submersions. In other respects, H. M. Taştan, B. Şahin and Ş. Yanan introduced the notion of hemi-slant submersions as a generalization of invariant, anti-invariant, semi-invariant and slant submersions in [33]. Later, pointwise slant submersions which extend slant submersion were studied by J. W. Lee and B. Şahin [24]. Pointwise slant submersions from cosymplectic manifolds were studied by S. Aykurt Sepet and M. Ergüt [7]. On the other hand, S. Aykurt Sepet and H. Gün Bozok [8] studied pointwise semi-slant submersion from almost Hermitian manifolds onto Riemannian manifolds.

The most important Riemann invariants and the most natural invariants of Riemannian geometry are the curvature invariants. Chen, in 1993, found some relations between the extrinsic invariants and the intrinsic invariants of a submanifold in a real space form [12] and published a collection of the results in this direction as a book [13]. Recently, many studies have been conducted on these inequalities (see, [2–6, 9, 10, 14, 16–18, 23, 25, 28, 29]).

In this paper, we introduce pointwise hemi-slant submersions from cosymplectic manifolds. We investigate the geometry of leaves of the vertical distribution and the horizontal distribution and give necessary and sufficient condition to be totally geodesic of such submersions. Next, we discusses ϕ-pluriharmonicity of pointwise hemi-slant submersions from cosymplectic manifolds. Finally, we obtain some inequalities involving the Ricci curvature and the scalar curvature according to whether ξ is vertical or horizontal for pointwise hemi-slant submersions from cosymplectic space forms onto Riemannian manifolds.

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2. Basic Properties

In this part, we give a brief view basic properties of cosymplectic manifolds and Riemannian submersions between Riemannian manifolds.

2.1. Cosymplectic Manifolds. Let a $(2n+1)$-dimensional manifold M which having an almost contact structure (ϕ, ξ, η), where a $(1,1)$ tensor fiels ϕ, a vector field ξ and a 1-form η, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

Here I is the identity tensor, [7]. Then, it is said that (M, ϕ, ξ, η) is called an almost contact manifold. If there is a Riemannian metric g on almost contact manifold M such that

$$g_M(\phi U, \phi V) = g_M(U, V) - \eta(U)\eta(V), \quad \eta(U) = g_M(U, \xi) \tag{2.2}$$

for any vector fields $U, V \in \Gamma(TM)$. (ϕ, ξ, η, g_M) and $(M, \phi, \xi, \eta, g_M)$ are respectively called an almost contact metric structure and almost contact metric manifold. The almost contact metric structure (ϕ, ξ, η, g_M) is said to be normal if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The fundamental 2-form Φ of M is defined as $\Phi(U, V) = g_M(U, \phi Y)$ for any vector fields $U, V \in \Gamma(TM)$. An almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ is called a cosymplectic manifold if it has a normal almost contact metric structure and both Φ and η are closed, that is to say $d\Phi = 0$ and $d\eta = 0$. Then the structure equation of a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ is given by

$$(\nabla_U \phi)V = 0$$

for any $U, V \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of the metric g on M. Furthermore, for a cosymplectic manifold, we have

$$\nabla_U \xi = 0$$

for every vector field $U \in \Gamma(TM)$, [7].

Example 1. We consider R^{2n+1} with Cartesian coordinates $u_i, \ v_i, \ z, \ i = 1, \ldots, n$ and its usual contact form $\eta = dz$. The Reeb vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{n}((du_i)^2 + (dv_i)^2) + (dz)^2, \quad \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ \delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n.$$

This gives a cosymplectic manifold on R^{2n+1}. For simplicity, we assume that

$$\frac{\partial}{\partial u_i} = \partial u_i.$$

In this case, the vector fields $e_i = \partial v_i,\ e_{n+i} = \partial u_i,\ \xi$ form a ϕ-basis for the cosymplectic structure. On the other hand, it can be shown that $(R^{2n+1}, \phi, \xi, \eta, g)$ is a cosymplectic manifold, [28].

2.2. Riemannian Submersions. Let (M, g_M) and (N, g_N) be Riemannian manifolds with m and n-dimensional. A Riemannian submersion $\sigma : M \to N$ is a surjective map of M onto N satisfing the following conditions:

(1) σ has the maximal rank,

(2) the differential map σ_* preserves the lengths of horizontal vectors.

For each $q \in N$, $\sigma^{-1}(q)$ an $(m-n)$-dimensional submanifold of M is called fiber and is indicated by $\sigma^{-1}(q)$. If a vector field on M is always tangent (resp. orthogonal) to fibers, then it is said to be vertical (resp. horizontal) [31]. A vector field X on M is called to be basic if it is horizontal and σ-related to a vector field X_* on N, i.e., $\sigma_* X_p = X_{*\sigma(p)}$ for all $p \in M$. We will show the projection morphisms on the distributions $\ker \sigma_*$ and $(\ker \sigma_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined by

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.3}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.4}$$

where $E, F \in \Gamma(TM)$ and ∇ the Levi-Civita connection of (M, g_M), [27].

We now recall the following lemma from [27].

Lemma 1. *Let σ be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). If X and Y are basic vector fields of M, then*

(i) $g_M(X, Y) = g_N(X_*, Y_*) \circ \sigma$,

(ii) the horizontal part $[X, Y]^{\mathcal{H}}$ of $[X, Y]$ is a basic vector field and $\sigma_*\left([X, Y]^{\mathcal{H}}\right) = [X_*, Y_*]$,

(iii) $[V, X] \in \Gamma(\{\ker \sigma_*)$ *for* $V \in \Gamma(\ker \sigma_*)$,

(iv) $\left(\nabla^M_X Y\right)^{\mathcal{H}}$ *is the basic vector σ-related to* $\nabla^N_{X_*} Y_*$,

where ∇^M and ∇^N the Levi-Civita connections on M and N, respectively.

