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Decompositions of local Morrey-type spaces

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Abstract We develop and apply a decomposition theory for generic local Morrey-type spaces. Our result is nonsmooth decomposition, which follows from the fact that local Morrey-type spaces are isomorphic to Hardy local Morrey-type spaces in the generic case. As an application of our results, we consider the Hardy operator.

Keywords Morrey-type spaces · Atomic decomposition · Molecular decomposition · Maximal operators

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1 Introduction

We obtain and apply a nonsmooth decomposition result for local Morrey-type spaces in this paper. The definition of local Morrey-type spaces is as follows. Here and in

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the sequel we write $B(r) \equiv \{y \in \mathbb{R}^n : |y| < r\}$ for $r > 0$. Let $1 < p < \infty$, $0 < q \le \infty$ and $0 \le \lambda < \frac{n}{p}$. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ one defines the norm $\|f\|_{LM^\lambda_{pq}}$ by:

$$\|f\|_{LM^\lambda_{pq}} \equiv \left\{ \left(\int_0^\infty \frac{1}{r^\lambda} \left(\int_{B(r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^q \frac{dr}{r} \right\}^{\frac{1}{q}}$$

when $q < \infty$ and

$$\|f\|_{LM^\lambda_{pq}} \equiv \sup_{r>0} \frac{1}{r^\lambda} \left(\int_{B(r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}$$

when $q = \infty$. One defines the space $LM^\lambda_{pq}(\mathbb{R}^n)$ as the set of all measurable functions f for which the norm $\|f\|_{LM^\lambda_{pq}}$ is finite. We also denote by $\mathcal{Q}$ the set of all cubes whose axes are parallel to the coordinate axes. The indicator function of a set E is denoted by χ_E.

In this paper, we shall establish and apply the following two theorems.

Theorem 1.1 *Let $1 < p < \infty$, $1 < q \le \infty$ and $0 \le \lambda < \frac{n}{p}$. Suppose that a real parameter s satisfies*

$$\frac{n}{s} < \frac{n}{p} - \lambda. \tag{1.1}$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset L^s(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ and

$$\|a_j\|_{L^s} \le \|\chi_{Q_j}\|_{L^s} = |Q_j|^{1/s}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{LM^\lambda_{pq}} < \infty. \tag{1.2}$$

Then the series $f \equiv \sum_{j=1}^\infty \lambda_j a_j$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and in the Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions and satisfies the estimate

$$\|f\|_{LM^\lambda_{pq}} \le C \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{LM^\lambda_{pq}}, \tag{1.3}$$

where $C > 0$ depends only on n, p, q, λ and s.

We are interested in the converse of Theorem 1.1. To this end, we need to have some quantitative information of the function space.

As the following proposition shows, $LM^\lambda_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ in the sense of continuous embedding for all admissible p, q and s.

Proposition 1.1 *Let $1 \le p < \infty$, $0 < q \le \infty$ and $0 \le \lambda < \frac{n}{p}$. Then for all $\kappa \in \mathcal{S}(\mathbb{R}^n)$ and $f \in LM^\lambda_{pq}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |\kappa(x) f(x)|\, dx \leq C \|f\|_{LM^{\lambda}_{pq}} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{2n+\lambda+1} |\kappa(x)|, \tag{1.4}$$

where C does not depend on κ and f. In particular, $LM^{\lambda}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ in the sense of continuous embedding.

With Proposition 1.1, we state the second main theorem.

Theorem 1.2 *Let $L \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $1 < p < \infty$, $1 < q \leq \infty$ and $0 \leq \lambda < \frac{n}{p}$. Let $f \in LM^{\lambda}_{pq}(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n)$ such that $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^1_{\rm loc}(\mathbb{R}^n)$, that*

$$|a_j| \leq \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^{\alpha} a_j(x)\, dx = 0, \tag{1.5}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq L$ and, that for all $v > 0$

$$\left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{LM^{\lambda}_{pq}} \leq C_v \|f\|_{LM^{\lambda}_{pq}}. \tag{1.6}$$

Here the constant $C_v > 0$ is independent of f.

When $q = \infty$, Theorems 1.1 and 1.2 are proved in [7]. Our results above are available in the weighted setting described below.

Recall that in 1994 in the doctoral thesis [24, pp. 75–76], (see also [25, pp. 123] as well as [26,27]) Guliyev introduced the local Morrey-type space $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and complementary local Morrey-type spaces ${}^{\complement}LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ given by

$$\|f\|_{LM_{p\theta,w(\cdot)}} \equiv \left\| w(r) \|\chi_{B(r)} f\|_{L^p} \right\|_{L^{\theta}(0,\infty)} < \infty,$$

and

$$\|f\|_{{}^{\complement}LM_{p\theta,w(\cdot)}} \equiv \left\| w(r) \|\chi_{\mathbb{R}^n \setminus B(r)} f\|_{L^p} \right\|_{L^{\theta}(0,\infty)} < \infty,$$

respectively, where w is a positive measurable function defined on $(0, \infty)$. In [24] (see also [25–27]) the author found the sufficient conditions for the boundedness of the singular and potential operators in the local Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and the complementary local Morrey-type spaces ${}^{\complement}LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.

During the last decades various classical operators, such as maximal, singular and potential operators were widely investigated in both in classical and local Morrey-type spaces. In [10, pp. 157], Burenkov and Guliyev introduced the space $GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Here and below we denote $B(x, r) \equiv \{x + y \,:\, y \in B(r)\}$.

Definition 1 Let $0 < p, \theta \le \infty$ and let w be a non-negative Lebesgue measurable function on $(0, \infty)$.

1. [24, pp. 75–76] Denote by $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ the local Morrey-type space, the space of, all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{LM_{p\theta,w(.)}} = \left\| w(r) \|f\chi_{B(r)}\|_{L^p} \right\|_{L^\theta(0,\infty)}.$$

2. [10–12] Denote by $GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ the global Morrey-type space, the space of all functions f Lebesgue measurable on $\mathbb{R}^n$ with finite quasi-norm

$$\|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \left\| w(r) \|f\chi_{B(x,r)}\|_{L^p} \right\|_{L^\theta(0,\infty)}. \tag{1.7}$$

The spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, $GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ are mostly aimed at describing the behaviour of $\|f\chi_{B(r)}\|_{L^p}$, $\|f\chi_{B(x,r)}\|_{L^p}$ respectively, for small $r > 0$ in a very general setting. Note that if $w(r) = 1$, then $LM_{p\infty,1}(\mathbb{R}^n) = GM_{p\infty,1}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Furthermore,

$$GM_{p\infty,r^{-\lambda}}(\mathbb{R}^n) \equiv M_p^\lambda(\mathbb{R}^n), \quad 0 < p \le \infty, \quad 0 \le \lambda \le \frac{n}{p}.$$

Next, we introduce classes so that $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and $GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ are not equal to Θ. Here Θ stands for the set of all functions which is almost everywhere zero.

Definition 2 Let $0 < p, \theta \le \infty$.

1. Denote by Ω_θ the set of all functions w which are non-negative, Lebesgue measurable on $(0, \infty)$, not equivalent to 0, and such that for some $t > 0$

$$\|w(r)\|_{L^\theta(t,\infty)} < \infty.$$

2. Denote by $\Omega_{p\theta}$, the set of all functions w which are non-negative, Lebesgue measurable on $(0, \infty)$, not equivalent to 0, and such that for all $t > 0$

$$\left\| w(r) r^{n/p} \right\|_{L^\theta(0,t)} < \infty, \quad \|w(r)\|_{L^\theta(t,\infty)} < \infty,$$

or, which is equivalent to,

$$\left\| w(r) \left(\frac{r}{t+r} \right)^{n/p} \right\|_{L^\theta(0,\infty)} < \infty$$

for all $t > 0$.

The next lemma explains why the above classes of weights are natural.

Lemma 1.1 [10,11] *Let* $0 < p, \theta \le \infty$ *and let* w *be a non-negative Lebesgue measurable function on* $(0, \infty)$*, which is not equivalent to* 0.

1. *Then the space $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is non-trivial, in the sense that $LM_{p\theta,w(\cdot)}(\mathbb{R}^n) \neq \Theta$, if and only if $w \in \Omega_\theta$, and the space $GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is non-trivial if and only if $w \in \Omega_{p\theta}$.*
2. *Assume $w \in \Omega_\theta$ and write $\tau \equiv \inf\left\{s > 0 : \|w\|_{L^\theta(s,\infty)} < \infty\right\}$. Then the space $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ contains all functions $f \in L^p$ such that $f = 0$ on $B(0,t)$ for some $t > \tau$. If $w \in \Omega_{p\theta}$, then*

$$L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset GM_{p\theta,w(\cdot)}(\mathbb{R}^n).$$

Keeping in mind this statement it will always be assumed that $w \in \Omega_\theta$ for the case of local Morrey-type spaces and that $w \in \Omega_{p\theta}$ for the case of global Morrey-type spaces.

