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The Two-Weighted Inequalities for Sublinear Operators Generated by B Singular Integrals in Weighted Lebesgue Spaces

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Abstract In this paper, the authors establish several general theorems for the boundedness of sublinear operators (B sublinear operators) satisfies the condition (1.2), generated by B singular integrals on a weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, where $B=\sum_{i=1}^k(\frac{\partial^2}{\partial x_k^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i})$. The condition (1.2) are satisfied by many important operators in analysis, including B maximal operator and B singular integral operators. Sufficient conditions on weighted functions ω and ω_1 are given so that B sublinear operators satisfies the condition (1.2) are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

Keywords Weighted Lebesgue space · B sublinear operator · B maximal operator · B singular integral operator · Two-weighted inequality

Mathematics Subject Classification (2000) 42B20 · 42B25 · 42B35

1 Introduction

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x=(x_1,\dots,x_n)$, $\xi=(\xi_1,\dots,\xi_n)$ are vectors in $\mathbb{R}^n$, $(x,\xi)=x_1\xi_1+\dots+x_n\xi_n$, $|x|=(x,x)^{\frac{1}{2}}$, $x=(x',x'')$, $x'=(x_1,\dots,x_k)$,

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$x'' = (x_{k+1}, \dots, x_n)$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $(x')^\gamma = x_1^{\gamma_1} \dots x_k^{\gamma_k}$. Let $\mathbb{R}^k_{++} = \{x' \in \mathbb{R}^k : x_1 > 0, \dots, x_k > 0\}$, $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0\}$, $1 \le k \le n$, $S_{k,+} = \{x \in \mathbb{R}^n_{k,+} : |x| = 1\}$.

For measurable set $E \subset \mathbb{R}^n_{k,+}$ let $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,k,\gamma) \times r^{n+|\gamma|}$, where $\omega(n,k,\gamma) = |E(0,1)|_\gamma$.

Let $f \in L^{\mathrm{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$ and B be the Laplace-Bessel differential operator:

$$\Delta_B = B + \Delta_{x''}, \quad B = \sum_{i=1}^k B_i, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \Delta_{x''} = \sum_{j=k+1}^n \frac{\partial^2}{\partial x_j^2}.$$

The B maximal function (see [9–12]) is defined by

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{\omega(n,k,\gamma) r^{n+|\gamma|}} \int_{E(0,r)} T^y |f(x)| \big(y'\big)^\gamma dy$$

and the B singular integral (see [2, 7, 15, 16, 19, 20]) defined by

$$K_\gamma f(x) = \text{p.v.} \int_{\mathbb{R}^n_{k,+}} K(y)\, T^y f(x)\, \big(y'\big)^\gamma dy,$$

where $K(rx) = r^{-n-|\gamma|} K(x)$ for each $r > 0$, $x \in \mathbb{R}^n_{k,+}$; $d\sigma$ is the element of area of the $S_{k,+}$; $\sup_{\theta \in S_{k,+}} |K(\theta)| < \infty$ and

$$\int_{S_{k,+}} K(x) \big(x'\big)^\gamma d\sigma(x) = 0. \tag{1.1}$$

We also consider the high order Riesz-Bessel transformations (see [19], pp. 60, see also [6])

$$R_B^{(k)} f(x) = \text{p.v.}\, c_k \int_{\mathbb{R}^n_{k,+}} \frac{P_k(y)}{|y|^{n+k+|\gamma|}} T^y f(x) \big(y'\big)^\gamma dy,$$

where $P_k(x) = P_k(x_1^2, \dots, x_k^2, x_{k+1}, \dots, x_n)$ is a homogeneous polynomial with order k which holds $\Delta_B P_k = 0$ and satisfies the cancellation condition (1.1).

The B maximal function has introduced and L_p boundedness investigated by V.S. Guliyev [9] (in the case $n = k = 1$ by K. Stempak [22], see also [10–12]). The singular integral operators that have been considered by S. Mihlin [17] and A. Calderon and A. Zygmund [3] are playing an important role in the theory Harmonic Analysis and in particular, in the theory partial differential equations. M. Klyuchantsev [16] and I. Kipriyanov and M. Klyuchantsev [15] have firstly introduced and investigated by the boundedness in L_p-spaces of multidimensional singular integrals, generated by the $B_{1,n}$ Laplace-Bessel differential operator ($B_{1,n}$ singular integrals). I.A. Aliev and A.D. Gadjiev [2], A.D. Gadjiev and E.V. Guliyev [7] and E.V. Guliyev [8] have studied the boundedness of $B_{1,n}$ singular integrals in weighted L_p-spaces with radial and general weights consequently.

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ with compact support and $x \notin \operatorname{supp} f$

$$|Tf(x)| \le c_0 \int_{\mathbb{R}^n_{k,+}} T^y |x|^{-n-|\gamma|} |f(y)| \big(y'\big)^\gamma dy, \tag{1.2}$$

where $T^y f(x)$ is the generalized shift operator (see below) and c_0 is independent of f and x.

In the paper, we shall prove the boundedness of the sublinear operators T satisfies the condition (1.2) (B sublinear operators), generated by B singular integral operators on a weighted $L_{p,\gamma}$ spaces. Sufficient conditions on weighted functions ω and ω_1 are given so that the sublinear operators T satisfies the condition (1.2) is bounded from the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ into $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

We point out that the condition (1.2) (see below) was first introduced by Emin Guliyev in [8], which in the case $\gamma=(0,\dots,0)$ and the generalized shift operator $T^y f(x)$ is the usual shift operator $f(x-y)$ was first introdused by Soria and Weiss in [21]. The condition (1.2) is satisfied by many interesting operators in harmonic analysis, such as the B singular integral operators (for example, for $k=1$ see [2, 7, 8, 15, 16, 19, 20]), the B Hardy–Littlewood maximal operator ([12], for $n=k=1$ see [22], for $k=1$ see [11] and for $k=n$ see [9]), the high order Riesz-Bessel transformations (see [6, 19]) and so on.

2 Notations and Preliminary Results

For $x\in\mathbb{R}^n_{k,+}$ and $r>0$, we denote by $E(x,r)=\{y\in\mathbb{R}^n_{k,+}:|x-y|<r\}$ the open ball centered at x of radius r, and by ${}^{\complement}E(x,r)=\mathbb{R}^n_{k,+}\setminus E(x,r)$ denote its complement, $E'(x',r)=\{y'\in\mathbb{R}^k_{++}:|x'-y'|<r\}$, ${}^{\complement}E'(x',r)=\mathbb{R}^k_{++}\setminus E'(x',r)$.