Considering equations (2.3) and (2.4), we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.8}$$

for $X, Y \in \Gamma((\ker \sigma_*)^\perp)$ and $V, W \in \Gamma(\ker \sigma_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is a basic vector field, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$. Furthermore, for any $E \in \Gamma(TM)$, it is said that $\mathcal{T}$ is vertical, i.e., $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, i.e., $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. The tensor fields $\mathcal{T}$ and $\mathcal{A}$ that are skew-symmetric on tangent bundle of M satisfy the following equations

$$\mathcal{T}_U W = \mathcal{T}_W U, \tag{2.9}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y] \tag{2.10}$$

for $U, W \in \Gamma(\ker \sigma_*)$ and $X, Y \in \Gamma((\ker \sigma_*)^{\perp})$. On the other hand, it is said that a Riemannian submersion $\sigma\colon M \to N$ has totally geodesic fibers if and only if $\mathcal{T}$ identically vanishes.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $\sigma\colon M \to N$ is a smooth mapping between them. Then the second fundamental form of σ is given by

$$\nabla\sigma_*(X, Y) = \nabla^{\sigma}_X \sigma_*(Y) - \sigma_*(\nabla_X Y) \tag{2.11}$$

for $X, Y \in \Gamma(TM)$, where ∇^{σ} is the pullback connection and ∇ the Riemannian connection of the metrics g_M and g_N. If σ is a Riemannian submersion, then we can write

$$(\nabla\sigma_*)(X, Y) = 0 \tag{2.12}$$

for $X, Y \in \Gamma\left((\ker \sigma_*)^{\perp}\right)$. Also, from [11], σ is said to be totally geodesic map if

$$(\nabla\sigma_*)(X, Y) = 0 \tag{2.13}$$

for $X, Y \in \Gamma(TM)$. A smooth map $\sigma\colon M \to N$ is said to be harmonic if $trace \nabla\sigma_* = 0$, [11].

Denote by $R, R', \hat{R}, R^*$ the Riemannian curvature tensor of Riemannian manifolds M, N, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, respectively. Then, the Gauss–Codazzi-type equations are given by

$$R(U, V, F, W) = \hat{R}(U, V, F, W) + g(\mathcal{T}_U W, \mathcal{T}_V F) - g(\mathcal{T}_V W, \mathcal{T}_U F), \tag{2.14}$$

$$R(X, Y, Z, H) = R^*(X, Y, Z, H) - 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) + g(\mathcal{A}_Y Z, \mathcal{A}_X H) - g(\mathcal{A}_X Z, \mathcal{A}_Y H), \tag{2.15}$$

$$R(X, Y, V, W) = g((\nabla_X \mathcal{T})_V W, Y) - g(\mathcal{T}_V X, \mathcal{T}_W Y) + g((\nabla_V \mathcal{A})_X Y, W) + g(\mathcal{A}_Y W, \mathcal{A}_X V), \tag{2.16}$$

where

$$\sigma_*(R^*(X, Y, Z)) = R'(\sigma_* X, \sigma_* Y, \sigma_* Z)$$

for any $U, V, F, W \in \Gamma(\ker \sigma_*)$ and $X, Y, Z, H \in \Gamma((\ker \sigma_*)^{\perp})$, [27].

Moreover, the mean curvature vector field $\mathcal{H}$ of any fiber of Riemannian submersion σ is given by

$$H = nN, \quad N = \sum_{j=1}^{n} \mathcal{T}_{U_j} U_j$$

where $\{U_1, \ldots, U_n\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$. Furthermore, σ has totally geodesic fibers it $\mathcal{T}$ vanishes on $\ker \sigma_*$ and $(\ker \sigma_*)^{\perp}$.

A plane section σ in TM is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A cosymplectic manifold with constant ϕ-sectional curvature c is said to be a cosymplectic space form [10] and is denoted by $M(c)$. The curvature tensor R of $M(c)$ is expressed by

$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \tag{2.17}$$

3. Pointwise Hemi-Slant Submersions for Cosymplectic Manifolds

In this section, we will give some characterizations for $\xi \in \Gamma(\ker \sigma_*)$ by defining pointwise hemi-slant submersions from cosymplectic manifolds onto Riemannian manifolds.

Definition 1. *Let (M,ϕ,ξ,η,g_M) be a cosymplectic manifold and (N,g_N) be a Riemannian manifold. A Riemannian submersion $\sigma\colon M \to N$ is said to be a pointwise hemi-slant submersion if there exists a pair of orthogonal distributions $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ on $\ker\sigma_*$ such that*

(1) the space $\ker\sigma_*$ admits the orthogonal direct decomposition $\mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \xi$ (if ξ is horizontal, $\ker\sigma_* = \mathcal{D}^\theta \oplus \mathcal{D}^\perp$),

(2) the distribution $\mathcal{D}^\perp$ is anti-invariant,

(3) the distribution $\mathcal{D}^\theta$ is pointwise slant with slant function θ.

In this case, the angle θ is called the hemi-slant angle of the submersion. If the angle θ is constant, σ submersion becomes hemi-slant submersion. We call the pointwise hemi-slant submersion $\sigma\colon M \to N$ proper if $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0,\ \frac{\pi}{2}$.

Example 2. Let R^7 be a cosymplectic manifold as in Example 1. Define a map $\sigma\colon R^7 \to R^3$ by

$$\sigma(x_1,x_2,\ldots,x_7) = (x_2, \tan h\,\varphi x_1 - \sec h\,\varphi x_3, x_6) \quad \text{for} \quad \varphi\colon R^7 \to R.$$

Then,

$$\ker\sigma_* = \operatorname{span}\left\{V_1 = \sec h\,\varphi\frac{\partial}{\partial x_1} + \tan h\,\varphi\frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_5}, V_4 = \xi = \frac{\partial}{\partial x_7}\right\}$$

and

$$(\ker\sigma_*)^\perp = \operatorname{span}\left\{H_1 = \tan h\,\varphi\frac{\partial}{\partial x_1} - \sec h\,\varphi\frac{\partial}{\partial x_3}, H_2 = \frac{\partial}{\partial x_2}, H_3 = \frac{\partial}{\partial x_6}\right\}.$$

Then, σ is a pointwise hemi-slant submersion with the hemi-slant function θ satisfying $\cos\theta = \sec \mathrm{h}\varphi$ such that

$$\mathcal{D}^\theta = \operatorname{span}\{V_1, V_2\}, \quad \text{and} \quad \mathcal{D}^\perp = \operatorname{span}\{V_3\}.$$

Example 3. R^7 has a cosymplectic structure as in Example 1. Let (R^3, g_{R^3}) be a Riemannian manifold. Let $\sigma\colon R^7 \to R^3$ be a map defined by

$$\sigma(x_1, x_2, \ldots, x_7) = (\sin\alpha x_1 + \cos\alpha x_3, \sin\beta x_4 - \cos\beta x_6, x_2) \quad \text{for} \quad \alpha, \beta\colon R^7 \to R.$$