We shall also use the notation $GM^\lambda_{p\theta}(\mathbb{R}^n)$ respectively, for the particular case in which $w(r) = r^{-\lambda-\frac{1}{\theta}}$. In this case

$$\|f\|_{GM^\lambda_{p\theta}} = \sup_{x\in\mathbb{R}^n} \left(\int_0^\infty \left(\frac{\|f(\cdot+x)\|_{L^p(B(r))}}{r^\lambda} \right)^\theta \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty.$$

By Lemma 1.1 the space $LM^\lambda_{p\theta}(\mathbb{R}^n)$ is non-trivial if and only if $\lambda > 0$ for $\theta < \infty$ and $\lambda \geq 0$ for $\theta = \infty$, and the space $GM^\lambda_{p\theta}(\mathbb{R}^n)$ is non-trivial if and only if $0 < \lambda < \frac{n}{p}$ for $\theta < \infty$ and $0 \leq \lambda \leq \frac{n}{p}$ for $\theta = \infty$.

We define H and H^* to be the operator and its dual, given by:

$$Hg(r) = \int_0^r g(t)\,dt \text{ and } H^*g(r) = \int_r^\infty g(t)\,dt.$$

Denote by $L_{p,v}(0,\infty)$ the weighted Lebesgue space for $1 \leq p < \infty$ and a measurable function $v : (0,\infty) \to (0,\infty)$, whose norm is given by

$$\|f\|_{L_{p,v}(0,\infty)} \equiv \left(\int_0^\infty |f(t)v(t)|^p\,dt \right)^{\frac{1}{p}}.$$

We extend and prove Theorems 1.1 and 1.2 to the weighted setting:

Theorem 1.3 *Let $1 < p < \infty$, $1 < \theta \leq \infty$ and $w \in \Omega_\theta$. Assume that w satisfies the doubling condition; $C^{-1}w(r) \leq w(2r) \leq Cw(r)$ for all $r > 0$. We define*

$$\hat{v}_1(r) \equiv r^{\frac{-n}{p}-1} w(r), \quad \hat{v}_2(r) = r^{\frac{-n}{p}} w(r). \tag{1.8}$$

Assume that H^* is bounded from $L_{\theta,\hat{v}_1}(0,\infty)$ to $L_{\theta,\hat{v}_2}(0,\infty)$. Suppose that a real parameter s satisfies

$$\int_r^\infty \frac{t^{1/s-1/p-1}}{w(rt)}\,dt \leq C\frac{r^{1/p-1/s}}{w(r)} \tag{1.9}$$

for all $r > 0$*. Assume that we are given* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$*,* $\{a_j\}_{j=1}^{\infty} \subset L^s(\mathbb{R}^n)$*,* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ *satisfying*

$$\|a_j\|_{L^s} \le \|\chi_{Q_j}\|_{L^s} = |Q_j|^{1/s}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{LM_{p\theta,w(\cdot)}} < \infty. \tag{1.10}$$

Then the series $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and in the Schwartz space* $\mathcal{S}'(\mathbb{R}^n)$ *of tempered distributions and satisfies the estimate*

$$\|f\|_{LM_{p\theta,w(\cdot)}} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{LM_{p\theta,w(\cdot)}}, \tag{1.11}$$

where $C > 0$ *depends only on* n, p, q, w *and* s*.*

We have a counter part of Proposition 1.1 to $LM_{p\theta,w(\cdot)}$.

Proposition 1.2 *Let* $1 < p < \infty$*,* $0 < \theta \le \infty$*,* $w \in \Omega_\theta$*. We define* $\hat{v}_1, \hat{v}_2$ *by* (1.8)*. Assume that* H^* *is bounded on* $L_{\theta,\hat{v}_1}(0, \infty)$ *to* $L_{\theta,\hat{v}_2}(0, \infty)$*. Then* $LM_{p\theta,w(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ *in the sense of continuous embedding.*

With Proposition 1.2, we extend the second main theorem.

Theorem 1.4 *Let* $L \in \mathbb{N}_0$*,* $1 < p < \infty$*,* $1 < \theta \le \infty$ *and* $w \in \Omega_\theta$*. Assume that* w *satisfies the doubling condition;* $C^{-1}w(r) \le w(2r) \le Cw(r)$ *for all* $r > 0$*. We define* $\hat{v}_1, \hat{v}_2$ *by* (1.8)*. Assume that* H *is bounded on* $L_{\theta,\hat{v}_1}(0, \infty)$ *to* $L_{\theta,\hat{v}_2}(0, \infty)$*. Suppose that a real parameter* s *satisfies* (1.9) *for all* $r > 0$*. Let* $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$*. Then there exist* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$*,* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *and* $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n)$ *such that* $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$*, that*

$$|a_j| \le \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^{\alpha} a_j(x)\, dx = 0, \tag{1.12}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *with* $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n \le L$ *and, that for all* $v > 0$

$$\left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{LM_{p\theta,w(\cdot)}} \le C_v \|f\|_{LM_{p\theta,w(\cdot)}}. \tag{1.13}$$

Here the constant $C_v > 0$ *is independent of* f*.*

In view of (1.1) and Theroem 3.2, we can say that all the assumptions in Theorems 1.3 and 1.4 can be covered. So, Theorems 1.1 and 1.2 are consequences of Theorems 1.3 and 1.4.

We organize the remaining part of this paper as follows: In Sect. 2, we collect some preliminary facts. In Sects. 3–5 we aim to collect some tools needed to prove Theorems

1.3 and 1.4. In particular, the key observation to pave the way of specifying the predual space of $LM^{\lambda}_{p\theta}(\mathbb{R}^n)$ is the following norm equivalence: for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$,

$$\|f\|_{LM^{\lambda}_{p\theta}} \sim \left\{ \sum_{j=-\infty}^{\infty} \left(2^{-\lambda j} \left(\int_{|y|<2^j} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^q \right\}^{\frac{1}{q}}. \tag{1.14}$$

In Sect. 6 we prove Theorems 1.3 and 1.4. In Sect. 7, we consider the boundedness of the Hardy operator given by:

$$Hf(x) \equiv \frac{1}{|B(|x|)|} \int_{B(|x|)} f(y)\, dy \quad (x \in \mathbb{R}^n). \tag{1.15}$$

Finally, we overview the role of related function spaces in Appendix, where we compare local Morrey spaces with Herz spaces and we show that these two things are the same and we compare local Morrey spaces with many other related spaces.

2 Preliminaries

Our idea is to convert the norm of $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ to one of Hardy type. To this end, we start by recalling the definition of the grand maximal function $\mathcal{M}f$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ as well as the topology of $\mathcal{S}(\mathbb{R}^n)$.

Definition 3 (1) The topology on $\mathcal{S}(\mathbb{R}^n)$ is defined by the norms $\{\rho_N\}_{N\in\mathbb{N}}$ where

$$\rho_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)| \quad (\varphi \in \mathcal{S}(\mathbb{R}^n)).$$

Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \rho_N(\varphi) \le 1\}$ for $N \in \mathbb{N}_0$.

(2) The space $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}(\mathbb{R}^n)$.

(3) Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ of f is defined by

$$\mathcal{M}f(x) = \mathcal{M}_N f(x) \equiv \sup\{|t^{-n}\varphi(t^{-1}\cdot) * f(x)| : t > 0, \varphi \in \mathcal{F}_N\}$$

for $x \in \mathbb{R}^n$.

Next, we recall the following lemma, which will be key to this paper: We refer to [41] for the proof. By $C^\infty_{\rm comp}(\mathbb{R}^n)$, we denote the set of all compactly supported infinitely continously differentiable functions in $\mathbb{R}^n$. The set of all polynomials of degree less than or equal to d is denoted by $\mathcal{P}_d(\mathbb{R}^n)$.

Lemma 2.1 *Let $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\rm loc}(\mathbb{R}^n)$, $d \in \mathbb{N}_0$ and $j \in \mathbb{Z}$. Then there exist an index set K_j, collections of cubes $\{Q_{j,k}\}_{k\in K_j}$ and functions $\{\eta_{j,k}\}_{k\in K_j} \subset C^\infty_{\rm comp}(\mathbb{R}^n)$, which are all indexed by K_j for every j, and a decomposition*

$$f = g_j + b_j, \quad b_j = \sum_{k\in K_j} b_{j,k},$$

such that the following properties hold:

(1) $g_j, b_j, b_{j,k} \in \mathcal{S}'(\mathbb{R}^n)$.