Let $1\le m<k\le n$, we put $\mathbb{R}^{n-m}_{k-m,+}=\{x_{m+1,n}=(x_{m+1},\dots,x_n):x_{m+1},\dots,x_k>0\}$, $x'\equiv x_{1,k}=(x_{1,m},x_{m+1,k})$, where $x_{1,m}=(x_1,\dots,x_m)\in\mathbb{R}^m_{++}$, $x_{m+1,k}=(x_{m+1},\dots,x_k)\in\mathbb{R}^{k-m}_{++}$. In this case we write $\gamma=(\gamma_{1,m},\gamma_{m+1,k})$, $\gamma_{1,m}=(\gamma_1,\dots,\gamma_m)$, $\gamma_{m+1,k}=(\gamma_{m+1},\dots,\gamma_k)$, $x_{1,m}^{\gamma_{1,m}}=x_1^{\gamma_1}\dots x_m^{\gamma_m}$, $x_{m+1,k}^{\gamma_{m+1,k}}=x_{m+1}^{\gamma_{m+1}}\dots x_k^{\gamma_k}$, $E_m(x_{1,m},r)=\{y_{1,m}\in\mathbb{R}^m_{++}:|x_{1,m}-y_{1,m}|<r\}$.

Note that $E_k(x_{1,k},r)=E'(x',r)$ and $E_n(x_{1,n},r)=E(x,r)$.

An almost everywhere positive and locally integrable function $\omega:\mathbb{R}^n_{k,+}\to\mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ the set of all measurable function f on $\mathbb{R}^n_{k,+}$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})}\equiv\|f\|_{p,\omega,\gamma;\mathbb{R}^n_{k,+}}=\left(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x)\big(x'\big)^\gamma dx\right)^{1/p},\quad 1\le p<\infty$$

is finite. For $\omega=1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_{k,+})$, and the norm $\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})}$ by $\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})}$.

The operator of generalized shift (B-shift operator) is defined by the following way (see [12, 18–20]):

$$T^y f(x)=C_{\gamma,k}\int_0^\pi\dots\int_0^\pi f\big((x',y')_\beta,x''-y''\big)d\nu(\beta),$$

where $C_{\gamma,k}=\pi^{-\frac{k}{2}}\Gamma^{-1}(\frac{|\gamma|}{2})\prod_{i=1}^k\Gamma(\frac{\nu_i+1}{2})$, $(x',y')_\beta=((x_1,y_1)_{\beta_1}\cdots(x_k,y_k)_{\beta_k})$, $(x_i,y_i)_{\beta_i}=(x_i^2-2x_iy_i\cos\beta_i+y_i^2)^{1/2}$, $1\le i\le k$.

Note that this shift operator is closely connected with B-Laplace-Bessel singular differential operators (see [12, 19]).

The following generalized Hardy inequalities have an important role in proofs of our main results see [5], Chap. 1 (see also [1, 4, 14]).

Lemma 2.1 *Suppose that $1\le p\le q\le\infty$, $p'=p/(p-1)$ and $\omega(x)$ and $v(x)$ are positive functions defined on $\mathbb{R}^n$.*

1. *For the n-dimensional Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|<|x|/2}|f(y)|dy\right)^q\omega(x)dx\right)^{1/q}\le C_5\left(\int_{\mathbb{R}^n}|f(x)|^p\upsilon(x)dx\right)^{1/p}$$

with a constant C_5, independent on f, to hold, it is necessary and sufficient that the following condition be satisfied:

$$\sup_{r>0}\left(\int_{|x|>2r}\omega(x)dx\right)^{1/q}\left(\int_{|x|<r}\upsilon^{1-p'}(x)dx\right)^{1/p'}<\infty.$$

2. *For the n-dimensional* (*dual*) *Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|>2|x|}|f(y)|dy\right)^q\omega(x)dx\right)^{1/q}\le C_6\left(\int_{\mathbb{R}^n}|f(x)|^p\upsilon(x)dx\right)^{1/p}$$

with a constant C_6, independent on f, to hold, it is necessary and sufficient that the following condition be satisfied:

$$\sup_{r>0}\left(\int_{|x|<r}\omega(x)dx\right)^{1/q}\left(\int_{|x|>2r}\upsilon^{1-p'}(x)dx\right)^{1/p'}<\infty.$$

This lemma could be directly deduced from results proved by P. Drabek, H. Heinig and A. Kufner (see Theorem 2.1, p. 4 and Theorem 2.2, p. 7 in [13]).

3 Main Results

In the following theorems we give the boundedness in weighted $L_{p,\gamma}$ spaces for the B sublinear operators satisfies the condition (1.2).

Theorem 3.1 *Let $p\in[1,\infty)$, T be a B sublinear operator satisfies condition* (1.2), *and for $p>1$ bounded on $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and for $p=1$ bounded from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to the weak $WL_{1,\gamma}(\mathbb{R}^n_{k,+})$. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and the following three conditions are satisfied*:

($a_{k,n}$) *there exist $b>0$ such that*

$$\sup_{|x|/8<|y|\le 8|x|}\omega_1(y)\le b\,\omega(x)\quad \text{for a.e. } x\in\mathbb{R}^n_{k,+},$$

$$(b_{k,n})\quad \mathcal{A}_{k,n}\equiv\sup_{r>0}\left(\int_{\complement E(0,2r)}\omega_1(x)|x|^{-(n+|\gamma|)p}\left(x'\right)^\gamma dx\right)\left(\int_{E(0,r)}\omega^{1-p'}(x)\left(x'\right)^\gamma dx\right)^{p-1}<\infty,$$

$$(c_{k,n})\quad \mathcal{B}_{k,n}\equiv\sup_{r>0}\left(\int_{E(0,r)}\omega_1(x)\left(x'\right)^\gamma dx\right)\left(\int_{\complement E(0,2r)}\omega^{1-p'}(x)|x|^{-(n+|\gamma|)p'}\left(x'\right)^\gamma dx\right)^{p-1}<\infty.$$

Then, for $p>1$ the operator T is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ and for $p=1$ from $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$. Moreover, there exists a constant c, independent of f, such that for all $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ for $p>1$

$$\int_{\mathbb{R}^n_{k,+}}|Tf(x)|^p\omega_1(x)\left(x'\right)^\gamma dx\le c\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x)\left(x'\right)^\gamma dx \tag{3.1}$$

and for $p = 1$

$$\int_{\{x\in\mathbb{R}^n_{k,+}:|Tf(x)|>\lambda\}} \omega_1(x)\big(x'\big)^\gamma dx \le \frac{c}{\lambda^p}\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(x)\big(x'\big)^\gamma dx. \tag{3.2}$$