Then,

$$\ker\sigma_* = \operatorname{span}\left\{V_1 = \cos\alpha\frac{\partial}{\partial x_1} - \sin\alpha\frac{\partial}{\partial x_3}, V_2 = \cos\beta\frac{\partial}{\partial x_4} + \sin\beta\frac{\partial}{\partial x_6}, V_3 = \frac{\partial}{\partial x_5}, V_4 = \xi = \frac{\partial}{\partial x_7}\right\}$$

and

$$(\ker\sigma_*)^\perp = \operatorname{span}\left\{H_1 = \sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_3}, H_2 = \cos\beta\frac{\partial}{\partial x_4} - \sin\beta\frac{\partial}{\partial x_6}, H_3 = \frac{\partial}{\partial x_2}\right\}.$$

Thus, σ is a pointwise hemi-slant submersion such that $\ker\sigma_* = \mathcal{D}^\theta \oplus \mathcal{D}^\perp$,

$$\mathcal{D}^\theta = \operatorname{span}\{V_1, V_2\} \text{ and } \mathcal{D}^\perp = \operatorname{span}\{V_3\},$$

where $\mathcal{D}^\theta$ is a poinwise slant distribution with the hemi-slant function $\theta = \alpha + \beta$ and $\mathcal{D}^\perp$ is the anti-invariant distribution.

Let $\sigma\colon (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then, we have

$$TM = (\ker\sigma_*) \oplus (\ker\sigma_*)^\perp.$$

For any $U \in \Gamma(\ker\sigma_*)$, we put

$$U = PU + QU + \eta(U)\xi, \tag{3.1}$$

where $PU \in \Gamma(\mathcal{D}^\theta)$ and $QU \in \Gamma(\mathcal{D}^\perp)$. For any $U \in \Gamma(\ker\sigma_*)$, we have

$$\phi U = \psi U + \omega U, \tag{3.2}$$

where ψU and ωU are vertical and horizontal components of ϕU, respectively. Similarly, for any $X \in \Gamma((\ker\sigma_*)^\perp)$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{3.3}$$

where $\mathcal{B}X$ and $\mathcal{C}X$ are vertical and horizontal components of ϕX, respectively.

We can write the following theorem, which has the proof in a similar way as in the article in [33]:

Theorem 1. *Let σ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with hemi-slant function θ. Then, we have*

$$\psi^2 U = -\left(\cos^2\theta\right) U$$

for $U \in \Gamma(\mathcal{D}^\theta)$.

Theorem 2. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then, the distribution $\mathcal{D}^{\perp}$ is always integrable.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.13 in [33].

Considering the case where ξ is vertical, we can give the following theorems that we will use later:

Theorem 3. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then, the distribution $\mathcal{D}^{\theta}$ is integrable if and only if we have,*

$$g_M(\mathcal{T}_Z \omega\psi W, U) - g_M(\mathcal{T}_W \omega\psi Z, U) = g_M(\mathcal{H}\nabla_Z \omega W, \phi U) - g_M(\mathcal{H}\nabla_W \omega Z, \phi U),$$

where $W, Z \in \Gamma\left(\mathcal{D}^{\theta}\right)$ and $U \in \Gamma\left(\mathcal{D}^{\perp}\right)$.

Proof. The proof of this theorem is similar to the proof of Theorem 3.11 in [33].

As can be seen from the Definition 1, the notion of pointwise hemi-slant submersion is a natural generalization of pointwise slant, hemi-slant, slant, anti-invariant, invariant submersions. Then, if we take the dimensions of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ as m_1 and m_2, respectively, we can analyze the following theorems with respect to the values of m_1, m_1 and θ.

Theorem 4. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with the hemi-slant function θ. Then, the distribution $\mathcal{D}^{\theta}$ is a totally geodesic foliation on M if and only if we have,*

$$g_M(\mathcal{T}_W \omega\psi Z, U) = g_M(\mathcal{H}\nabla_W \omega Z, \phi U)$$

and

$$\sin 2\theta X[\theta] g_M(W, Z) = -g_M(\mathcal{A}_X \omega\psi W, Z) + g_M(\mathcal{A}_X \omega W, \psi Z) + g_M(\mathcal{H}\nabla_X \omega W, \omega Z),$$

where $W, Z \in \Gamma(\mathcal{D}^{\theta})$, $U \in \Gamma(\mathcal{D}^{\perp})$, and $X \in \Gamma((\ker \sigma_*)^{\perp})$.

Proof. For $W, Z \in \Gamma\left(\mathcal{D}^{\theta}\right)$ and $U \in \Gamma\left(\mathcal{D}^{\perp}\right)$, by using (3.2), we write

$$\begin{aligned} g_M(\nabla_W Z, U) &= g_M(\phi\nabla_W Z, \phi U) \\ &= g_M(\phi\nabla_W \psi Z, \phi^2 U) + g_M(\nabla_W \omega Z, \phi U). \end{aligned}$$

Considering (2.1), we get

$$g_M(\nabla_W Z, U) = -g_M(\nabla_W \psi^2 Z, U) - g_M(\nabla_W \omega\psi Z, U) + g_M(\nabla_W \omega Z, \phi U).$$

From Theorem 1, the above equation is obtained as follows

$$g_M(\nabla_W Z, U) = \cos^2\theta g_M(\nabla_W Z, U) - g_M(\nabla_W \omega\psi Z, U) + g_M(\nabla_W \omega Z, \phi U).$$

Using (2.6), we have

$$\sin^2\theta g_M(\nabla_W Z, U) = -g_M(\mathcal{T}_W \omega\psi Z, U) + g_M(\mathcal{H}\nabla_W \omega Z, \phi U).$$

Similarly, for $W, Z \in \Gamma\left(\mathcal{D}^\theta\right)$ and $X \in \Gamma((\ker\sigma_*)^\perp)$, we get

$$g_M(\nabla_W Z, X) = -g_M([W,X], Z) - g_M(\phi\nabla_X W, \phi Z).$$

By using (3.2), we write

$$g_M(\nabla_W Z, X) = -g_M([W,X], Z) + g_M(\nabla_X \psi^2 W, Z) + g_M(\nabla_X \omega\psi W, Z) - g_M(\nabla_X \omega W, \phi Z).$$