(2) Define $\mathcal{O}_j \equiv \{y \in \mathbb{R}^n : \mathcal{M}f(y) > 2^j\}$ *and consider its Whitney decomposition. Then the cubes* $\{Q_{j,k}\}_{k\in K_j}$ *have the bounded intersection property, and*

$$\mathcal{O}_j = \bigcup_{k\in K_j} Q_{j,k}. \tag{2.1}$$

(3) Consider the partition of unity $\{\eta_{j,k}\}_{k\in K_j}$ *with respect to* $\{Q_{j,k}\}_{k\in K_j}$*. Then each function* $\eta_{j,k}$ *is supported in* $Q_{j,k}$ *and*

$$\sum_{k\in K_j} \eta_{j,k} = \chi_{\{y\in\mathbb{R}^n : \mathcal{M}f(y)>2^j\}}, \quad 0 \le \eta_{j,k} \le 1.$$

(4) g_j *is an* $L^\infty(\mathbb{R}^n)$*-function satisfying* $\|g_j\|_{L^\infty} \le 2^{-j}$.

(5) Each distribution $b_{j,k}$ *is given by* $b_{j,k} = (f - c_{j,k})\eta_{j,k}$ *with a certain polynomial* $c_{j,k} \in \mathcal{P}_d(\mathbb{R}^n)$ *satisfying*

$$\langle f - c_{j,k}, \eta_{j,k} \cdot P\rangle = 0 \text{ for all } q \in \mathcal{P}_d(\mathbb{R}^n),$$

and

$$\mathcal{M}b_{j,k}(x) \le C\left(\mathcal{M}f(x)\chi_{Q_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}}\chi_{\mathbb{R}^n\setminus Q_{j,k}}(x)\right) \tag{2.2}$$

for all $x \in \mathbb{R}^n$.

In the above, $x_{j,k}$ *and* $\ell_{j,k}$ *denote the center and the edge-length of* $Q_{j,k}$*, respectively, and* C_1 *and* C_2 *depend only on n.*

We now prove Propositions 1.1 and 1.2.

Proof of Propositon 1.1 We decompose the left-hand side as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} |\kappa(x)f(x)|dx &= \int_{B(1)} |\kappa(x)f(x)|dx + \sum_{j=1}^{\infty} \int_{B(j+1)\setminus B(j)} |\kappa(x)f(x)|dx \\
&\le \|\kappa\|_{L^\infty(B(1))}\|f\|_{L^1(B(1))} + \sum_{j=1}^{\infty} \int\limits_{B(j+1)\setminus B(j)} \frac{|x|^{2n+\lambda+1}}{j^{2n+\lambda+1}}|\kappa(x)f(x)|dx \\
&\le C\|f\|_{LM^\lambda_{pq}}\left(\sup_{x\in\mathbb{R}^n}(1+|x|)^{2n+\lambda+1}|\kappa(x)|\right),
\end{aligned}$$

where C depends on n, p and λ. Here we used the Hölder inequality for the last line. □

Proof of Propositon 1.2 Denote by $\mathcal{B}_x$ the set of all open balls in $\mathbb{R}^n$ which contain x. We know that the Hardy–Littlewood maximal operator M, which is given by

$$Mf(x) \equiv \sup_{B \in \mathcal{B}_x} \frac{1}{|B|} \int_B |f(y)|\, dy \quad (x \in \mathbb{R}^n),$$

is bounded on $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ [10, Theorem 4, pp. 170]. Therefore,

$$\frac{1}{|B(R)|} \int_{B(R)} |f(y)|\, dy \leq Mf(x)$$

for all $|x| \leq 1$ ane for all $R > 1$. This means that

$$\frac{\alpha}{|B(R)|} \int_{B(R)} |f(y)|\, dy \leq \|\chi_{B(1)} Mf\|_{LM_{p\theta,w(\cdot)}} \leq \|Mf\|_{LM_{p\theta,w(\cdot)}} \leq C\|f\|_{LM_{p\theta,w(\cdot)}},$$

where $\alpha \equiv \|\chi_{B(1)}\|_{LM_{p\theta,w(\cdot)}}$. Thus, it follows that

$$\int_{\mathbb{R}^n} |\kappa(x) f(x)| dx \leq C\|f\|_{LM_{p\theta,w(\cdot)}} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{2n+1} |\kappa(x)|.$$

□

3 Vector valued maximal inequalities

In this section, we consider the Hardy–Littlewood maximal operator M.

The aim of this section is to extend the well-known inequalities

$$\int_{\mathbb{R}^n} Mf(x)^p\, dx \leq c_{p,n} \int_{\mathbb{R}^n} |f(x)|^p\, dx \tag{3.1}$$

and

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} Mf_j(x)^q \right)^{\frac{p}{q}} dx \leq c_{p,q,n} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{p}{q}} dx, \tag{3.2}$$

where $c_{p,n}$ and $c_{p,q,n}$ are independent of f and f_j, $j \in \mathbb{N}$, respectively. Here the parameters p and q satisfy $1 < p, q < \infty$. When $1 < p < q = \infty$, we have a counterpart to (3.2);

$$\int_{\mathbb{R}^n} \left(M\left[\sup_{j \in \mathbb{N}} |f_j| \right](x) \right)^p dx \leq c_{p,n} \int_{\mathbb{R}^n} \left(\sup_{j \in \mathbb{N}} |f_j(x)| \right)^p dx. \tag{3.3}$$

Note that (3.3) is a direct consequence of (3.1) and the pointwise estimate

$$M\left[\sup_{j \in \mathbb{N}} |f_j| \right](x) \leq \sup_{j \in \mathbb{N}} |f_j(x)|. \tag{3.4}$$

We aim to obtain the counterpart of (3.1)–(3.4) to $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.

Our main result in this section is as follows:

Theorem 3.1 *Let* $1 < p < \infty$, $1 < \theta \le \infty$, $1 < v < \infty$. *We define weights* $\hat{v}_1, \hat{v}_2$ *by* (1.8). *Assume that* H^* *is bounded from* $L_{\theta,\hat{v}_1}(0,\infty)$ *to* $L_{\theta,\hat{v}_2}(0,\infty)$. *Then we have*

$$\|Mf\|_{LM_{p\theta,w(\cdot)}} \le C\|f\|_{LM_{p\theta,w(\cdot)}} \tag{3.5}$$

and

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^v\right)^{\frac{1}{v}}\right\|_{LM_{p\theta,w(\cdot)}} \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^v\right)^{\frac{1}{v}}\right\|_{LM_{p\theta,w(\cdot)}}. \tag{3.6}$$

Here, the constant C *in* (3.5) *depends only on* p, *and* n *and the one in* (3.6) *depends only on* p, q, v *and* n. *In particular,*

$$\left\|M\left[\sup_{j\in\mathbb{N}}|f_j|\right]\right\|_{LM_{p\theta,w(\cdot)}} \le C\left\|\sup_{j\in\mathbb{N}}|f_j|\right\|_{LM_{p\theta,w(\cdot)}}. \tag{3.7}$$

Note that (3.5) is due to [10, Theorem 4, pp. 170].

Proof We can deduce (3.7) by using (3.4) and (3.5). By setting $f_1 = f$, $f_2 = f_3 = \cdots = 0$ in (3.6), we can obtain (3.5). Hence, we concentrate on proving (3.6). We follow the line of [4, Theorem 4.1].

We suppose $\theta < \infty$; the case $\theta = \infty$ can be dealt similarly.

As we have proved in [4, Lemma 4.4]

$$\left\|\chi_{B(x,r)}\left(\sum_{j=1}^{\infty}Mf_j{}^v\right)^{\frac{1}{v}}\right\|_{L^p} \le Cr^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}-1}\left\|\chi_{B(x,t)}\left(\sum_{j=1}^{\infty}|f_j|^v\right)^{\frac{1}{v}}\right\|_{L^p}dt.$$

Thus, using the boundedness of the dual Hardy operator, we obtain the desired result.

It remains to go through the same argument as [15, Theorem 5]. □

We supplement the case of $LM^{\lambda}_{pq}(\mathbb{R}^n)$, where we supply a direct proof without using the Hardy operator.

Theorem 3.2 *Let* $1 < p < \infty$, $0 < q \le \infty$, $1 < v < \infty$ *and* $0 \le \lambda < \frac{n}{p}$. *Then we have*

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^v\right)^{\frac{1}{v}}\right\|_{LM^{\lambda}_{pq}} \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^v\right)^{\frac{1}{v}}\right\|_{LM^{\lambda}_{pq}}. \tag{3.8}$$

Proof We define $f_{j,1} \equiv f_j \chi_{B(5r)}$ and $f_{j,2} \equiv f_j - f_{j,1}$ as before. We shall prove two inequalities;

$$\int_0^\infty \frac{1}{r^{\lambda q}} \left(\int_{B(r)} \left(\sum_{j=1}^\infty M f_{j,1}(y)^v \right)^{\frac{p}{v}} dy \right)^{\frac{q}{p}} \frac{dr}{r} \le C \left\| \left(\sum_{j=1}^\infty |f_j|^v \right)^{\frac{1}{v}} \right\|_{LM^\lambda_{pq}}^q, \tag{3.9}$$

$$\int_0^\infty \frac{1}{r^{\lambda q}} \left(\int_{B(r)} \left(\sum_{j=1}^\infty M f_{j,2}(y)^v \right)^{\frac{p}{v}} dy \right)^{\frac{q}{p}} \frac{dr}{r} \le C \left\| \left(\sum_{j=1}^\infty |f_j|^v \right)^{\frac{1}{v}} \right\|_{LM^\lambda_{pq}}^q. \tag{3.10}$$

The estimate (3.9) follows analogously to (3.8), which we omit.