Remark 3.1 Note that, the condition $(a_{k,n})$ can be replaced by the condition $(a_{k,n}{}')$ there exist $b > 0$ such that

$$\omega_1(x)\left(\sup_{|x|/8<|y|\le 8|x|}\frac{1}{\omega(y)}\right)\le b \quad \text{for a.e. } x\in\mathbb{R}^n.$$

Corollary 3.1 *Let* $p \in [1,\infty)$ *and* T *be the* B *maximal operator or the* B *singular integral operator. Moreover, let* $\omega(x)$, $\omega_1(x)$ *be weight functions on* $\mathbb{R}^n_{k,+}$ *and conditions* $(a_{k,n})$, $(b_{k,n})$, $(c_{k,n})$ *be satisfied. Then, for* $p > 1$ *the operator* T *is bounded from* $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ *and for* $p = 1$ *from* $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

Remark 3.2 Note that, the Theorem 3.1 in the case $k = 1$ was proved in [8] and the Corollary 3.1 for the high order Riesz-Bessel transformations was proved in [6].

Theorem 3.2 *Let* $p \in [1,\infty)$, $1 \le m < k \le n$, T *be the* B *sublinear operator satisfying* (1.2) *and for* $p > 1$ *bounded on* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and for* $p = 1$ *bounded from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to the weak* $WL_{1,\gamma}(\mathbb{R}^n_{k,+})$. *Moreover, let* $\omega(x_{1,m})$, $\omega_1(x_{1,m})$ *be weight functions on* $\mathbb{R}^m_{++}$ *and the following three conditions are satisfied*:

$(a_{k,m})$ *there exists a constant* $b > 0$ *such that*

$$\sup_{|x_{1,m}|/8<|y_{1,m}|<8|x_{1,m}|}\omega_1(y_{1,m}) \le b\,\omega(x_{1,m}) \quad \textit{for a.e. } x_{1,m}\in\mathbb{R}^m_{++},$$

$$(b_{k,m})\quad \mathcal{A}_{m,k} \equiv \sup_{r>0}\left(\int_{\complement E_m(0,2r)}\omega_1(x_{1,m})|x_{1,m}|^{-(m+|\gamma_{1,m}|)p}x_{1,m}^{\gamma_{1,m}}\,dx_{1,m}\right) \times\left(\int_{E_m(0,r)}\omega^{1-p'}(x_{1,m})x_{1,m}^{\gamma_{1,m}}\,dx_{1,m}\right)^{p-1}<\infty,$$

$$(c_{k,m})\quad \mathcal{B}_{m,k} \equiv \sup_{r>0}\left(\int_{E_m(0,r)}\omega_1(x_{1,m})\,x_{1,m}^{\gamma_{1,m}}\,dx_{1,m}\right) \times\left(\int_{\complement E_m(0,2r)}\omega^{1-p'}(x_{1,m})|x_{1,m}|^{-(m+|\gamma_{1,m}|)p'}x_{1,m}^{\gamma_{1,m}}\,dx_{1,m}\right)^{p-1}<\infty.$$

Then, for $p > 1$ *the operator* T *is bounded from* $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ *and for* $p = 1$ *from* $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$. *Moreover, there exists a constant* c, *independent of* f, *such that for all* $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *for* $p > 1$

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p\omega_1(x_{1,m})\big(x'\big)^\gamma dx \le c\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(x_{1,m})\big(x'\big)^\gamma dx \tag{3.3}$$

and for $p = 1$

$$\int_{\{x\in\mathbb{R}^n_{k,+}:|Tf(x)|>\lambda\}} \omega_1(x_{1,m})\big(x'\big)^\gamma dx \le \frac{c}{\lambda^p}\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(x_{1,m})\big(x'\big)^\gamma dx. \tag{3.4}$$

Remark 3.3 Note that, the condition $(a_{k,m})$ can be replaced by the condition $(a_{k,m}{}')$ there exist $b > 0$ such that

$$\omega_1(x_{1,m})\bigg(\sup_{|x_{1,m}|/8<|y_{1,m}|<8|x_{1,m}|} \frac{1}{\omega(y_{1,m})}\bigg) \le b \quad \text{for a.e. } x_{1,m} \in \mathbb{R}^m.$$

Corollary 3.2 *Let $p \in [1,\infty)$, $1 \le m < k \le n$ and T be the B maximal operator or the B singular integral operator. Moreover, let $\omega(x_{1,m})$, $\omega_1(x_{1,m})$ be weight functions on $\mathbb{R}^m_{++}$ and conditions $(a_{k,m})$, $(b_{k,m})$, $(c_{k,m})$ be satisfied. Then, for $p > 1$ the operator T is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ and for $p = 1$ from $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

Theorem 3.3 *Let $p \in [1,\infty)$, $1 \le m < k \le n$, T be a B sublinear operator satisfying (1.2) and for $p > 1$ bounded on $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and for $p = 1$ bounded from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to the weak $WL_{1,\gamma}(\mathbb{R}^n_{k,+})$. Moreover, let $\omega(x_{m+1,n})$, $\omega_1(x_{m+1,n})$ be weight functions on $\mathbb{R}^{n-m}_{k-m,+}$ and the following three conditions are satisfied:*

$(a_{m+1,k})$ *there exists a constant $b > 0$ such that*

$$\sup_{|x_{m+1,n}|/8<|y_{m+1,n}|<8|x_{m+1,n}|} \omega_1(x_{m+1,n}) \le b\,\omega(x_{m+1,n}) \quad \textit{for a.e. } x_{m+1,n} \in \mathbb{R}^{n-m}$$

$$(b_{m+1,k}) \quad \mathcal{A}_{m+1,k} \equiv \sup_{r>0}\bigg(\int_{E_{n-m}(0,r)} \omega^{1-p'}(x_{m+1,n}) x_{m+1,k}^{\gamma_{m+1,k}}\, dx_{m+1,n}\bigg)^{p-1} \times \bigg(\int_{\complement E_{n-m}(0,2r)} \omega_1(x_{m+1,n})|x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p} x_{m+1,k}^{\gamma_{m+1,k}}\, dx_{m+1,n}\bigg) < \infty,$$

$$(c_{m+1,k}) \quad \mathcal{B}_{m+1,k} \equiv \sup_{r>0}\bigg(\int_{E_{n-m}(0,r)} \omega_1(x_{m+1,n}) x_{m+1,k}^{\gamma_{m+1,k}}\, dx_{m+1,n}\bigg) \times \bigg(\int_{\complement E_{n-m}(0,2r)} \omega^{1-p'}(x_{m+1,n})|x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p'} x_{m+1,k}^{\gamma_{m+1,k}}\, dx_{m+1,n}\bigg)^{p-1} < \infty.$$