Considering Lie operation $[W,X] = \nabla_W X - \nabla_X W$, we write

$$g_M(\nabla_X W, Z) = -g_M(\nabla_X \psi^2 W, Z) - g_M(\nabla_X \omega\psi W, Z) + g_M(\nabla_X \omega W, \phi Z).$$

On the other hand, from Theorem 1, we arrive

$$g_M(\nabla_X W, Z) = \cos^2\theta g_M(\nabla_X W, Z) - \sin 2\theta X[\theta] g_M(W, Z)$$
$$- g_M(\nabla_X \omega\psi W, Z) + g_M(\nabla_X \omega W, \phi Z).$$

If we consider the Lie operation again, we have

$$\sin^2\theta g_M(\nabla_X W, Z) = -\sin 2\theta X[\theta] g_M(W, Z) - g_M(\nabla_X \omega\psi W, Z) + g_M(\nabla_X \omega W, \phi Z).$$

By using (2.8), we arrive

$$\sin^2\theta g_M(\nabla_X W, Z) = -\sin 2\theta X[\theta] g_M(W, Z) - g_M(\mathcal{A}_X \omega\psi W, Z)$$
$$+ g_M(\mathcal{A}_X \omega W, \psi Z) + g_M(\mathcal{H}\nabla_X \omega W, \omega Z).$$

This completes the proof of the theorem.

Theorem 5. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then, the distribution $\ker\sigma_*$ defines a totally geodesic foliation if and only if we have,*

$$\sin^2\theta g_M([U,X], V) = \sin 2\theta X[\theta] g_M(U, V) + g_M(\mathcal{A}_X \omega\psi U, V)$$
$$- g_M(\mathcal{A}_X \omega U, \phi V) - g_M(\mathcal{H}\nabla_X \omega U, \phi V),$$

where $U, V \in \Gamma(\ker\sigma_*)$ and $X \in \Gamma((\ker\sigma_*)^\perp)$.

Proof. Given $U, V \in \Gamma(\ker\sigma_*)$ and $X \in \Gamma((\ker\sigma_*)^\perp)$. Then, we derive

$$g_M(\nabla_U V, X) = -g_M(\nabla_U X, V)$$

$$= -g_M([U,X],V) - g_M(\phi\nabla_X U, \phi V).$$

From (3.2), we write

$$g_M(\nabla_U V, X) = -g_M([U,X],V) + g_M(\nabla_X \psi^2 U, V) + g_M(\nabla_X \omega\psi U, V) - g_M(\nabla_X \omega U, \phi V).$$

From Theorem 1, we have

$$g_M(\nabla_U V, X) = -g_M([U,X],V) + \cos^2\theta g_M([U,X],V) - \cos^2\theta g_M(\nabla_U X, V)$$

$$+ \sin 2\theta X[\theta] g_M(U,V) + g_M(\nabla_X \omega\psi U, V) - g_M(\nabla_X \omega U, \phi V).$$

Then, we have

$$\sin^2\theta g_M(\nabla_U X, V) = \sin^2\theta g_M([U,X],V) - \sin 2\theta X[\theta] g_M(U,V)$$

$$- g_M(\nabla_X \omega\psi U, V) + g_M(\nabla_X \omega U, \phi V).$$

By using (2.7) and (2.8), we obtain

$$\sin^2\theta g_M(\nabla_U X, V) = \sin^2\theta g_M([U,X],V) - \sin 2\theta X[\theta] g_M(U,V)$$

$$- g_M(\mathcal{A}_X \omega\psi U, V) + g_M(\mathcal{A}_X \omega U, \phi V) + g_M(\mathcal{H}\nabla_X \omega U, \phi V).$$

Considering $\ker\sigma_*$ as being totally geodesic, we obtain the formula given in the theorem.

Theorem 6. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then, σ is a totally geodesic map if and only if we have,*

$$\sin^2\theta g_M([U,X],V) = \cos^2\theta g_M(\nabla_X QU, V) + \sin 2\theta X[\theta] g_M(PU, V)$$

$$+ g_N(\nabla^\sigma_X(\sigma_*(\omega\psi PU)), \sigma_* V) - g_M(\mathcal{A}_X \omega PU, \phi V)$$

$$- g_M(\mathcal{H}\nabla_X \omega PU, \phi V) - g_M(\mathcal{A}_X \phi QU, \phi V) - g_M(\mathcal{H}\nabla_X \phi QU, \phi V)$$

and

$$\cos^2\theta g_M(\nabla_X QU, Y) = -g_N(\nabla^\sigma_X(\sigma_*(\omega\psi PU)), \sigma_* Y) + g_N(\nabla^\sigma_X(\sigma_*(\omega PU)), \sigma_* CY)$$

$$+ g_M(\mathcal{A}_X \omega PU, BY) + g_M(\mathcal{A}_X \phi QU, BY) + g_M(\mathcal{H}\nabla_X \phi QU, CY),$$

where $U, V \in \Gamma(\ker\sigma_*)$ and $X, Y \in \Gamma((\ker\sigma_*)^\perp)$.

Proof. By definition, it follows that σ is totally geodesic if and only if

$$(\nabla\sigma_*)(X,Y)=0 \quad \text{for} \quad X,Y\in\Gamma((\ker\sigma_*)^{\perp}), \qquad (\nabla\sigma_*)(X,U)=0 \quad \text{for} \quad U\in\Gamma(\ker\sigma_*)$$

and

$$(\nabla\sigma_*)(U,V)=0 \quad \text{for} \quad U,V\in\Gamma(\ker\sigma_*).$$

From (2.12), it follows that $(\nabla\sigma_*)(X,Y)=0$. Since σ is a Riemannian submersion, now we will consider cases of $(\nabla\sigma_*)(X,U)=0$ and $(\nabla\sigma_*)(U,V)=0$. Then we can write

$$(\nabla\sigma_*)(U,V)=0$$

for $U,V\in\Gamma\,(\ker\sigma_*)$ and $X\in\Gamma((\ker\sigma_*)^{\perp})$. Then,

$$g_N((\nabla\sigma_*)(U,V),\sigma_*X)=g_M([U,X],V)+g_M(\phi\nabla_XU,\phi V).$$

By using (3.1) and (3.2), we write

$$g_N((\nabla\sigma_*)(U,V),\sigma_*X)=g_M([U,X],V)-g_M(\nabla_X\psi^2PU,\phi V)-g_M(\nabla_X\omega\psi PU,\phi V)$$
$$+g_M(\nabla_X\omega PU,\phi V)+g_M(\phi\nabla_XQU,\phi V).$$