As for (3.10), we have:

$$\begin{aligned}
&\frac{1}{r^\lambda} \left(\int_{B(r)} \left(\sum_{j=1}^\infty M f_{j,2}(x)^v \right)^{\frac{p}{v}} dx \right)^{\frac{1}{p}} \\
&\le C r^{n/p-\lambda} \sum_{k=1}^\infty \frac{1}{|B(2^k r)|} \int_{B(2^k r)} \left(\sum_{j=1}^\infty |f_j(y)|^v \right)^{\frac{1}{v}} dy \\
&\le C r^{n/p-\lambda} \sum_{k=1}^\infty \left(\frac{1}{|B(2^k r)|} \int_{B(2^k r)} \left(\sum_{j=1}^\infty |f_j(y)|^v \right)^{\frac{p}{v}} dy \right)^{\frac{1}{p}}.
\end{aligned}$$

By the change of variables $2^k r \mapsto r$, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{r^\lambda} \left(\int_{B(r)} \left(\sum_{j=1}^\infty M f_{j,2}(x)^v \right)^{\frac{p}{v}} dx \right)^{\frac{1}{p}} \right)^q \frac{dr}{r} \\
&\le C \int_0^\infty \left(r^{n/p-\lambda} \sum_{k=1}^\infty \left(\frac{1}{|B(2^k r)|} \int_{B(2^k r)} \left(\sum_{j=1}^\infty |f_j(y)|^v \right)^{\frac{p}{v}} dy \right)^{\frac{1}{p}} \right)^q \frac{dr}{r} \\
&= C \int_0^\infty \left(\sum_{k=1}^\infty 2^{-k(n/p-\lambda)} r^{n/p-\lambda} \left(\frac{1}{|B(2^k r)|} \int_{B(2^k r)} \left(\sum_{j=1}^\infty |f_j(y)|^v \right)^{\frac{p}{v}} dy \right)^{\frac{1}{p}} \right)^q \frac{dr}{r}.
\end{aligned}$$

When $0 < q \le 1$, we use $(a+b)^q \le a^q + b^q$ for $a, b \ge 0$ and when $q > 1$, we use the Hölder inequality. The result is;

$$\int_0^\infty \left(\frac{1}{r^\lambda}\left(\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,2}(x)^v\right)^{\frac{p}{v}} dx\right)^{\frac{1}{p}}\right)^q \frac{dr}{r}$$

$$\le C\int_0^\infty \sum_{k=1}^\infty 2^{-k\delta(n/p-\lambda)}\left(r^{n/p-\lambda}\left(\frac{1}{|B(2^k r)|}\int_{B(2^k r)}\left(\sum_{j=1}^\infty |f_j(y)|^v\right)^{\frac{p}{v}} dy\right)^{\frac{1}{p}}\right)^q \frac{dr}{r}$$

for some $\delta > 0$. Since $\lambda < \frac{n}{p}$, the most right-hand side is summable over k and we obtain

$$\left\{\int_0^\infty \left(\frac{1}{r^\lambda}\left(\int_{B(r)}\left(\sum_{j=1}^\infty Mf_{j,2}(x)^v\right)^{\frac{p}{v}} dx\right)^{\frac{1}{p}}\right)^q \frac{dr}{r}\right\}^{\frac{1}{q}} \le C\left\|\left(\sum_{j=1}^\infty |f_j|^v\right)^{\frac{1}{v}}\right\|_{LM_{pq}^\lambda},$$

as was to be shown. □

4 Predual space

This section is oriented to a Banach space Y such that Y^* is isomorphic to $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ when the parameters p, θ and w satisify

$$1 < p < \infty, \quad 1 < \theta \le \infty, \quad w \in \Omega_\theta.$$

One says that a function A is a (p', w, R)-block, if $\text{supp}(A) \subset B(R)$ and

$$\|A\|_{L^{p'}} \le w(R). \tag{4.1}$$

With the notion of (p', w, R)-blocks in mind, we give a candidate of Y. The local block space $LH_{p'\theta',w(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions g for which there exists a decomposition

$$g(x) = \sum_{j=-\infty}^\infty \lambda_j A_j(x),$$

where each A_j is a $(p', w, 2^j)$-block and $\{\lambda_j\}_{j=-\infty}^\infty \in \ell^{\theta'}(\mathbb{Z})$ and the convergence takes place for almost all $x \in \mathbb{R}^n$. The norm of g is given by:

$$\|g\|_{LH_{p'\theta',w(\cdot)}} \equiv \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{\theta'} \right)^{\frac{1}{\theta'}},$$

where $\{\lambda_j\}_{j=-\infty}^{\infty}$ runs over all the admissible expressions above.

In this section, we aim to prove the following result:

Theorem 4.1 *Let $1 < p < \infty$, $1 < \theta \le \infty$ and $w \in \Omega_\theta$. Assume that w satisfies the doubling condition; $C^{-1}w(r) \le w(2r) \le Cw(r)$ for all $r > 0$. Define $\tilde{w}$ by $\tilde{w}(r) = w(r)r^{-1/q}$. Then $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the dual of $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$ in the following sense:*

(1) Let $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Then, for any $g \in LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$, we have $f \cdot g \in L^1(\mathbb{R}^n)$ and the mapping

$$g \in LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{C}$$

defines a continuous linear functional L_f on $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$.

(2) Conversely, any continuous linear functional L on $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$ can be realized as $L = L_f(\mathbb{R}^n)$ with a certain $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. In addition, if f_1 and $f_2 \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ define the same functional, then $f_1 = f_2$ almost everywhere.

Furthermore, the operator norm of L_f is equivalent to $\|f\|_{LM_{p\theta,w(\cdot)}}$; there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{LM_{p\theta,w(\cdot)}} \le \|L_f\|_{LH_{p'\theta',\tilde{w}(\cdot)} \to \mathbb{C}} \le C\|f\|_{LM_{p\theta,w(\cdot)}} \tag{4.2}$$

for all $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.

Proof of Theorem 4.1 (1) Let g be such that

$$g = \sum_{j=-\infty}^{\infty} \lambda_j A_j,$$

where each A_j is a $(p', w, 2^j)$-block and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^{\theta'}(\mathbb{Z})$ satisfies

$$\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{\theta'} \right)^{\frac{1}{\theta'}} \le 2\|g\|_{LH_{p'\theta',\tilde{w}(\cdot)}}.$$

Then we have

$$\begin{aligned} \|f \cdot g\|_{L^1} &\le \sum_{j=-\infty}^{\infty} |\lambda_j| \int_{B(2^j)} |f(x)A_j(x)|\,dx \\ &\le \sum_{j=-\infty}^{\infty} |\lambda_j| \left(\int_{B(2^j)} |f(x)|^p\,dx \right)^{\frac{1}{p}} \left(\int_{B(2^j)} |A_j(x)|^{p'}\,dx \right)^{\frac{1}{p'}} \end{aligned}$$

by the Hölder inequality for Lebesgue spaces and (4.1). Again by the Hölder inequality for sequences, we obtain

$$\begin{aligned}\|f\cdot g\|_{L^1} &\le \sum_{j=-\infty}^{\infty}|\lambda_j|w(2^j)\left(\int_{B(2^j)}|f(x)|^p\,dx\right)^{\frac{1}{p}}\\ &\le \left(\sum_{j=-\infty}^{\infty}|\lambda_j|^{\theta'}\right)^{\frac{1}{\theta'}}\left(\sum_{j=-\infty}^{\infty}\left(w(2^j)\left(\int_{B(2^j)}|f(x)|^p\,dx\right)^{\frac{1}{p}}\right)^{\theta}\right)^{\frac{1}{\theta}}\\ &\le C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^{\theta'}\right)^{\frac{1}{\theta'}}\left(\int_0^{\infty}\left(w(r)\left(\int_{B(r)}|f(x)|^p\,dx\right)^{\frac{1}{p}}\right)^{\theta}\frac{dr}{r}\right)^{\frac{1}{\theta}}\\ &\le C\|f\|_{LM_{p\theta,w(\cdot)}}\|g\|_{LH_{p'\theta',\tilde{w}(\cdot)}},\end{aligned}$$

which proves (1) and the right inequality in (4.2).