Then, for $p > 1$ the operator T is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ and for $p = 1$ from $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$. Moreover, there exists a constant c, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ for $p > 1$

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx \le c\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})\big(x'\big)^\gamma dx \tag{3.5}$$

and for $p = 1$

$$\int_{\{x\in\mathbb{R}^n_{k,+}\,:\,|Tf(x)|>\lambda\}} \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx \le \frac{c}{\lambda^p}\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})\big(x'\big)^\gamma dx. \tag{3.6}$$

Remark 3.4 Note that, the condition $(a_{m+1,k})$ can be replaced by the condition $(a_{m+1,k}{}')$ there exist $b > 0$ such that

$$\omega_1(x_{m+1,n})\bigg(\sup_{|x_{m+1,n}|/8<|y_{m+1,n}|<8|x_{m+1,n}|} \frac{1}{\omega(y_{m+1,n})}\bigg) \le b \quad \text{for a.e. } x_{m+1,n} \in \mathbb{R}^{n-m}_{k-m,+}.$$

Corollary 3.3 *Let $p \in [1,\infty)$, $1 \le m < k \le n$ and let T be the B maximal operator or the B singular integral operator. Moreover, let $\omega(x_{m+1,n})$, $\omega_1(x_{m+1,n})$ be weight functions on $\mathbb{R}^{n-m}_{k-m,+}$ and conditions $(a_{m+1,k})$, $(b_{m+1,k})$, $(c_{m+1,k})$ be satisfied. Then, for $p>1$ the operator T is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ and for $p=1$ from $L_{1,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

4 Proof of the Theorems

Note that by the same argument in Theorem 3.2 we may be proved Theorem 3.1.

Proof of Theorem 3.2 For $l \in Z$ we define $\widetilde{E}_l = \{x \in \mathbb{R}^n_{k,+} : 2^l < |x_{1,m}| \le 2^{l+1}\}$, $\widetilde{E}_{l,1} = \{x \in \mathbb{R}^n_{k,+} : |x_{1,m}| \le 2^{l-1}\}$, $\widetilde{E}_{l,2} = \{x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x_{1,m}| \le 2^{l+2}\}$, $\widetilde{E}_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x_{1,m}| > 2^{l+2}\}$. Then $\widetilde{E}_{l,2} = \widetilde{E}_{l-1} \cup \widetilde{E}_l \cup \widetilde{E}_{l+1}$ and the multiplicity of the covering $\{\widetilde{E}_{l,2}\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$\begin{aligned}
|Tf(x)| &= \sum_{l\in Z} |Tf(x)|\chi_{\widetilde{E}_l}(x) \\
&\le \sum_{l\in Z} \big|Tf_{l,1}(x)\big|\chi_{\widetilde{E}_l}(x) + \sum_{l\in Z} \big|Tf_{l,2}(x)\big|\chi_{\widetilde{E}_l}(x) + \sum_{l\in Z} \big|Tf_{l,3}(x)\big|\chi_{\widetilde{E}_l}(x) \\
&\equiv T_1 f(x) + T_2 f(x) + T_3 f(x), \qquad (4.1)
\end{aligned}$$

where $\chi_{\widetilde{E}_l}$ is the characteristic function of the set $\widetilde{E}_l$, $f_{l,i} = f\chi_{\widetilde{E}_{l,i}}$, $i=1,2,3$. We shall estimate $\|T_1 f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,1}$ we have $|y_{1,m}| \le 2^{l-1} \le |x_{1,m}|/2$. Moreover, $\widetilde{E}_l \cap \operatorname{supp} f_{l,1} = \emptyset$ and $|x_{1,m} - y_{1,m}| \ge |x_{1,m}|/2$. Hence by (1.2)

$$\begin{aligned}
T_1 f(x) &\le c_4 \sum_{l\in Z} \left(\int_{\mathbb{R}^n_{k,+}} |f_{l,1}(y)| T^y |x|^{-n-|\gamma|} \big(y'\big)^\gamma dy \right) \chi_{\widetilde{E}_l} \\
&\le c_4 \int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{E_m(0,|x_{1,m}|/2)} T^y |x|^{-n-|\gamma|} |f(y)| \big(y'\big)^\gamma dy \\
&\le c_5 \int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{E_m(0,|x_{1,m}|/2)} \big(|x_{1,m}| + |x_{m+1,n} - y_{m+1,n}|\big)^{-n-|\gamma|} |f(y)| \big(y'\big)^\gamma dy_{1,m} dy_{m+1,n}
\end{aligned}$$

for any $x \in \widetilde{E}_l$. Using this last inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \\
&\quad\le c_5^p \int_{\mathbb{R}^n_{k,+}} \left(\int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{E_m(0,|x_{1,m}|/2)} \big(|x_{1,m}| + |x_{m+1,n} - y_{m+1,n}|\big)^{-n-|\gamma|} \right. \\
&\qquad \left. \times |f(y)| y_{1,m}^{\gamma_{1,m}} dy_{1,m}\, y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx.
\end{aligned}$$

For $x=(x_{1,m},x_{m+1,n})\in\mathbb{R}^n_{k,+}$ let

$$
\begin{aligned}
I(x_{1,m}) &= \int_{\mathbb{R}^{n-m}_{k-m,+}}\left(\int_{\mathbb{R}^{n-m}_{k-m,+}}\int_{E_m(0,|x_{1,m}|/2)}(|x_{1,m}|+|x_{m+1,n}-y_{m+1,n}|)^{-n-|\gamma|}\right.\\
&\qquad\left.\times|f(y_{1,m},y_{m+1,n})|y_{1,m}^{\gamma_{1,m}}dy_{1,m}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\right)^p(x_{m+1,k})^{\gamma_{m+1,k}}dx_{m+1,n}\\
&= \int_{\mathbb{R}^{n-m}_{k-m,+}}\left(\int_{E_m(0,|x_{1,m}|/2)}\left(\int_{\mathbb{R}^{n-m}_{k-m,+}}(|x_{1,m}|+|x_{m+1,n}-y_{m+1,n}|)^{-n-|\gamma|}\right.\right.\\
&\qquad\left.\left.\times|f(y_{1,m},y_{m+1,n})|y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\right)y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right)^p(x_{m+1,k})^{\gamma_{m+1,k}}dx_{m+1,n}.
\end{aligned}
$$