From Theorem 1, we get

$$g_N((\nabla\sigma_*)(U,V),\sigma_*X)=g_M([U,X],V)+\cos^2\theta g_M(\nabla_UX,V)-\cos^2\theta g_M([U,X],V)$$
$$-\cos^2\theta g_M(\nabla_XQU,V)-\sin 2\theta X[\theta]g_M(PU,V)$$
$$-g_M(\nabla_X\omega\psi PU,\phi V)+g_M(\nabla_X\omega PU,\phi V)+g_M(\phi\nabla_XQU,\phi V).$$

By using (2.7), (2.8) and (2.11), we find

$$\sin^2\theta g_N((\nabla\sigma_*)(U,V),\sigma_*X)=\sin^2\theta g_M([U,X],V)-\cos^2\theta g_M(\nabla_XQU,V)$$
$$-\sin 2\theta X[\theta]g_M(PU,V)-g_N(\nabla^{\sigma}_X(\sigma_*(\omega\psi PU)),\sigma_*V)$$
$$+g_M(\mathcal{A}_X\omega PU,\phi V)+g_M(\mathcal{H}\nabla_X\omega PU,\phi V)$$
$$+g_M(\mathcal{A}_X\phi QU,\phi V)+g_M(\mathcal{H}\nabla_X\phi QU,\phi V).$$

Similarly, for $U\in\Gamma(\ker\sigma_*)$ and $X,Y\in\Gamma((\ker\sigma_*)^{\perp})$, we can write $(\nabla\sigma_*)(X,U)=0$. Then, we get

$$\sin^2\theta g_N((\nabla\sigma_*)(X,U),\sigma_*Y)=-\cos^2\theta g_M(\nabla_XQU,Y)-g_N(\nabla^{\sigma}_X(\sigma_*(\omega\psi PU)),\sigma_*Y)$$

$$+ g_N(\nabla^\sigma_X(\sigma_*(\omega PU)), \sigma_* CY) + g_M(\mathcal{A}_X \omega PU, BY)$$

$$+ g_M(\mathcal{A}_X \phi QU, BY) + g_M(\mathcal{H}\nabla_X \phi QU, CY).$$

This concludes the proof.

4. ϕ-Pluriharmonicity of Pointwise Hemi-Slant Submersion

Recently, several authors studied the concept of pluriharmonic map by considering different structures. Y. Ohnita established J-pluriharmonicity from an almost Hermitian manifold in [26]. If on contact metric manifolds, Ianus and Pastore [20] obtained results regarding harmonic maps.

In this section, we investigate some theorems for pointwise hemi-slant submersions by applying the notion of ϕ-pluriharmonicity on certain distributions, for the characteristic vector field being vertical.

Let σ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then, pointwise hemi-slant submersion is ϕ-pluriharmonic, $\mathcal{D}^\theta - \phi$-pluriharmonic and $(\ker \sigma_*) - \phi$-pluriharmonic if

$$(\nabla\sigma_*)(U, V) + (\nabla\sigma_*)(\phi U, \phi V) = 0, \tag{4.1}$$

for any $U, V \in \Gamma(TM)$, for any $U, V \in \Gamma(\mathcal{D}^\theta)$ and for any $U, V \in \Gamma(\ker \sigma_*)$, [20].

Theorem 7. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Suppose that σ is $\mathcal{D}^\theta - \phi$-pluriharmonic. Then, $\mathcal{D}^\theta$ defines totally geodesic foliation if and only if*

$$\nabla^\sigma_{\phi U}\sigma_*(\omega\psi V) = -\cos^2\theta\sigma_*(\nabla_{\omega U}V) + \sigma_*(\nabla_{\psi U}\omega\psi V) + \sigma_*(\nabla_{\omega U}\omega\psi V)$$

for any $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof. For any $U, V \in \Gamma(\mathcal{D}^\theta)$, it is $\psi V \in \Gamma(\mathcal{D}^\theta)$ and since σ is $\mathcal{D}^\theta - \phi$-pluriharmonic, from (4.1), we write

$$\nabla\sigma_*(U, \psi V) + \nabla\sigma_*(\phi U, \phi\psi V) = 0.$$

By using the second fundamental form of σ, we get

$$-\sigma_*(\nabla_U \psi V) = -\nabla^\sigma_{\phi U}\sigma_*(\phi\psi V) + \sigma_*(\nabla_{\phi U}\phi\psi V).$$

Using (3.2), we obtain

$$-\sigma_*(\nabla_U \psi V) = -\nabla^\sigma_{\phi U}\sigma_*(\psi^2 V + \omega\psi V) + \sigma_*(\nabla_{\phi U}\psi^2 V + \omega\psi V).$$

Considering Theorem 1, we have

$$-\sigma_*(\nabla_U \psi V) = -\nabla^\sigma_{\phi U}\sigma_*(\omega\psi V) - \cos^2\theta\sigma_*(\nabla_{\psi U}V) - \cos^2\theta\sigma_*(\nabla_{\omega U}V)$$

$$+ \sigma_*(\nabla_{\psi U}\omega\psi V) + \sigma_*(\nabla_{\omega U}\omega\psi V).$$

If $\mathcal{D}^\theta$ is a totally geodesic, it becomes $\sigma_*(\nabla_U \psi V) = 0$ and $\sigma_*(\nabla_{\psi U} V) = 0$. Then, this completes the proof of theorem.

Theorem 8. *Let $\sigma: (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Suppose that σ is $(\ker \sigma_*) - \phi$-pluriharmonic. Then, the distribution $(\ker \sigma_*)$ defines totally geodesic foliation if and only if*

$$\nabla\sigma_*(\phi U, \omega PV) + \nabla_{\phi U}\sigma_*(\phi QV) = \sigma_*(\nabla_{\omega U}\psi PV) + \sigma_*(\omega[\psi U, QV])$$

$$+ \sigma_*(\nabla_{QV}\omega\psi PU) + \sigma_*(\nabla_{\omega U}\phi QV)$$

for any $U, V \in \Gamma(\ker \sigma_*)$.