(2) Let L be a bounded linear functional on $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$. Then since the mapping

$$g\in L^{p'}(\mathbb{R}^n)\mapsto L(g\chi_{B(2^j)})\in\mathbb{C}$$

is a bounded linear functional, we see that L is realized by an $L^p_{\mathrm{loc}}(\mathbb{R}^n)$-function f; $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is a function satisfying

$$L(g\chi_{B(2^j)})=\int_{B(2^j)}g(x)f(x)\,dx \tag{4.3}$$

for all $g\in L^{p'}(\mathbb{R}^n)$ and $j\in\mathbb{Z}$. We have to check $f\in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, or equivalently,

$$\left(\sum_{j=-\infty}^{\infty}\left(w(2^j)\left(\int_{B(2^j)}|f(x)|^p\,dx\right)^{\frac{1}{p}}\right)^{\theta}\right)^{\frac{1}{\theta}}<\infty.$$

To this end, choose a nonnegative $\ell^{\theta'}(\mathbb{Z})$-sequence $\{\rho_j\}_{j=-\infty}^{\infty}$ arbitrarily so that $\rho_j=0$ with $|j|\gg 1$ and we estimate

$$\sum_{j=-\infty}^{\infty}w(2^j)\rho_j\left(\int_{B(2^j)}|f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Let us set

$$g_j(x)\equiv\begin{cases}\|f\|^{1-p}_{L^p(B(2^j))}w(2^j)\cdot|f(x)|^{p-1}\chi_{B(2^j)}(x), & \text{if } \|f\|_{L^p(B(2^j))}>0,\\ 0, & \text{otherwise.}\end{cases}$$

Then each g_j is a $(p', \tilde{w}, R)$-block

$$g \equiv \sum_{j=-\infty}^{\infty} \rho_j g_j$$

belongs to $LH_{p'\theta', \tilde{w}(\cdot)}(\mathbb{R}^n)$ and satisfies

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx = \sum_{j=-\infty}^{\infty} \tilde{w}(2^j)\rho_j \left(\int_{B(2^j)} |f(x)|^p\, dx \right)^{\frac{1}{p}}.$$

Therefore, by letting $h(x) \equiv \overline{\mathrm{sgn}(f(x))} g(x)$ for $x \in \mathbb{R}^n$, since $\mathrm{supp}(h) \subset B(2^J)$ for some large J,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx = L(h)$$

thanks to (4.3) and the fact that $\rho_j = 0$ if $j \gg 1$. Thus, we obtain

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \tilde{w}(2^j)\rho_j \left(\int_{B(2^j)} |f(x)|^p\, dx \right)^{\frac{1}{p}} &= \int_{\mathbb{R}^n} |f(x)g(x)|\, dx \\ &= L(h) \\ &\leq \|L\|_{LH_{p'\theta', \tilde{w}(\cdot)} \to \mathbb{C}} \|h\|_{LH_{p'\theta', \tilde{w}(\cdot)}} \\ &\leq \|L\|_{LH_{p'\theta', \tilde{w}(\cdot)} \to \mathbb{C}} \left(\sum_{j=-\infty}^{\infty} |\rho_j|^{\theta'} \right)^{\frac{1}{\theta'}}. \end{aligned}$$

Thus, $f \in LM_{p\theta, w(\cdot)}(\mathbb{R}^n)$. From (4.3), we have $L_f = L$ at least for blocks. Finally, if f_1 and $f_2 \in LM_{p\theta, w(\cdot)}(\mathbb{R}^n)$ define the same continuous linear functional on $LH_{p'\theta', \tilde{w}(\cdot)}(\mathbb{R}^n)$, then

$$\int_{B(x_0,r)} f_1(x)\, dx = L_{f_1}(\chi_{B(x_0,r)}) = L_{f_2}(\chi_{B(x_0,r)}) = \int_{B(x_0,r)} f_2(x)\, dx$$

and by the Lebesgue differentiation theorem, we have $f_1(x) = f_2(x)$ for almost all $x \in \mathbb{R}^n$. □

5 Characterization of Hardy local Morrey-type spaces in terms of the grand maximal operator and the heat kernel

Let $1 < p < \infty$, $1 < \theta \leq \infty$ and $w \in \Omega_\theta$. We characterize the space $LM_{p\theta, w(\cdot)}(\mathbb{R}^n)$ in terms of the heat kernel. Let $t > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ and define

$$e^{t\Delta} f(x) \equiv \left\langle f, \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-\cdot|^2}{4t}\right)\right\rangle \quad (x \in \mathbb{R}^n).$$

The Hardy local Morrey-type space $HLM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ collects all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\sup_{t>0} |e^{t\Delta} f| \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. We define

$$\|f\|_{HLM_{p\theta,w(\cdot)}} \equiv \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{LM_{p\theta,w(\cdot)}}.$$

Let us show that $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and $HLM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ are isomorphic by showing the following proposition:

Proposition 5.1 *Let* $1 < p < \infty$, $1 < \theta \le \infty$ *and* $w \in \Omega_\theta$. *We define* $\hat{v}_1, \hat{v}_2$ *by (1.8). Assume that* H^* *is bounded from* $L_{\theta,\hat{v}_1}(0,\infty)$ *to* $L_{\theta,\hat{v}_2}(0,\infty)$.

(1) If $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, *then* $f \in HLM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.
(2) If $f \in HLM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, *then* f *is represented by a measurable function* g *which belongs to* $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.

If $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, *then*

$$\|f\|_{LM_{p\theta,w(\cdot)}} \le \|f\|_{HLM_{p\theta,w(\cdot)}} \le C\|f\|_{LM_{p\theta,w(\cdot)}}. \tag{5.1}$$

Proof (1) We can easily verify that $LM_{p\theta,w(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ in the sense of continuous embedding by using Proposition 1.2. Also, we have

$$\sup_{t>0} |e^{t\Delta} f| \le Mf.$$

From (3.5), the $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$-boundedness of the Hardy–Littlewood maximal operator, we see that $f \in HLM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and that the right inequality in (5.1) follows.

(2) Due to Theorem 4.1, the dual of $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$ is isomorphic to $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Let $L : h \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n) \mapsto L_h \in (LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n))^*$ be an isomorphism in Theorem 4.1. By the Banach-Alaoglu theorem, there exists a positive decreasing sequence $\{t_j\}_{j=1}^\infty \subset (0,1)$ such that $L_{e^{t_j\Delta} f}$ is convergent to $G = L_g \in (LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n))^*$ for some $g \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ in the weak-* sense. Observe that

$$\begin{aligned} \|g\|_{LM_{p\theta,w(\cdot)}} &\sim \|L_g\|_{(LH_{p'\theta',\tilde{w}(\cdot)})^*} \\ &\le \liminf_{j\to\infty} \|L_{e^{t_j\Delta} f}\|_{(LH_{p'\theta',\tilde{w}(\cdot)})^*} \\ &\sim \liminf_{j\to\infty} \|e^{t_j\Delta} f\|_{LM_{p\theta,w(\cdot)}} \le \|f\|_{HLM_{p\theta,w(\cdot)}}. \end{aligned} \tag{5.2}$$

Meanwhile, since $f \in \mathcal{S}'(\mathbb{R}^n)$, $e^{t_j\Delta} f$ is convergent to $f \in \mathcal{S}'(\mathbb{R}^n)$. Thus, we conclude $\mathcal{S}'(\mathbb{R}^n) \ni f = g \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.

The left inequality in (5.1) follows since the space $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is isomorphic to the dual of $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$. Thus, from Lebesgue's differentiation theorem,

$$\|f\|_{LM_{p\theta,w(\cdot)}} \le \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{LM_{p\theta,w(\cdot)}} = \|f\|_{HLM_{p\theta,w(\cdot)}}, \tag{5.3}$$

as was to be shown. □

In terms of the grand maximal opetator defined in Definition 3, we can rephrase Proposition 5.1 as follows:

Proposition 5.2 *Let $1 < p < \infty$, $1 < \theta \le \infty$ and $w \in \Omega_\theta$. We define $\hat{v}_1, \hat{v}_2$ by (1.8). Assume that H^* is bounded from $L_{\theta,\hat{v}_1}(0,\infty)$ to $L_{\theta,\hat{v}_2}(0,\infty)$.*

(1) If $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, then $\mathcal{M}f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.
(2) Let $f \in \mathcal{S}'(\mathbb{R}^n)$. If $\mathcal{M}f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, then f is represented by a measurable function g which belongs to $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$.

If $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$, then $C^{-1}\|f\|_{LM_{p\theta,w(\cdot)}} \le \|\mathcal{M}f\|_{LM_{p\theta,w(\cdot)}} \le C\|f\|_{LM_{p\theta,w(\cdot)}}$.