Using the Minkowski and Young inequalities we obtain

$$
\begin{aligned}
I(x_{1,m}) &\le \left[\int_{E_m(0,|x_{1,m}|/2)}\left(\int_{\mathbb{R}^{n-m}_{k-m,+}}|f(y_{1,m},y_{m+1,n})|^p y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\right)^{1/p}\right.\\
&\qquad\left.\times\left(\int_{\mathbb{R}^{n-m}_{k-m,+}}\frac{(x_{m+1,k})^{\gamma_{m+1,k}}dx_{m+1,n}}{(|x_{1,m}|+|x_{m+1,n}|)^{n+|\gamma|}}\right)y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right]^p\\
&= \left(\int_{E_m(0,|x_{1,m}|/2)}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right)^p\\
&\qquad\times\left(\int_{\mathbb{R}^{n-m}_{k-m,+}}\frac{(x_{m+1,k})^{\gamma_{m+1,k}}dx_{m+1,n}}{(|x_{1,m}|+|x_{m+1,n}|)^{n+|\gamma|}}\right)^p\\
&= |x_{1,m}|^{-(m+|\gamma_{1,m}|)p}\left(\int_{E_m(0,|x_{1,m}|/2)}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right)^p\\
&\qquad\times\left(\int_{\mathbb{R}^{n-m}_{k-m,+}}\frac{dx_{m+1,n}}{(|x_{m+1,n}|+1)^{n+|\gamma|}}\right)^p\\
&= c_6|x_{1,m}|^{-(m+|\gamma_{1,m}|)p}\left(\int_{E_m(0,|x_{1,m}|/2)}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right)^p.
\end{aligned}
$$

Integrating in $\mathbb{R}^m_{++}$ we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n_{k,+}}|T_1f(x)|^p\omega_1(x_{1,m})\big(x'\big)^{\gamma}dx\\
&\quad\le c_7\int_{\mathbb{R}^m_{++}}\omega_1(x_{1,m})|x_{1,m}|^{-(m+|\gamma_{1,m}|)p}\\
&\qquad\times\left(\int_{E_m(0,|x_{1,m}|/2)}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right)^p x_{1,m}^{\gamma_{1,m}}dx_{1,m}.
\end{aligned}
$$

Since $\mathcal{A}_{m,k}<\infty$, the Hardy inequality

$$
\int_{\mathbb{R}^m_{++}}\left(\int_{E_m(0,|x_{1,m}|/2)}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\right)^p|x_{1,m}|^{-(m+|\gamma_{1,m}|)p}\omega_1(x_{1,m})x_{1,m}^{\gamma_{1,m}}\,dx_{1,m}
$$

$$\leq C \int_{\mathbb{R}^m_{++}} \|f(x_{1,m},\cdot)\|^p_{p,\mathbb{R}^{n-m}_{k-m,+}} \omega(x_{1,m})\, x_{1,m}^{\gamma_{1,m}}\, dx_{1,m}$$

holds and $C \leq c' \mathcal{A}_{m,k}$, where c' depends only on n and p. In fact the condition $\mathcal{A}_{m,k} < \infty$ is necessary and sufficient for the validity of this inequality (see Lemma 2.1). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \leq c_9 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m}) \big(x'\big)^\gamma dx. \tag{4.2}$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As is easy to verify, for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,3}$ we have $|y_{1,m}| > 2|x_{1,m}|$ and $|x_{1,m} - y_{1,m}| \geq |y_{1,m}|/2$. Since $\widetilde{E}_l \cap \operatorname{supp} f_{l,3} = \emptyset$, for $x \in \widetilde{E}_l$ by (1.2) we obtain

$$\begin{aligned} T_3 f(x) \leq c_5 \int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{\complement E_m(0,2|x_{1,m}|)} |f(y)| (|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|)^{-n-|\gamma|} \\ \times y_{1,m}^{\gamma_{1,m}} dy_{1,m}\, y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}. \end{aligned}$$

Using this last inequality we have

$$\begin{aligned} &\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \\ &\leq c_5^p \int_{\mathbb{R}^n_{k,+}} \bigg(\int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{\complement E_m(0,2|x_{1,m}|)} (|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|)^{-n-|\gamma|} \\ &\quad \times |f(y)| y_{1,m}^{\gamma_{1,m}} dy_{1,m} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \bigg)^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx. \end{aligned}$$

For $x = (x_{1,m}, x_{m+1,n}) \in \mathbb{R}^n_{k,+}$ let

$$\begin{aligned} I_1(x_{1,m}) = \int_{\mathbb{R}^{n-m}_{k-m,+}} \bigg(\int_{\complement E_m(0,2|x_{1,m}|)} \int_{\mathbb{R}^{n-m}_{k-m,+}} |f(y)| (|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|)^{-n-|\gamma|} \\ \times y_{1,m}^{\gamma_{1,m}} dy_{1,m} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \bigg)^p (x_{m+1,k})^{\gamma_{m+1,k}} dx_{m+1,n}. \end{aligned}$$

Using the Minkowski and Young inequalities we obtain

$$\begin{aligned} I_1(x_{1,m}) &\leq \Bigg[\int_{\complement E_m(0,2|x_{1,m}|)} \bigg(\int_{\mathbb{R}^{n-m}_{k-m,+}} |f(y)|^p y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \bigg)^{1/p} \\ &\quad \times \bigg(\int_{\mathbb{R}^{n-m}_{k-m,+}} \frac{y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}}{(|y_{1,m}| + |y_{m+1,n}|)^{n+|\gamma|}} \bigg) y_{1,m}^{\gamma_{1,m}} dy_{1,m} \Bigg]^p \\ &= c_6 \bigg(\int_{\complement E_m(0,2|x_{1,m}|)} |y_{1,m}|^{-m-|\gamma_{1,m}|} \|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \bigg)^p \\ &\quad \times \bigg(\int_{\mathbb{R}^{n-m}_{k-m,+}} \frac{y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}}{(|y_{m+1,n}| + 1)^{n+|\gamma|}} \bigg)^p \\ &= c_7 \bigg(\int_{\complement E_m(0,2|x_{1,m}|)} |y_{1,m}|^{-m-|\gamma_{1,m}|} \|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \bigg)^p. \end{aligned}$$