Proof. For $U, V \in \Gamma(\ker \sigma_*)$, since σ is $(\ker \sigma_*) - \phi$-pluriharmonic, then by using (4.1), we have

$$\nabla\sigma_*(U, V) + \nabla\sigma_*(\phi U, \phi V) = 0.$$

According to the second fundamental form of σ, we obtain

$$\nabla^\sigma_U \sigma_* V - \sigma_*(\nabla_U V) = -\nabla^\sigma_{\phi U}\sigma_*(\phi V) + \sigma_*(\nabla_{\phi U}\phi V).$$

From (3.1) and (3.2), we get

$$-\sigma_*(\nabla_U V) = -\nabla_{\phi U}\sigma_*(\omega PV) - \nabla_{\phi U}\sigma_*(\phi QV) + \sigma_*(\nabla_{\psi U}\psi PV$$

$$+ \nabla_{\omega U}\psi PV + \nabla_{\phi U}\omega PV + \nabla_{\psi U}\phi QV + \nabla_{\omega U}\phi QV),$$

where

$$\nabla_{\phi U}\sigma_*(\omega PV) - \sigma_*(\nabla_{\phi U}\omega PV) = \nabla\sigma_*(\phi U, \omega PV).$$

Then, we write

$$-\sigma_*(\nabla_U V) = -\nabla\sigma_*(\phi U, \omega PV) - \nabla_{\phi U}\sigma_*(\phi QV) + \sigma_*(\nabla_{\psi U}\psi PV)$$

$$+ \sigma_*(\nabla_{\omega U}\psi PV) + \sigma_*(\nabla_{\psi U}\phi QV) + \sigma_*(\nabla_{\omega U}\phi QV)$$

$$= -\nabla\sigma_*(\phi U, \omega PV) - \nabla_{\phi U}\sigma_*(\phi QV) + \sigma_*(\nabla_{\psi U}\psi PV) + \sigma_*(\nabla_{\omega U}\psi PV)$$

$$+ \sigma_*(\omega[\psi U, QV]) + \sigma_*(\nabla_{QV}\psi^2 PU) + \sigma_*(\nabla_{QV}\omega\psi PU) + \sigma_*(\nabla_{\omega U}\phi QV).$$

Considering Theorem 1, we have

$$-\sigma_*(\nabla_U V) = -\nabla\sigma_*(\phi U, \omega PV) - \nabla_{\phi U}\sigma_*(\phi QV) + \sigma_*(\nabla_{\psi U}\psi PV) + \sigma_*(\nabla_{\omega U}\psi PV)$$

$$+ \sigma_*(\omega[\psi U, QV]) + \cos^2\theta\sigma_*(\nabla_{QV} PU) + \sigma_*(\nabla_{QV}\omega\psi PU) + \sigma_*(\nabla_{\omega U}\phi QV).$$

If $(\ker \sigma_*)$ is a totally geodesic, it becomes

$$\nabla\sigma_*(\phi U, \omega PV) + \nabla_{\phi U}\sigma_*(\phi QV) = \sigma_*(\nabla_{\omega U}\psi PV) + \sigma_*(\omega[\psi U, QV]) + \sigma_*(\nabla_{QV}\omega\psi PU) + \sigma_*(\nabla_{\omega U}\phi QV).$$

Then, this completes the proof of theorem.

Theorem 9. *Let $\sigma : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a pointwise hemi-slant submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Suppose that σ is $\mathcal{D}^\theta - \phi$-pluriharmonic. Then, the distribution $\mathcal{D}^\theta$ is integrable if and only if*

$$g_M(\nabla_U\omega\psi V, W) = -g_N(\sigma_*\omega V, \nabla_U\sigma_*\phi W) + g_N(\sigma_*\omega V, \nabla\sigma_*(\phi U, W)) + g_M(\nabla_V\omega\psi U, W) + g_N(\sigma_*\omega U, \nabla_V\sigma_*\phi W) - g_N(\sigma_*\omega U, \nabla\sigma_*(\phi V, W)),$$

where $U, V \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$.

Proof. For $U, V \in \Gamma(\mathcal{D}^\theta)$, this must be $[U, V] \in \Gamma(\mathcal{D}^\theta)$. As examined in Theorem 3, the following equation is obtained for $W \in \Gamma(\mathcal{D}^\perp)$:

$$\sin^2\theta g_M([U, V], W) = -g_M(\nabla_U\omega\psi V, W) + g_M(\nabla_U\omega V, \phi W) + g_M(\nabla_V\omega\psi U, W) - g_M(\nabla_V\omega U, \phi W).$$

Considering equation (2.11), we write

$$\sin^2\theta g_M([U, V], W) = -g_M(\nabla_U\omega\psi V, W) - g_N(\sigma_*\omega V, \nabla_U\sigma_*\phi W) + g_N(\sigma_*\omega V, \nabla\sigma_*(U, \phi W)) + g_M(\nabla_V\omega\psi U, W) + g_N(\sigma_*\omega U, \nabla_V\sigma_*\phi W) - g_N(\sigma_*\omega U, \nabla\sigma_*(V, \phi W)).$$

Since σ is ϕ-pluriharmonic, we get

$$\sin^2\theta g_M([U, V], W) = -g_M(\nabla_U\omega\psi V, W) - g_N(\sigma_*\omega V, \nabla_U\sigma_*\phi W) + g_N(\sigma_*\omega V, \nabla\sigma_*(\phi U, W)) + g_M(\nabla_V\omega\psi U, W) + g_N(\sigma_*\omega U, \nabla_V\sigma_*\phi W) - g_N(\sigma_*\omega U, \nabla\sigma_*(\phi V, W)).$$

This completes the proof.

5. Inequalities for Pointwise Hemi-Slant Submersions

In this section, we obtain some inequalities involving the Ricci curvature and the scalar curvature according to whether ξ is vertical or horizontal for pointwise hemi-slant submersions from cosymplectic space forms onto Riemannian manifolds.

Let $\sigma : M(c) \to N$ be a pointwise hemi-slant submersion from a cosymplectic space form $(M(c), g)$ onto a Riemannian manifold (N, g_N).