Proof The implication (1) $\Longrightarrow$ (2) immediately follows from the pointwise inequality $\mathcal{M}f(x) \le CMf(x)$. The converse implication (2) $\Longrightarrow$ (1) follows from the pointwise estimatee $|e^{t\Delta}f(x)| \le C\mathcal{M}f(x)$. Indeed, from this pointwise estimate, we conclude $\sup_{t>0}|e^{t\Delta}f| \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Thus, we are in the position of applying Proposition 5.1 to have $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. □

6 Proof of Theorems 1.3 and 1.4

6.1 Norm estimate (Proof of Theorem 1.3)

We use the following lemma:

Lemma 6.1 *Let $1 < p < \infty$, $1 < \theta \le \infty$ and $w \in \Omega_\theta$. Let each A_j be a $(p', \tilde{w}, 2^j)$-block and $\{\rho_j\}_{j=-\infty}^{\infty} \in \ell^{\theta'}(\mathbb{Z})$. Suppose $s \in (p,\infty)$ satisfies (1.9) for all $r > 0$. Then*

$$h \equiv \sum_{j=-\infty}^{\infty} \rho_j (M[|A_j|^{s'}])^{1/s'} \in LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$$

where $\tilde{w}$ is given by $\tilde{w}(r) = r^{-1/q}w(r)$, and it satisfies

$$\|h\|_{LH_{p'\theta',\tilde{w}(\cdot)}} \le C \left(\sum_{j=-\infty}^{\infty} |\rho_j|^{\theta'} \right)^{1/\theta'}.$$

Proof By the $L^{s'}(\mathbb{R}^n)$-boundedness of the Hardy–Littlewood maximal operator and $\theta' < \infty$, we have

$$\sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+1})}(M[|A_j|^{s'}])^{1/s'} \in LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$$

and

$$\left\| \sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+1})}(M[|A_j|^{s'}])^{1/s'} \right\|_{LH_{p'\theta',\tilde{w}(\cdot)}} \le C \left(\sum_{j=-\infty}^{\infty} |\rho_j|^{\theta'} \right)^{1/\theta'}.$$

Meanwhile, we have $\|A_j\|_{L^{s'}} \le |B(2^j)|^{\frac{1}{p}-\frac{1}{s}} \|A_j\|_{L^{p'}} \le C 2^{\frac{jn}{p}-\frac{jn}{s}} \tilde{w}(2^j)$. Therefore,

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+1})^c}(M[|A_j|^{s'}])^{1/s'} \\
&= \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+k+2})\setminus B(2^{j+k+1})}(M[|A_j|^{s'}])^{1/s'} \\
&\le C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{1}{2^{(j+k)n/s'}} \left(\int_{B(2^j)} |A_j(z)|^{s'}\, dz \right)^{1/s'} \chi_{B(2^{j+k+2})\setminus B(2^{j+k+1})} \\
&\le C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{1}{2^{(j+k)n/s'}} \left(\int_{B(2^j)} |A_j(z)|^{s'}\, dz \right)^{1/s'} \chi_{B(2^{j+k+2})\setminus B(2^{j+k+1})} \\
&\le C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{2^{jn/p-jn/s}}{2^{(j+k)n/s'}} \left(\int_{B(2^j)} |A_j(z)|^{p'}\, dz \right)^{1/p'} \chi_{B(2^{j+k+2})\setminus B(2^{j+k+1})}.
\end{aligned}$$

Since A_j is a $(p', \tilde{w}, 2^j)$-atom, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+1})^c}(M[|A_j|^{s'}])^{1/s'} \\
&\le C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{2^{jn/p-jn/s} \tilde{w}(2^j)}{2^{(j+k)n/s'}} \chi_{B(2^{j+k+2})\setminus B(2^{j+k+1})} \\
&= C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{2^{k/p'-k/s'} \tilde{w}(2^j)}{\tilde{w}(2^{j+k+2})} 2^{-(j+k+2)/p'} \tilde{w}(2^{j+k+2}) \chi_{B(2^{j+k+2})\setminus B(2^{j+k+1})}.
\end{aligned}$$

Since we are assuming (1.9),

$$\sum_{k=1}^{\infty}\sum_{j=-\infty}^{\infty}\rho_j\frac{2^{k/p'-k/s'}w(2^j)}{w(2^{j+k+2})}\le C\sum_{j=-\infty}^{\infty}\rho_j.$$

Inserting this estimate, we complete the proof. □

We shall now prove Theorem 1.3.

Proof To prove (1.3), we resort to the duality obtained in Theorem 4.1:

$$\|f\|_{LM_{p\theta,w(\cdot)}}\sim\sup\left\{\left|\int_{\mathbb{R}^n}f(x)g(x)\,dx\right|\,:\,\|g\|_{LH_{p'\theta',\tilde{w}(\cdot)}}=1\right\}.$$

We can assume that $\{\lambda_j\}_{j=1}^{\infty}$ is finitely supported thanks to the monotone convergence theorem. Let us assume in addition that the a_j's are non-negative without loss of generality. We write

$$g\equiv\sum_{k=-\infty}^{\infty}\rho_k A_k,\quad G\equiv\sum_{k=-\infty}^{\infty}|\rho_k|M[|A_k|^{s'}]^{\frac{1}{s'}},$$

where each A_k is a $(p',\tilde{w},2^k)$-block and

$$\sum_{k=-\infty}^{\infty}|\rho_k|^{\theta'}\le 1.$$

Then we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}f(x)g(x)\,dx\right|&\le\sum_{(j,k)\in\mathbb{N}\times\mathbb{Z}}\lambda_j|\rho_k|\int_{B(2^k)\cap Q_j}a_j(x)|A_k(x)|\,dx\\&\le\sum_{(j,k)\in\mathbb{N}\times\mathbb{Z}}\lambda_j|\rho_k|\cdot\|a_j\|_{L^s(Q_j)}\|A_k\|_{L^{s'}(Q_j)}\\&\le\sum_{(j,k)\in\mathbb{N}\times\mathbb{Z}}\lambda_j|\rho_k|\cdot\int_{Q_j}M[A_k{}^{s'}](x)^{\frac{1}{s'}}\,dx\\&=\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}(x)\right)G(x)\,dx.\end{aligned}$$

Since G belongs to $LH_{p'\theta',\tilde{w}(\cdot)}(\mathbb{R}^n)$ and $\|G\|_{LH_{p'\theta',\tilde{w}(\cdot)}}\le C$ from Lemma 6.1, where C depends only on p, q, w and s, we obtain the desired estimate. □

6.2 Nonsmooth decomposition of functions (Proof of Theorem 1.2)

The following lemma is the key to the decomposition of local Morrey-type spaces as is mentioned in Sect. 1; the structure of local Morrey-type spaces comes into play here. We invoke the following estimate from [7].

Lemma 6.2 *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. With the same notation as Lemma 2.1, we have*

$$|\langle b_j, \varphi\rangle| \le C_\varphi \left\{ \sum_{l=0}^{\infty} \left(\frac{1}{2^{ln}} \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} \right\|_{L^1(B(2^l))} \right)^{\mu} \right\}^{1/\mu}, \tag{6.1}$$

where $\mu \equiv \frac{n+d+1}{n}$ and the constant C_φ in (6.1) depends on φ but not on j or k.

In the next lemma, we verify what happens in Lemma 2.1 if $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ with $1 < p < \infty$, $1 < \theta \le \infty$ and $w \in \Omega_\theta$.

Lemma 6.3 *Let $1 < p < \infty$, $1 < \theta \le \infty$ and $w \in \Omega_\theta$. We define $\hat{v}_1, \hat{v}_2$ by (1.8). Assume that H is bounded on $L_{\theta,\hat{v}_1}(0,\infty)$ to $L_{\theta,\hat{v}_2}(0,\infty)$. Suppose that a real parameter $s > 0$ satisfies (1.9) for all $r > 0$. Assume $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. In the notation of Lemma 2.1, in the topology of $\mathcal{S}'(\mathbb{R}^n)$, we have $g_j \to 0$ as $j \to -\infty$ and $b_j \to 0$ as $j \to \infty$. In particular,*

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j)$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$.

Proof Observe that

$$\begin{aligned}
\frac{1}{2^{ln}} \|\mathcal{M}f \cdot \chi_{\mathcal{O}_j}\|_{L^1(B(2^l))} &\le \frac{C}{2^{ln}} \|\mathcal{M}f\|_{L^1(B(2^l))} \\
&\le \frac{C}{2^{ln/p}} \|\mathcal{M}f\|_{L^p(B(2^l))} \\
&\le \frac{C}{2^{ln/p} w(2^l)} \|f\|_{HLM_{p\theta,w(\cdot)}} \\
&\le \frac{C}{2^{ln/p} w(2^l)} \|f\|_{LM_{p\theta,w(\cdot)}}.
\end{aligned}$$

Note that (1.9) yields

$$\sum_{l=1}^{\infty} \left(\frac{1}{2^{ln/p} w(2^l)} \right)^{\mu} < \infty.$$

Consequently, we may use the Lebesgue convergence theorem to conclude that $b_j \to 0$ as $j \to \infty$. Hence, it follows that $f = \lim_{j\to\infty} g_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Consequently, it follows from Lemma 2.1(4) that $f = \lim_{j\to\infty} g_j = \lim_{j,k\to\infty} \sum_{l=-k}^{j} (g_{l+1} - g_l)$ in $\mathcal{S}'(\mathbb{R}^n)$. □

If we mimic the proof of [7], then Theorem 1.2 follows thanks to Lemma 6.3.