Integrating over $\mathbb{R}^m_{++}$ we get

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx$$

$$\le c_8 \int_{\mathbb{R}^m_{++}} \left(\int_{\complement E_m(0,2|x_{1,m}|)} |y_{1,m}|^{-m-|\gamma_{1,m}|} \|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right)^p$$

$$\times \omega_1(x_{1,m}) x_{1,m}^{\gamma_{1,m}} dx_{1,m}.$$

Since $\mathcal{B}_{m,k} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^m_{++}} \omega_1(x_{1,m}) \left(\int_{\complement E_m(0,2|x_{1,m}|)} |y_{1,m}|^{-m-|\gamma_{1,m}|} \|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right)^p x_{1,m}^{\gamma_{1,m}} dx_{1,m}$$

$$\le C \int_{\mathbb{R}^m_{++}} \|f(x_{1,m},\cdot)\|^p_{p,\mathbb{R}^{n-m}_{k-m,+}} |x_{1,m}|^{-(m+|\gamma_{1,m}|)p} \omega(x_{1,m}) |x_{1,m}|^{(m+|\gamma|)p} x_{1,m}^{\gamma_{1,m}} dx_{1,m}$$

$$= C \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m}) \big(x'\big)^\gamma dx$$

holds and $C \le c' \mathcal{B}_{m,k}$, where c' depends only on n, γ and p. In fact the condition $\mathcal{B}_{m,k} < \infty$ is necessary and sufficient for the validity of this inequality (see Lemma 2.1). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \le c_{10} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m}) \big(x'\big)^\gamma dx. \tag{4.3}$$

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of T and condition $(a_{k,m})$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx &= \int_{\mathbb{R}^n_{k,+}} \left(\sum_{l\in Z} \big|T f_{l,2}(x)\big| \chi_{\widetilde{E}_l}(x) \right)^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\sum_{l\in Z} \big|T f_{l,2}(x)\big|^p \chi_{\widetilde{E}_l}(x) \right) \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \\
&= \sum_{l\in Z} \int_{\widetilde{E}_l} \big|T f_{l,2}(x)\big|^p \omega_1(x_{1,m}) \big(x'\big)^\gamma dx \\
&\le \sum_{l\in Z} \sup_{y\in \widetilde{E}_l} \omega_1(y_{1,m}) \int_{\mathbb{R}^n_{k,+}} \big|T f_{l,2}(x)\big|^p \big(x'\big)^\gamma dx \\
&\le \|T\|^p \sum_{l\in Z} \sup_{y\in \widetilde{E}_l} \omega_1(y_{1,m}) \int_{\mathbb{R}^n_{k,+}} \big|f_{l,2}(x)\big|^p \big(x'\big)^\gamma dx \\
&= \|T\|^p \sum_{l\in Z} \sup_{y\in \widetilde{E}_l} \omega_1(y_{1,m}) \int_{\widetilde{E}_{l,2}} |f(x)|^p \big(x'\big)^\gamma dx,
\end{aligned}$$

where $\|T\| \equiv \|T\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})\to L_{p,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in \widetilde{E}_{l,2}$, $2^{l-1} < |x_{1,m}| \le 2^{l+2}$, we have by condition $(a_{k,m})$

$$\sup_{y\in\widetilde{E}_l} \omega_1(y_{1,m}) = \sup_{2^{l-1}<|y_{1,m}|\le 2^{l+2}} \omega_1(y_{1,m})$$
$$\le \sup_{|x_{1,m}|/8<|y_{1,m}|<8|x_{1,m}|} \omega_1(y_{1,m}) \le b\,\omega(x_{1,m})$$

for almost all $x \in \widetilde{E}_{l,2}$. Therefore

$$\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x_{1,m})\big(x'\big)^\gamma dx \le b\,\|T\|^p \sum_{l\in Z} \int_{\widetilde{E}_{l,2}} |f(x)|^p \omega(x_{1,m}) dx$$
$$\le c_{11} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m})\big(x'\big)^\gamma dx, \tag{4.4}$$

where $c_{11} = 3\|T\|^p b$, since the multiplicity of covering $\{\widetilde{E}_{l,2}\}_{l\in Z}$ is equal to 3. Inequalities (4.1), (4.2), (4.3), (4.4) imply (3.3) which completes the proof.

Similarly we prove the following weak variant Theorem 3.2.

Thus Theorem 3.2 is proved. □

Proof of Theorem 3.3 For $l \in Z$ we define $\widetilde{E}_l = \{x \in \mathbb{R}^n_{k,+} : 2^l < |x_{m+1,n}| \le 2^{l+1}\}$, $\widetilde{E}_{l,1} = \{x \in \mathbb{R}^n_{k,+} : |x_{m+1,n}| \le 2^{l-1}\}$, $\widetilde{E}_{l,2} = \{x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x_{m+1,n}| \le 2^{l+2}\}$, $\widetilde{E}_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x_{m+1,n}| > 2^{l+2}\}$. Then $\widetilde{E}_{l,2} = \widetilde{E}_{l-1} \cup \widetilde{E}_l \cup \widetilde{E}_{l+1}$ and the multiplicity of the covering $\{\widetilde{E}_{l,2}\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$|Tf(x)| = \sum_{l\in Z} |Tf(x)|\chi_{\widetilde{E}_l}(x)$$
$$\le \sum_{l\in Z}\big|Tf_{l,1}(x)\big|\chi_{\widetilde{E}_l}(x) + \sum_{l\in Z}\big|Tf_{l,2}(x)\big|\chi_{\widetilde{E}_l}(x) + \sum_{l\in Z}\big|Tf_{l,3}(x)\big|\chi_{\widetilde{E}_l}(x)$$
$$\equiv T_1 f(x) + T_2 f(x) + T_3 f(x), \tag{4.5}$$

where $\chi_{\widetilde{E}_l}$ is the characteristic function of the set $\widetilde{E}_l$, $f_{l,i} = f\chi_{\widetilde{E}_{l,i}}$, $i = 1,2,3$. We shall estimate $\|T_1 f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,1}$ we have $|y_{m+1,n}| \le 2^{l-1} \le |x_{m+1,n}|/2$. Moreover, $\widetilde{E}_l \cap \operatorname{supp} f_{l,1} = \emptyset$ and $|x_{m+1,n} - y_{m+1,n}| \ge |x_{m+1,n}|/2$. Hence by (1.2)

$$T_1 f(x) \le c_4 \sum_{l\in Z}\left(\int_{\mathbb{R}^n_{k,+}} |f_{k,1}(y)| T^y |x|^{-n-|\gamma|}\big(y'\big)^\gamma dy\right)\chi_{\widetilde{E}_l}$$
$$\le c_4 \int_{\mathbb{R}^m_{++}} \int_{E_{n-m}(0,|x_{m+1,n}|/2)} T^y |x|^{-n-|\gamma|} |f(y)|\big(y'\big)^\gamma dy$$
$$\le c_5 \int_{\mathbb{R}^m_{++}} \int_{E_{n-m}(0,|x_{m+1,n}|/2)} (|x_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|} |f(y)|\big(y'\big)^\gamma dy_{1,m} dy_{m+1,n}$$

for any $x \in \widetilde{E}_l$. Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x_{m+1,n}) (x')^\gamma dx$$
$$\leq c_5^p \int_{\mathbb{R}^n_{k,+}} \left(\int_{\mathbb{R}^m_{++}} \int_{E_{n-m}(0,|x_{m+1,n}|/2)} (|x_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|} \right.$$
$$\left. \times |f(y)| (y')^\gamma dy_{1,m} dy_{m+1,n} \right)^p \omega_1(x_{m+1,n}) (x')^\gamma dx.$$

