Tech Know Learn (2012) 17:109–114
DOI 10.1007/s10758-012-9194-5

Exploring the Mathematical Model of the Thumbaround Motion by GeoGebra

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Published online: 21 September 2012

Have you ever watched somebody in class or at the office skillfully twirl a pencil around his or her thumb, and wondered how that person did it? Have you tried to do it on your own and found that it's not as easy as it looks? Such real life situations can be motivating for connecting to mathematics. In this snapshot, we examine this motion by using dynamic mathematics software.

GeoGebra is a dynamic mathematics software whose fundamental idea is producing multiple representations (Hohenwarter and Preiner 2007). GeoGebra can provide both algebraic and geometric representations synchronously through its algebra and geometry windows. With the help of this feature, clues for abstraction can be obtained when a visual model is created geometrically. That is, it will be easier to read the visual model, and to express it in a more mathematical way as is described by Doerr and Pratt (2008). GeoGebra offers deeper understanding of the mathematical models by providing the opportunity to examine algebraic and geometric representations together (Duval 1999).

Based on the features of GeoGebra described above, Kabaca and Aktumen (2010) described the abstraction process of cycloid curves' parametric equation from its visual model. Aktumen et al. (2011) described a real life situation concerning the change in surface area of the water in a cylinder, half full and positioned perpendicularly to the ground. In that work, they generated partial functions which calculate the surface area of the water depending on the angle between the cylinder and the floor. This process is

This column will publish short (from just a few paragraphs to ten or so pages), lively and intriguing computer-related mathematics vignettes. These vignettes or snapshots should illustrate ways in which computer environments have transformed the practice of mathematics or mathematics pedagogy. They could also include puzzles or brain teasers involving the use of computers or computational theory. Snapshots are subject to peer review. From the Column Editor Uri Wilensky, Northwestern University. Email: uri@northwestern.edu.

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Fig. 1 Initial position of the pencil before thumbaround motion

visualized using GeoGebra. In another work, the concept of parametric equation was used (Aktumen et al. 2010). That example featured a pen placed with its point on a barrier and its back on the floor. They examined the trace of the tip-point while pushing the back of the pen through a line towards the barrier, and generated the parametric equations. This process was visualized using GeoGebra.

In this study, we demonstrate the process of mathematical abstraction grounded in a well-known situation taken from daily life. Specifically, we investigate the movement of the writing end (tip point) of the pencil shown in Fig. 1 as one executes a thumbaround motion. To model this motion, we designed a GeoGebra construction and algebraically derived the parametric equations for the movement of the end of the pencil.

1 Methodology of the Study

This study describes an abstraction process to show how a relation is constructed between a geometric model and an algebraic model of a real life situation. First, the visual model, created by GeoGebra, is described. Then, a preliminary description of the mathematical model in terms of the components of the visual model is given. Third, the process of obtaining the mathematical model, which is a parametric equation, is described in detail. Last, the mathematical model and the visual model are compared again in the GeoGebra environment.

2 Abstraction Process

2.1 Description of Visual Model

In modeling the situation, we represent the cross-section of the thumb as a circle and the pencil as a line segment. The length of the pencil, the radius of the thumb, and the point on the pencil where it initially meets the thumb, are each changeable via sliders in the GeoGebra construction. An animation simulates the thumbaround action, in which the pencil rotates around the thumb. During the animation, the tip point of the segment that represents our pencil traces a curve. In the discussion below, we derive the parametric equation of this curve. Figure 2 shows the final position of the counter-clockwise rotation; Fig. 1 shows the initial position.

2.2 From Visual Model to Mathematics

Let's assume that the circle, which represents the cross-section of the fingertip, is on the x-axis and the initial position of the pen is also along the x-axis. Let's define the tip point of