7 Application to the Hardy inequality

As an application of the main results, we shall prove the boundedness of the Hardy operator defined by (1.15). We note that the Hardy operators in Morrey type spaces were studied for instance in [33,37,39].

Theorem 7.1 *Suppose* $1 < p < \infty$, $1 \le \theta \le \infty$, $w \in \Omega_\theta$. *Then*

$$\|Hf\|_{LM_{p\theta,w(\cdot)}} \le C\|f\|_{LM_{p\theta,w(\cdot)}}.$$

Proof We let $f = \sum_{j=1}^{\infty} \lambda_j a_j$ as in Theorem 1.4 with $L = 0$. Let us define the operator S by

$$Sf(x) \equiv \int_{\mathrm{SO}(n)} f(Ax)\, d\mu(A),$$

where μ stands for the Haar measure of $\mathrm{SO}(n)$. Note that

$$S : LM_{p\theta,w(\cdot)}(\mathbb{R}^n) \to LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$$

is a bounded linear opeator. Since

$$\begin{aligned}
Hf(x) &\sim \frac{1}{|x|^n} \int_{B(|x|)} f(y)\, dy \\
&= \int_{\mathrm{SO}(n)} \frac{1}{|Ax|^n} \int_{B(|Ax|)} f(y)\, dy\, d\mu(A) \\
&= \int_{\mathrm{SO}(n)} \frac{1}{|Ax|^n} \int_{B(|Ax|)} f(Ay)\, dy\, d\mu(A) \\
&= \int_{\mathrm{SO}(n)} \frac{1}{|x|^n} \int_{B(|x|)} f(Ay)\, dy\, d\mu(A) \\
&= HSf(x),
\end{aligned}$$

we have

$$Hf = HSf = \sum_{j=1}^{\infty} \lambda_j HSa_j.$$

Observe also that

$$|HSa_j| \le CS\chi_{Q_j},$$

since a_j is compactly supported. Thus,

$$\begin{aligned}\|Hf\|_{LM_{p\theta,w(\cdot)}} &\le \left\| \sum_{j=1}^{\infty} \lambda_j HSa_j \right\|_{LM_{p\theta,w(\cdot)}} \\ &\le C \left\| \sum_{j=1}^{\infty} \lambda_j S\chi_{Q_j} \right\|_{LM_{p\theta,w(\cdot)}} \\ &\le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{LM_{p\theta,w(\cdot)}}\end{aligned}$$

and we can use Theorem 1.4. □

Remark 1 See [17] for another approach.

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Appendix

We can relate local Morrey-type spaces with Herz spaces.

Lemma 8.1 *Let* $1 < p < \infty$, $1 \le \theta \le \infty$ *and* $0 < \lambda < \frac{n}{p}$. *Then*

$$\|f\|_{LM^{\lambda}_{p\theta}} \sim \left\{ \sum_{j=-\infty}^{\infty} \left(2^{-\lambda j} \left(\int_{2^{j-1}<|y|<2^j} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}}$$

for all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$.

The right-hand side above is called the Herz norm.

Proof It is clear from (1.14) that

$$\|f\|_{LM^{\lambda}_{p\theta}} \gtrsim \left\{ \sum_{j=-\infty}^{\infty} \left(2^{-\lambda j} \left(\int_{2^{j-1}<|y|<2^j} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}}.$$

To prove the reverse estimate, we have

$$\|f\|_{LM^{\lambda}_{p\theta}} \sim \left\{ \sum_{j=-\infty}^{\infty} \left(2^{-\lambda j} \left(\int_{|y|<2^j} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}}$$

$$= \left\{ \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j} 2^{-\lambda j} \left(\int_{2^{k-1}<|y|<2^k} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}}$$

$$= \left\{ \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \chi_{(-\infty,j]}(k) 2^{-\lambda j} \left(\int_{2^{k-1}<|y|<2^k} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}}$$

$$\leq \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} \left(\chi_{(-\infty,j]}(k) 2^{-\lambda j} \left(\int_{2^{k-1}<|y|<2^k} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}}$$

$$= \sum_{k=-\infty}^{\infty} \left\{ \left(\frac{1}{1-2^{-\lambda}} \cdot 2^{-\lambda k} \left(\int_{2^{k-1}<|y|<2^k} |f(y)|^p \, dy \right)^{\frac{1}{p}} \right)^{\theta} \right\}^{\frac{1}{\theta}},$$

as was to be shown. □

Comparison of various Morrey spaces and Nikolskii spaces

Global weighted Morrey type spaces

Theorems 1.3 and 1.4 are translated into the following results on global Morrey spaces.

Theorem 9.1 *Let* $1 < p < \infty$, $1 < \theta \leq \infty$ *and* $w \in \Omega_\theta$. *Assume that* w *satisfies the doubling condition;* $C^{-1} w(r) \leq w(2r) \leq C w(r)$ *for all* $r > 0$. *We define* $\hat{v}_1, \hat{v}_2$ *by* (1.8). *Assume that* H^* *is bounded on* $L_{\theta,\hat{v}_1}(0,\infty)$ *to* $L_{\theta,\hat{v}_2}(0,\infty)$. *Suppose that a real parameter* s *satisfies* (1.9) *for all* $r > 0$. *Assume that we are given* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset L^s(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ *satisfying*

$$\|a_j\|_{L^s} \leq \|\chi_{Q_j}\|_{L^s} = |Q_j|^{1/s}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{GM_{p\theta,w(\cdot)}} < \infty. \tag{9.1}$$

Then the series $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and in the Schwartz space* $\mathcal{S}'(\mathbb{R}^n)$ *of tempered distributions and satisfies the estimate*

$$\|f\|_{GM_{p\theta,w(\cdot)}} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{GM_{p\theta,w(\cdot)}}, \tag{9.2}$$

where $C > 0$ *depends only on* n, p, q, w *and* s.

Proof Just use (1.7) and reexamine the proof of Theorem 1.4. We omit the further detail. □

Theorem 9.2 *Let $L \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $1 < p < \infty$, $1 < \theta \le \infty$ and $w \in \Omega_\theta$. Assume that w satisfies the doubling condition; $C^{-1} w(r) \le w(2r) \le C w(r)$ for all $r > 0$. We define $\hat{v}_1, \hat{v}_2$ by (1.8). Assume that H^* is bounded from $L_{\theta,\hat{v}_1}(0,\infty)$ to $L_{\theta,\hat{v}_2}(0,\infty)$. Suppose that a real parameter s satisfies (1.9) for all $r > 0$. Let $f \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f \equiv \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, that a_j satisfies (1.12) for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n \le L$ and, that for all $v > 0$*

$$\left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{GM_{p\theta,w(\cdot)}} \le C_v \|f\|_{GM_{p\theta,w(\cdot)}}. \tag{9.3}$$

Here the constant $C_v > 0$ is independent of f.

Proof Use the method of Theorem 9.1 to obtain $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f \equiv \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, that a_j satisfies (1.12) and (1.13) for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n \le L$ and for all $v > 0$. We need to check (9.3). To this end, we have only to show

$$\left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j}(\cdot + x))^v \right)^{1/v} \right\|_{LM_{p\theta,w(\cdot)}} \le C_v \|f(\cdot + x)\|_{LM_{p\theta,w(\cdot)}}.$$

However, since

$$f = \sum_{j=1}^\infty \lambda_j a_j$$

is the atomic decomposition of $f \in LM_{p\theta,w(\cdot)}$, we can say that

$$f(\cdot + x) = \sum_{j=1}^\infty \lambda_j a_j(\cdot + x)$$

is the atomic decomposition of f. Thus, we obtain

$$\begin{aligned} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j}(\cdot + x))^v \right)^{1/v} \right\|_{LM_{p\theta,w(\cdot)}} &\le C_v \|\mathcal{M}[f(\cdot + x)]\|_{LM_{p\theta,w(\cdot)}} \\ &= C_v \|\mathcal{M} f(\cdot + x)\|_{LM_{p\theta,w(\cdot)}} \\ &\le C_v \|f(\cdot + x)\|_{LM_{p\theta,w(\cdot)}}, \end{aligned}$$

as was to be shown. □

Classical Morrey spaces

The classical Morrey spaces $M_p^\lambda(\mathbb{R}^n)$ were first introduced by Morrey in [34] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [1,20,38]. For the properties and applications of classical Morrey spaces, see [21,23] and references therein.