For $x = (x_{1,m}, x_{m+1,n}) \in \mathbb{R}^n_{k,+}$ let

$$I(x_{m+1,n}) = \int_{\mathbb{R}^m_{++}} \left(\int_{\mathbb{R}^m_{++}} \int_{E_{n-m}(0,|x_{m+1,n}|/2)} (|x_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|} \right.$$
$$\left. \times |f(y)| (y')^\gamma dy_{m+1,n} dy_{1,m} \right)^p x_{1,m}^{\gamma_{1,m}} dx_{1,m}$$
$$= \int_{\mathbb{R}^m_{++}} \left(\int_{E_{n-m}(0,|x_{m+1,n}|/2)} \left(\int_{\mathbb{R}^m_{++}} (|x_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|} \right. \right.$$
$$\left. \left. \times |f(y_{1,m}, y_{m+1,n})| y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right) y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p x_{1,m}^{\gamma_{1,m}} dx_{1,m}.$$

Using the Minkowski and Young inequalities we obtain

$$I(x_{m+1,n}) \leq \left[\int_{E_{n-m}(0,|x_{m+1,n}|/2)} \left(\int_{\mathbb{R}^m_{++}} |f(y_{1,m}, y_{m+1,n})|^p y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right)^{1/p} \right.$$
$$\left. \times \left(\int_{\mathbb{R}^m_{++}} \frac{x_{1,m}^{\gamma_{1,m}} dx_{1,m}}{(|x_{1,m}| + |x_{m+1,n}|)^{n+|\gamma|}} \right) y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right]^p$$
$$= \left(\int_{E_{n-m}(0,|x_{m+1,n}|/2)} \| f(y_{1,m}, \cdot) \|_{p,\mathbb{R}^m_{++}} (y')^\gamma y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p$$
$$\times \left(\int_{\mathbb{R}^m_{++}} \frac{x_{1,m}^{\gamma_{1,m}} dx_{1,m}}{(|x_{1,m}| + |x_{m+1,n}|)^{n+|\gamma|}} \right)^p$$
$$= |x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p}$$
$$\times \left(\int_{E_{n-m}(0,|x_{m+1,n}|/2)} \| f(\cdot, y_{m+1,n}) \|_{p,\mathbb{R}^m_{++}} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p$$
$$\times \left(\int_{\mathbb{R}^m_{++}} \frac{x_{1,m}^{\gamma_{1,m}} dx_{1,m}}{(|x_{1,m}| + 1)^{n+|\gamma|}} \right)^p |x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p}$$
$$= c_6 \left(\int_{E_{n-m}(0,|x_{m+1,n}|/2)} \| f(\cdot, y_{m+1,n}) \|_{p,\mathbb{R}^m_{++}} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p.$$

Integrating in $\mathbb{R}^{n-m}$ we get

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x_{m+1,n}) \big(x'\big)^\gamma dx$$
$$\le c_7 \int_{\mathbb{R}^{n-m}} \omega_1(x_{m+1,n}) |x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p}$$
$$\times \left(\int_{E_{n-m}(0,|x_{1,m}|/2)} \|f(\cdot, y_{m+1,n})\|_{p,\mathbb{R}^m_{++}} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}.$$

Since $\mathcal{A}_{m+1,k} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^{n-m}} \left(\int_{E_{n-m}(0,|x_{m+1,n}|/2)} \|f(\cdot, y_{m+1,n})\|_{p,\mathbb{R}^m_{++}} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p$$
$$\times \omega_1(x_{m+1,n}) |x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p} x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}$$
$$\le C \int_{\mathbb{R}^{n-m}} \|f(\cdot, x_{m+1,n})\|^p_{p,\mathbb{R}^m_{++}} \omega(x_{m+1,n}) x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}$$

holds and $C \le c' \mathcal{A}_{m+1,k}$, where c' depends only on n and p. In fact the condition $\mathcal{A}_{m+1,k} < \infty$ is necessary and sufficient for the validity of this inequality (see Lemma 2.1). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x_{m+1,n}) \big(x'\big)^\gamma dx \le c_9 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n}) \big(x'\big)^\gamma dx. \tag{4.6}$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As is easy to verify, for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,3}$ we have $|y_{m+1,n}| > 2|x_{m+1,n}|$ and $|x_{m+1,n} - y_{m+1,n}| \ge |y_{m+1,n}|/2$. Since $\widetilde{E}_l \cap \operatorname{supp} f_{k,3} = \emptyset$, for $x \in \widetilde{E}_l$ by (1.2) we obtain

$$T_3 f(x) \le c_5 \int_{\mathbb{R}^m_{++}} \int_{\complement E_{n-m}(0,2|x_{m+1,n}|)} |f(y)| (|y_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|}$$
$$\times y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} y_{1,m}^{\gamma_{1,m}} dy_{1,m}.$$

Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x_{m+1,n}) \big(x'\big)^\gamma dx$$
$$\le c_5^p \int_{\mathbb{R}^n_{k,+}} \left(\int_{\mathbb{R}^m_{++}} \int_{\complement E_{n-m}(0,2|x_{m+1,n}|)} (|y_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|} \right.$$
$$\left. \times |f(y)| y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \right)^p \omega_1(x_{m+1,n}) \big(x'\big)^\gamma dx.$$

For $x = (x_{1,m}, x_{m+1,n}) \in \mathbb{R}^n_{k,+}$ let

$$I_1(x_{m+1,n}) = \int_{\mathbb{R}^m_{++}} \left(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)} \int_{\mathbb{R}^m_{++}} |f(y)| (|y_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{-n-|\gamma|} \right.$$
$$\left. \times y_{1,m}^{\gamma_{1,m}} dy_{1,m} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \right)^p x_{1,m}^{\gamma_{1,m}} dx_{1,m}.$$