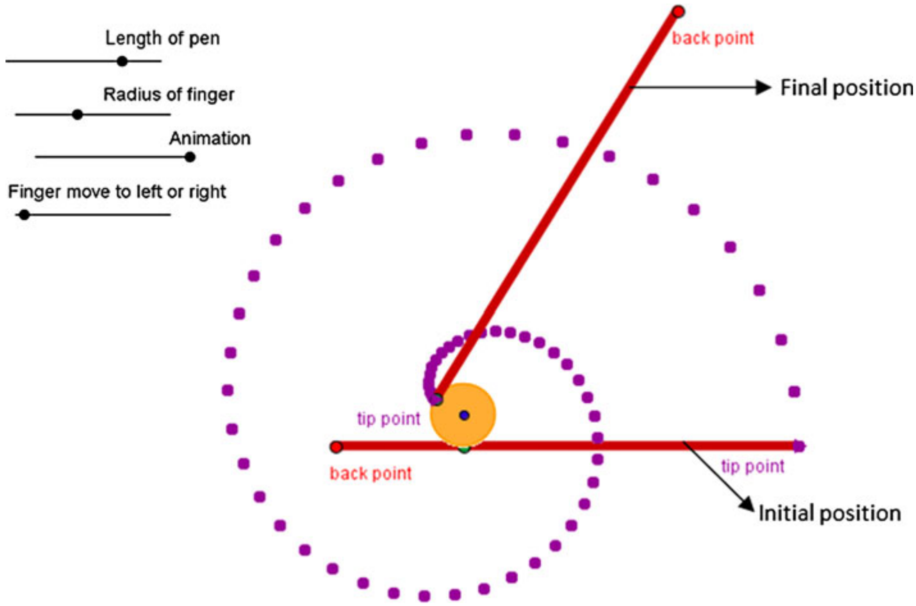

Fig. 2 The trace of the pencil's tip after the complete rotation

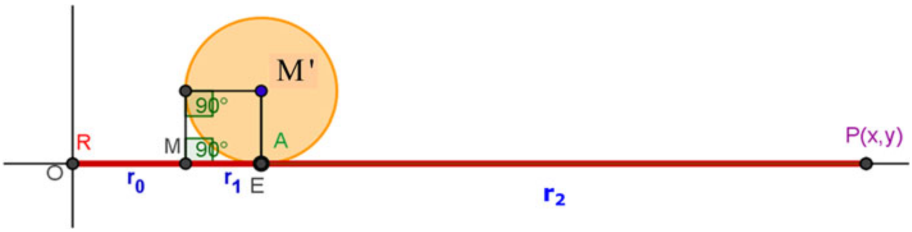

Fig. 3 GeoGebra Model of the first position of the pencil

the line segment as P(x, y). This point will be the producer of the curve. Let's call the back point of the line segment R and position it at the origin (Fig. 3). By defining the variables in Fig. 3 in terms of sliders of the visual model, we will try to obtain the parametric equation of the curve, which is produced by the point P(x, y). Lastly, the curve will also be drawn by using the GeoGebra command Curve[x-component, y-component, name of parameter, initial point, end point] to check if it is overlapped by the trace of the tip point P(x, y).

The length of the pen can be defined as $|PR| = r_0 + r_1 + r_2$, which corresponds to the slider "*length of pen*", where $|OM| = r_0, |ME| = r_1$, which corresponds to the slider "*Radius of finger*", and $|EP| = r_2$.

During the thumbaround movement, the pen will rotate around the circle whose center is M′. While the angle θ, which controls the rotation, corresponds to the slider "*Animation*", let's examine the mathematical situation at any position of the rotating pencil (Fig. 4).

The sum of the length of the circular arc EA and the length r_3 is equal to r_2 at the initial position of the pen. So, it can be written that $|PR| = r_0 + r_1 + r_3 + |\overset{\frown}{AE}|$ where $|OM| = r_0$,