From the equations (2.14), (2.15) and (2.17), we have

$$\begin{aligned}\hat{R}(U,V,F,W) = \frac{c}{4}\{&g(V,F)g(U,W) - g(U,F)g(V,W) + \eta(U)\eta(F)g(V,W)\\ &- \eta(V)\eta(F)g(U,W) + \eta(V)\eta(W)g(U,F) - \eta(U)\eta(W)g(V,F)\\ &+ g(\phi V,F)g(\phi U,W) - g(\phi U,F)g(\phi V,W) - 2g(\phi U,V)g(\phi F,W)\}\\ &- g(\mathcal{T}_U W,\mathcal{T}_V F) + g(\mathcal{T}_V W,\mathcal{T}_U F) \qquad (5.1)\end{aligned}$$

and

$$\begin{aligned}R^*(X,Y,Z,H) = \frac{c}{4}\{&g(Y,Z)g(X,H) - g(X,Z)g(Y,H) + \eta(X)\eta(Z)g(Y,H)\\ &- \eta(Y)\eta(Z)g(X,H) + \eta(Y)\eta(H)g(X,Z) - \eta(X)\eta(H)g(Y,Z)\\ &+ g(\phi Y,Z)g(\phi X,H) - g(\phi Y,H)g(\phi X,Z) - 2g(H,\phi Z)g(\phi X,Y)\}\\ &+ 2g(\mathcal{A}_X Y,\mathcal{A}_Z H) - g(\mathcal{A}_Y Z,\mathcal{A}_X H) + g(\mathcal{A}_X Z,\mathcal{A}_Y H), \qquad (5.2)\end{aligned}$$

where $U,V,F,W \in \Gamma(\ker\sigma_*)$ and $X,Y,Z,H \in \Gamma((\ker\sigma_*)^\perp)$.

Case 1. Assume that ξ is vertical.

Then, for every $q \in M$, we can write

$$\{U_1, U_2, \dots, U_r, U_{r+1}, \dots, U_{r+2m}, U_n = \xi\}$$

an orthonormal basis of $(\ker\sigma_*)$ and $\{X_1,\dots,X_k\}$ an orthonormal basis of $((\ker\sigma_*)^\perp)$, respectively, such that $\{U_1,U_2,\dots,U_r\}$ is an orthonormal basis of $\mathcal{D}^\perp$, while $\{U_{r+1},\dots,U_{r+2m}\}$ is an orthonormal basis of $\mathcal{D}^\theta$, where $n = r+2m+1$. Obviously, we have

$$g^2(\phi U_i, U_{i+1}) = \begin{cases} 0, & \text{for} \quad i \in \{1,\dots,r\},\\ \cos^2\theta, & \text{for} \quad i \in \{r+1,\dots,r+2m-1\}\end{cases}$$

and

$$\sum_{i,j=1}^{n} g^2(\phi U_i, U_j) = 2m\cos^2\theta. \qquad (5.3)$$

Theorem 10. *Let $\sigma\colon M(c) \to N$ be a pointwise hemi-slant submersions from a cosymplectic space form $(M(c),g)$ onto a Riemannian manifold (N,g_N) with $\xi \in (\ker\sigma_*)$. Then, we have*

(i) for a unit vector field $U \in \Gamma(\mathcal{D}^\theta)$

$$\widehat{Ric}(U) \geq \frac{c}{4}(n-2+3\cos^2\theta) - ng(\mathcal{T}_U U, H), \qquad (5.4)$$

(ii) for a unit vector field $U \in \Gamma(\mathcal{D}^{\perp})$

$$\widehat{Ric}(U) \geq \frac{c}{4}(n-2) - ng(\mathcal{T}_U U, H). \tag{5.5}$$

The equality cases of (5.4) and (5.5) holds identically respectively, for a unit vector field $U \in \Gamma(\mathcal{D}^{\theta})$ and a unit vector field $U \in \Gamma(\mathcal{D}^{\perp})$ if and only if each fiber is totally geodesic.

Proof. For a unit vector field $U \in \Gamma(\mathcal{D}^{\theta})$ and $\xi \in (\ker \sigma_*)$, using (5.1)

$$\widehat{Ric}(U) = \frac{c}{4}\{n-2+3\sum_{i=1}^{n} g^2(\phi U, U_i)\} - ng(\mathcal{T}_U U, H) + \sum_{i=1}^{n} \| \mathcal{T}_U U_i \|^2, \tag{5.6}$$

where

$$\widehat{Ric}(U) = \sum_{i=1}^{n} \hat{R}(U, U_i, U_i, U).$$

Using the equation

$$\sum_{i=1}^{n} g^2(\phi U, U_i) = \cos^2\theta$$

in (5.6), we get

$$\widehat{Ric}(U) = \frac{c}{4}\{n-2+3\cos^2\theta\} - ng(\mathcal{T}_U U, H) + \sum_{i=1}^{n} \| \mathcal{T}_U U_i \|^2 . \tag{5.7}$$

Therefore, we arrive the inequality (5.4) in (i). Similarly, for a unit vector field $U \in \Gamma(\mathcal{D}^{\perp})$, by using (5.1), we have

$$\widehat{Ric}(U) = \frac{c}{4}(n-2) - ng(\mathcal{T}_U U, H) + \sum_{i=1}^{n} \| \mathcal{T}_U U_i \|^2, \tag{5.8}$$

which implies (5.5).

Theorem 11. *Let $\sigma\colon M(c) \to N$ be a pointwise hemi-slant submersions from a cosymplectic space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with $\xi \in (\ker \sigma_*)$. Then, we have*

$$2\hat{\tau} \geq \frac{c}{4}(n^2 - 3n + 2 + 6m\cos^2\theta) - n^2 \| \mathcal{H} \|^2 .$$

The equality case of the inequality holds if and only if each fiber is totally geodesic.

Proof. For $U_i, U_j \in \Gamma(\ker \sigma_*)$ and $\xi \in (\ker \sigma_*)$, from equation (5.1)

$$2\hat{\tau} = \frac{c}{4}\{n^2 - 3n + 2 + 3\sum_{i,j=1}^{n} g^2(\phi U_i, U_j)\} - n^2 g(H, H) + \sum_{i,J=1}^{n} \| \mathcal{T}_{U_i} U_j \|^2,$$

where

$$\hat{\tau} = \sum_{i,j=1}^{n} \hat{R}(U_i, U_j, U_j, U_i)$$

and to be (5.3), the required statement is obtained.