Using the Minkowski and Young inequalities we obtain

$$\begin{aligned}
&I_1(x_{m+1,n})\\
&\quad\le \left[\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}\left(\int_{\mathbb{R}^m_{++}}|f(y)|^p y_{1,m}^{\gamma_{1,m}}\,dy_{1,m}\right)^{1/p}\right.\\
&\qquad\times\left.\left(\int_{\mathbb{R}^m_{++}}\frac{y_{1,m}^{\gamma_{1,m}}\,dy_{1,m}}{(|y_{m+1,n}|+|y_{1,m}|)^{n+|\gamma|}}\right)y_{m+1,k}^{\gamma_{m+1,k}}\,dy_{m+1,n}\right]^p\\
&\quad= c_6\left(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{-n+m-|\gamma_{m+1,k}|}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}\,y_{m+1,k}^{\gamma_{m+1,k}}\,dy_{m+1,n}\right)^p\\
&\qquad\times\left(\int_{\mathbb{R}^m_{++}}\frac{y_{1,m}^{\gamma_{1,m}}\,dy_{1,m}}{(|y_{1,m}|+1)^{n+|\gamma|}}\right)^p\\
&= c_7\left(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{-n+m-|\gamma_{m+1,k}|}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}\,y_{m+1,k}^{\gamma_{m+1,k}}\,dy_{m+1,n}\right)^p.
\end{aligned}$$

Integrating over $\mathbb{R}^{n-m}$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^n_{k,+}}|T_3f(x)|^p\omega_1(x_{m+1,n})\big(x'\big)^\gamma\,dx\\
&\quad\le c_8\int_{\mathbb{R}^{n-m}}\omega_1(x_{m+1,n})\,x_{m+1,k}^{\gamma_{m+1,k}}\,dx_{m+1,n}\\
&\qquad\times\left(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{-n+m-|\gamma_{m+1,k}|}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}\,y_{m+1,k}^{\gamma_{m+1,k}}\,dy_{m+1,n}\right)^p.
\end{aligned}$$

Since $\mathcal{B}_{m+1,k}<\infty$, the Hardy inequality

$$\begin{aligned}
&\int_{\mathbb{R}^{n-m}}\left(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{-n+m-|\gamma_{m+1,k}|}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}\,y_{m+1,k}^{\gamma_{m+1,k}}\,dy_{m+1,n}\right)^p\\
&\qquad\times\omega_1(x_{m+1,n})\,x_{m+1,k}^{\gamma_{m+1,k}}\,dx_{m+1,n}\\
&\quad\le C\int_{\mathbb{R}^{n-m}}\|f(\cdot,x_{m+1,n})\|^p_{p,\mathbb{R}^m_{++}}|x_{m+1,n}|^{-(n-m+|\gamma_{m+1,k}|)p}\\
&\qquad\times\omega(x_{m+1,n})|x_{m+1,n}|^{(n-m+|\gamma_{m+1,k}|)p}\,x_{m+1,k}^{\gamma_{m+1,k}}\,dx_{m+1,n}\\
&\quad= C\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x_{m+1,n})\,\big(x'\big)^\gamma\,dx
\end{aligned}$$

holds and $C\le c'\mathcal{B}_{m+1,k}$, where c' depends only on n, γ and p. In fact the condition $\mathcal{B}_{m+1,k}<\infty$ is necessary and sufficient for the validity of this inequality (see Lemma 2.1). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}}|T_3f(x)|^p\omega_1(x_{m+1,n})\big(x'\big)^\gamma\,dx\le c_{10}\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x_{m+1,n})\big(x'\big)^\gamma\,dx. \tag{4.7}$$

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of T and condition $(a_{m+1,k})$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx &= \int_{\mathbb{R}^n_{k,+}} \Big(\sum_{l\in Z}\big|Tf_{l,2}(x)\big|\chi_{\widetilde{E}_l}(x)\Big)^p \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \Big(\sum_{l\in Z}\big|Tf_{l,2}(x)\big|^p \chi_{\widetilde{E}_l}(x)\Big) \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx \\
&= \sum_{l\in Z}\int_{\widetilde{E}_l} \big|Tf_{l,2}(x)\big|^p \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx \\
&\le \sum_{l\in Z}\sup_{x\in\widetilde{E}_l}\omega_1(x_{m+1,n})\int_{\mathbb{R}^n_{k,+}} \big|Tf_{l,2}(x)\big|^p\big(x'\big)^\gamma dx \\
&\le \|T\|^p\sum_{l\in Z}\sup_{x\in\widetilde{E}_l}\omega_1(x_{m+1,n})\int_{\mathbb{R}^n_{k,+}} \big|f_{l,2}(x)\big|^p\big(x'\big)^\gamma dx \\
&= \|T\|^p\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y_{m+1,n})\int_{\widetilde{E}_{l,2}} |f(x)|^p\big(x'\big)^\gamma dx,
\end{aligned}$$

where $\|T\| \equiv \|T\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})\to L_{p,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in \widetilde{E}_{l,2}$, $2^{l-1} < |x_{1,m}| \le 2^{l+2}$, we have by condition $(a_{m+1,k})$

$$\begin{aligned}
\sup_{y\in\widetilde{E}_l}\omega_1(y_{m+1,n}) &= \sup_{2^{l-1}<|y_{m+1,n}|\le 2^{l+2}}\omega_1(y_{m+1,n}) \\
&\le \sup_{|x_{m+1,n}|/8<|y_{m+1,n}|<8|x_{m+1,n}|}\omega_1(y_{m+1,n}) \le b\omega(x_{m+1,n})
\end{aligned}$$

for almost all $x \in \widetilde{E}_{l,2}$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x_{m+1,n})\big(x'\big)^\gamma dx &\le \|T\|^p b\sum_{l\in Z}\int_{\widetilde{E}_{l,2}} |f(x)|^p\omega(x_{m+1,n})dx \\
&\le c_{11}\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(x_{m+1,n})\big(x'\big)^\gamma dx, \qquad (4.8)
\end{aligned}$$

where $c_{11} = 3\|T\|^p b$, since the multiplicity of covering $\{\widetilde{E}_{l,2}\}_{l\in Z}$ is equal to 3. Inequalities (4.5), (4.6), (4.7), (4.8) imply (3.5) which completes the proof of the Theorem 3.3. □

Similarly we prove the following weak variant Theorem 3.3.

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