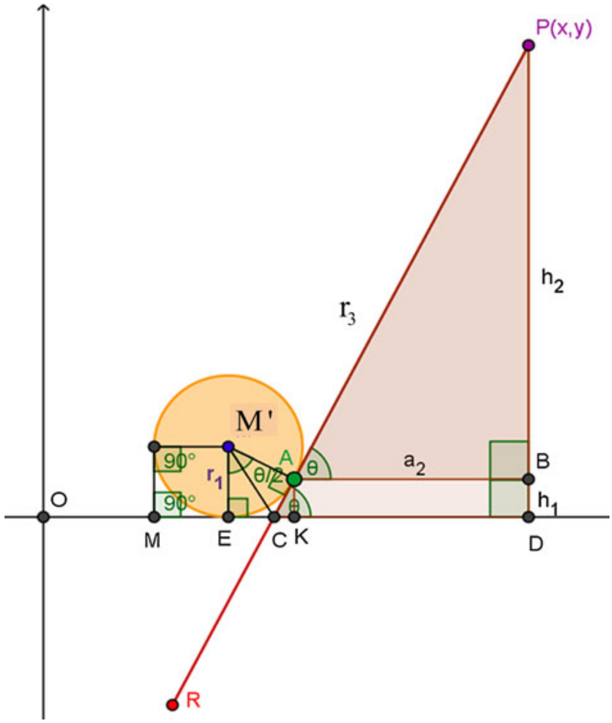

Fig. 4 The instant position of the pencil a moment later

$|ME| = r_1$ and $r_2 = r_3 + |\widehat{AE}|$. From here $r_3 = |PR| - r_0 - r_1 - |\widehat{AE}|$ is obtained. We can define the length of the arc $|\widehat{AE}|$ as $2\pi r_1\left(\frac{\theta}{2\pi}\right) = r_1\theta$ where θ is in radians. Now, we need to write x and y components of the point P in terms of θ; $x = |OM| + |ME| + |EC| + |CK| + |KD|$ where $|OM| = r_0$, $|ME| = r_1$ $|EC| = r_1\tan\left(\frac{\theta}{2}\right)$, $|CK| = r_1\tan\left(\frac{\theta}{2}\right)\cos(\theta)$ and $|KD| = r_3\cos(\theta)$. So, $x = r_0 + r_1 + r_1\tan\left(\frac{\theta}{2}\right) + r_1\tan\left(\frac{\theta}{2}\right)\cos(\theta) + r_3\cos(\theta)$.

Since, $y = h_1 + h_2$ where $h_1 = |AC|\sin(\theta) = r_1\tan\left(\frac{\theta}{2}\right)\sin(\theta)$ and $h_2 = r_3\sin(\theta)$, so, $y = r_1\tan\left(\frac{\theta}{2}\right)\sin(\theta) + r_3.\sin(\theta)$. At last, the parametric equation is obtained as

$$(x,y) = (f(\theta), g(\theta)) = \left(r_0 + r_1 + r_1\tan\left(\frac{\theta}{2}\right) + r_1\tan\left(\frac{\theta}{2}\right)\cos(\theta) + r_3\cos(\theta), r_1\tan\left(\frac{\theta}{2}\right)\sin(\theta) + r_3.\sin(\theta)\right).$$

2.3 Comparing the Mathematical Model with the Visual Model

The parametric equation above can be easily verified by showing that the curve obtained by the parametric equation and the trace of the tip point of the pencil overlap (Fig. 5).

We used "Curve code" from GeoGebra to draw the curve obtained by the parametric equation. We define r_o, r_1, r_2 and θ as in Figs. 3 and 4 and define $0 \le b \le 2\pi$ to be the value used to run the animation and $0 \le i \le 30$ to the position of the finger. Value of b "*run animation*" and value of i "*change position of finger*".

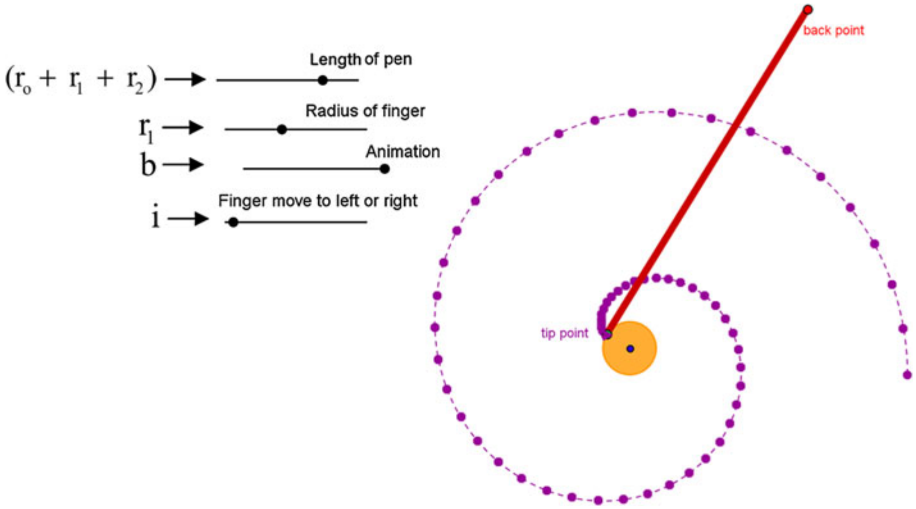

Fig. 5 Trace of the pencil's tip point and the curve obtained by the parametric equation

$$\mathrm{Curve}[(r_o + r_1 + r_2 - i - r_1 - 2\pi r_1\theta/360/^\circ)\cos(\theta) + i + r_1 + r_1\tan(\theta/2)\cos(\theta) + r_1\tan(\theta/2), \sin(\theta)(r_o + r_1 + r_2 - r_1 - i - 2\pi r_1\theta/360/^\circ) + r_1\tan(\theta/2)\sin(\theta), \theta, 0, (r_o + r_1 + r_2 - r_1 - i)/(2\pi r_1)b]$$

3 Conclusion

In this study, we obtained the parametric equation of the curve for a motion well-known to most students. In accordance with the literature, this study again brings up two important issues. First, a dynamic model of a real situation can help us to express and interpret the mathematical model (Doerr and Pratt 2008). Second, it is possible to support a better understanding of a mathematical model by showing its relationship with a graphical representation (Duval 1999). GeoGebra provides a suitable environment for creating such supports and for designing multiple represented models.

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