Theorem 12. *Let $\sigma\colon M(c) \to N$ be a pointwise hemi-slant submersions from a cosymplectic space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with $\xi \in (\ker \sigma_*)$. Then, we have*

$$2\tau^* \leq \frac{c}{4}(n^2 + 2n + 3 \parallel C \parallel^2).$$

The equality case of the inequality holds if and only if $(\ker \sigma_*)^\perp$ is integrable.

Proof. For $X_i, X_j \in \Gamma((\ker \sigma_*)^\perp)$ and $\xi \in (\ker \sigma_*)$, since $\mathcal{A}$ is anti-symmetric, using (5.2) we have

$$2\tau^* = \frac{c}{4}\left\{ n^2 + 2n + 3\sum_{i,j=1}^{n} g^2(CX_i, X_j) \right\} - 3\sum_{i,j=1}^{n} \parallel \mathcal{A}_{X_i} X_j \parallel^2,$$

where

$$\tau^* = \sum_{i,j=1}^{n} R^*(X_i, X_j, X_j, X_i).$$

Then, we arrive

$$2\tau^* = \frac{c}{4}\{n^2 + 2n + 3 \parallel C \parallel^2\} - 3\sum_{i,j=1}^{n} \parallel \mathcal{A}_{X_i} X_j \parallel^2,$$

where

$$\parallel C \parallel^2 = \sum_{i,j=1}^{n} g^2(CX_i, X_j).$$

Case 2. Assume that ξ is horizontal.

We can think of the basis of $(\ker \sigma_*)$ and $((\ker \sigma_*)^\perp)$ as $\{U_1, \ldots, U_{r+2m}\}$ and $\{X_1, \ldots, X_k = \xi\}$, respectively, such that $\dim(\mathcal{D}^\perp) = r$ and $\dim(\mathcal{D}^\theta) = 2m$, where $n = r + 2m$. Then, it can be easily seen that equality (5.3) is satisfied. By using this equality, we can write the following theorems.

Theorem 13. *Let $\sigma\colon M(c) \to N$ be a pointwise hemi-slant submersions from a cosymplectic space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with $\xi \in ((\ker \sigma_*)^\perp)$. Then, we have*

(i) for a unit vector field $U \in \Gamma(\mathcal{D}^\theta)$

$$\widehat{Ric}(U) \geq \frac{c}{4}(n - 1 + 3\cos^2\theta) - ng(\mathcal{T}_U U, H), \tag{5.9}$$

(ii) for a unit vector field $U \in \Gamma(\mathcal{D}^\perp)$

$$\widehat{Ric}(U) \geq \frac{c}{4}(n - 1) - ng(\mathcal{T}_U U, H). \tag{5.10}$$

The equality cases of (5.9) and (5.10) holds identically respectively, for a unit vector field $U \in \Gamma(\mathcal{D}^\theta)$ and a unit vector field $U \in \Gamma(\mathcal{D}^\perp)$ if and only if each fiber is totally geodesic.

Proof. For a unit vector field $U \in \Gamma(\mathcal{D}^\theta)$ and $\xi \in ((\ker\sigma_*)^\perp)$, using (5.1), we have

$$\widehat{Ric}(U) = \frac{c}{4}\left\{n - 1 + 3\sum_{i=1}^{n} g^2(\phi U, U_i)\right\} - ng(\mathcal{T}_U U, H) + \sum_{i=1}^{n} \parallel \mathcal{T}_U U_i \parallel^2 . \tag{5.11}$$

From here, we can write

$$\widehat{Ric}(U) = \frac{c}{4}\{n - 1 + 3\cos^2\theta\} - ng(\mathcal{T}_U U, H) + \sum_{i=1}^{n} \parallel \mathcal{T}_U U_i \parallel^2 .$$

Thus, the inequality (5.9) in (i) is satisfied. Similarly, for a unit vector field $U \in \Gamma(\mathcal{D}^\perp)$, we have

$$\widehat{Ric}(U) = \frac{c}{4}(n-1) - ng(\mathcal{T}_U U, H) + \sum_{i=1}^{n} \parallel \mathcal{T}_U U_i \parallel^2, \tag{5.12}$$

which implies (5.10).

Theorem 14. *Let $\sigma : M(c) \to N$ be a pointwise hemi-slant submersions from a cosymplectic space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with $\xi \in ((\ker\sigma_*)^\perp)$. Then, we have*

$$2\hat{\tau} \geq \frac{c}{4}(n^2 - n + 6m\cos^2\theta) - n^2 \parallel \mathcal{H} \parallel^2 .$$

The equality case of the inequality holds if and only if each fiber is totally geodesic.

Proof. For $U_i, U_j \in \Gamma(\ker\sigma_*)$ and $\xi \in ((\ker\sigma_*)^\perp)$, from (5.1), we obtain

$$2\hat{\tau} = \frac{c}{4}\{n^2 - n + 3\sum_{i,j=1}^{n} g^2(\phi U_i, U_j)\} - n^2 g(H, H) + \sum_{i,J=1}^{n} \parallel \mathcal{T}_{U_i} U_j \parallel^2 .$$

This completes the proof of the theorem.

Theorem 15. *Let $\sigma : M(c) \to N$ be a pointwise hemi-slant submersions from a cosymplectic space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with $\xi \in ((\ker\sigma_*)^\perp)$. Then, we have*

$$2\tau^* \leq \frac{c}{4}(n^2 - 1 + 3 \parallel C \parallel^2).$$

The equality case of the inequality holds if and only if $(\ker\sigma_*)^\perp$ is integrable.

Proof. For $X_i, X_j \in \Gamma((\ker\sigma_*)^\perp)$ and $\xi \in ((\ker\sigma_*)^\perp)$, since $\mathcal{A}$ is anti-symmetric, using (5.2) we have

$$2\tau^* = \frac{c}{4}\{n^2 - 1 + 3\sum_{i,j=1}^{n} g^2(CX_i, X_j)\} - 3\sum_{i,j=1}^{n} \parallel \mathcal{A}_{X_i} X_j \parallel^2 .$$

Then,

$$2\tau^* = \frac{c}{4}\{n^2 - 1 + 3 \parallel C \parallel^2\} - 3 \sum_{i,j=1}^{n} \parallel \mathcal{A}_{X_i} X_j \parallel^2,$$

where

$$\tau^* = \sum_{i,j=1}^{n} R^*(X_i, X_j, X_j, X_i).$$

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