Morrey spaces $M_p^\lambda(\mathbb{R}^n)$, named after Morrey, were based on his study of elliptic differential operators in 1938 [34] and they are defined as follows: For $\lambda \in \mathbb{R}$, $0 < p \le \infty$, $f \in M_p^\lambda(\mathbb{R}^n)$ if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda}\|f\|_{L^p(B(x,r))} < \infty,$$

where $B(x, r)$ is the open ball in $\mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$ of radius $r > 0$.

In other words $f \in M_p^\lambda(\mathbb{R}^n)$ if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ and there exists $c > 0$ (depending on f) such that for all $x \in \mathbb{R}^n$ and for all $r > 0$

$$\|f\|_{L^p(B(x,r))} \le cr^\lambda.$$

The minimal value of c in this inequality is $\|f\|_{M_p^\lambda}$.

If $\lambda = 0$, then

$$M_p^0(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

If $\lambda = \frac{n}{p}$, then

$$M_p^{\frac{n}{p}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n).$$

If $\lambda > \frac{n}{p}$ or $\lambda < 0$, then

$$M_p^\lambda(\mathbb{R}^n) = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

So the admissible range of the parameters is

$$0 < p \le \infty \quad \textit{and} \quad 0 \le \lambda \le \frac{n}{p}. \tag{9.4}$$

The cases $p = \infty$ forces λ to be 0 and $M_\infty^0(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. Under these assumptions, which will always be assumed in the sequel, the space $M_p^\lambda(\mathbb{R}^n)$ is a Banach space for $1 \le p \le \infty$ and a quasi-Banach space for $0 < p < 1$.

Also the space $M_p^\lambda(\mathbb{R}^n)$ does not coincide with a Lebesgue space, if and only if

$$0 < p < \infty \quad and \quad 0 < \lambda < \frac{n}{p}. \tag{9.5}$$

Furthermore,

$$L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \subset M_p^\lambda(\mathbb{R}^n).$$

If $f \in L^p$, then $f \in M_p^\lambda(\mathbb{R}^n)$ if and only if $\sup_{x\in\mathbb{R}^n, 0<r\le 1} r^{-\lambda}\|f\|_{L^p(B(x,r))} < \infty$, hence under this assumption only local properties of f are of importance.

Consider the Nikolskii space $H_p^\lambda \equiv H_p^\lambda(\mathbb{R}^n)$ of functions possessing common smoothness of order λ measured in the L^p metrics. For $\lambda > 0, 1 \le p \le \infty$ they are defined in the following way: fix an integer $\sigma > \lambda$. We say that $f \in H_p^\lambda(\mathbb{R}^n)$ if $f \in L^p(\mathbb{R}^n)$ and

$$\|f\|_{H_p^\lambda} = \|f\|_{L^p} + \sup_{h\in\mathbb{R}^n, h\neq 0} |h|^{-\lambda} \left\|\Delta_h^\sigma f\right\|_{L^p} < \infty,$$

where $\Delta_h^\sigma f$ is the difference of f of order $\sigma \in \mathbb{N}$ with step h. For different $\sigma > \lambda$ the definitions are equivalent.) One can prove that if $0 < \lambda < \frac{n}{p}$, then

$$H_p^\lambda(\mathbb{R}^n) \subset M_p^\lambda(\mathbb{R}^n).$$

We refer to [31] for $n = 1$ and [35,36] $n > 1$. Clearly the converse inclusion does not hold, because if $f \in M_p^\lambda(\mathbb{R}^n)$, then clearly $fg \in M_p^\lambda(\mathbb{R}^n)$ for any bounded measurable function g, which is not true for the case of the spaces $H_p^\lambda(\mathbb{R}^n)$. So, $M_p^\lambda(\mathbb{R}^n)$ is not a space of functions possessing any kind of common smoothness of order λ, but the expressions $\|f\|_{L^p(B(x,r))}$ behave like the ones for functions f possessing certain smoothness of order λ. Detailed exposition of properties of these spaces can be found in [9,36]. Note that the expression for $\|f\|_{LM_{p\theta}^\lambda}$ is very similar to the semi-norms $\|f\|_{b_{p\theta}^\lambda}$ of the Nikol'skii-Besov spaces $B_{p\theta}^\lambda$. In the latter case, we suppose $\lambda > 0, 1 \le p, \theta \le \infty$ and $\|f\|_{L^p(B(r))}$ should be replaced by the L^p modulus of continuity: $w^\sigma(f, r) = \sup_{|h|\le r} \left\|\Delta_h^\sigma f\right\|_{L^p(\mathbb{R}^n)}$ with $\sigma > \lambda$. Recall that $\|f\|_{B_{p\theta}^\lambda} = \|f\|_{L^p} + \|f\|_{b_{p\theta}^\lambda}$. If $\theta = \infty$ then $B_{p\infty}^\lambda(\mathbb{R}^n) \equiv H_p^\lambda(\mathbb{R}^n)$. There are several definitions, equivalent for these values of the parameters, of the spaces $B_{p\theta}^\lambda(\mathbb{R}^n)$. The definition mentioned above makes sense for a wider range of the parameters, namely for $\lambda > 0, 0 < p, \theta \le \infty$. For this range of the parameters the equivalence of the quasi-norms $\|\cdot\|_{B_{p\theta}^\lambda}$ for different $\sigma > \lambda$ was proved in [20]. If $\theta = p$ then

$$\|f\|_{LM_{pp}^\lambda} = (\lambda p)^{-\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{\lambda p}} dx\right)^{\frac{1}{p}}. \tag{9.6}$$

For $n = 1$, $1 \le p, \theta < \infty$, $0 < \lambda < \frac{1}{p}$ the inclusion

$$B^{\lambda}_{p\theta}(\mathbb{R}^n) \subset GM^{\lambda}_{p\theta}(\mathbb{R}^n) \tag{9.7}$$

was proved by Kuznetsov [32]. In the diagonal case $p = \theta$ (9.7) follows by equality (9.6) and the estimate of the right-hand side of (4.2) via $\|f\|_{b^{\lambda}_{pp}}$ for functions $f \in B^{\lambda}_{pp}$, proved by Yakovlev [42,43].

Let us recall some results on local Morrey-type spaces. In 1994 Guliyev initially introdused and studied the local Morrey-type spaces in his doctoral thesis [24]; see also [25]. The main purpose of [24,25] is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in the local Morrey-type spaces.

In a series of papers [2,3,10–16] by Burenkov, Husein Guliyev and Vagif Guliyev etc. some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ were given. The fractional maximal operator, the Hardy–Littlewood maximal operator, the fractional integral operator and the Marcinkiewicz operator are considered in [10,13,14,16,28], respectively. We refer to [40] for the two-weight estimates for the Hardy–Littlewood maximal operators.

Local Morrey-type spaces and interpolation

Investigating local Morrey-type spaces is not a mere quest to generality; it appears naturally in the context of real interpolation. In [18], Burenkov and Nursultanov established that

$$(L^p(\Omega, w^{\lambda_0}, \mu), L^p(\Omega, w^{\lambda_1}, \mu))_{\theta,q}$$

is a generalized local Morrey-type space, when we are given weights. More precisely, we can state the result as follows: We start with generalizing the space $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. Let $0 < p, q \le \infty$, $0 < \lambda < \infty$ if $q < \infty$, $0 \le \lambda < \infty$ if $q = \infty$. Let $\Omega \subset \mathbb{R}^n$ be a measurable set and μ be a σ-finite Borel measure on Ω. Moreover, let $G = (G_t)_{t>0}$ where all the G_t's are μ-measurable subsets for which $G_t \ne \Omega$ for some $t > 0$, $G_{t_1} \subset G_{t_2}$ if $t_1 < t_2$, and $\bigcup_{t>0} G_t = \Omega$. We define

$$\|f\|_{LM^{\lambda}_{p,q}(G,\mu)} \equiv \left(\int_0^{\infty} (t^{-\lambda}\|f\|_{L^p(G_t,\mu)})^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

To formulate the interpolation result, we consider a special case of G. Let w_0, w_1 be positive μ-measurable functions on $\Omega \subset \mathbb{R}^n$ and $0 < \lambda_0, \lambda_1 < \infty$. Let the family $G_{\lambda_0,\lambda_1} = (G_{t,\lambda_0,\lambda_1})_{t>0}$ be defined by:

$$G_{t,\lambda_0,\lambda_1} = \{x \in \Omega : w_0(x)^{\alpha_0} w_1(x)^{\alpha_1} < t\} \quad (t > 0).$$
$$d\nu_{\lambda_0,\lambda_1}(x) = (w_0(x)^{\beta_0} w_1(x)^{\beta_1})^p \, d\mu(x),$$

where

$$\alpha_0 = \frac{1}{\lambda_1 - \lambda_0}, \quad \alpha_1 = \frac{1}{\lambda_0 - \lambda_1}, \quad \beta_0 = \frac{\lambda_1}{\lambda_1 - \lambda_0}, \quad \beta_1 = \frac{\lambda_0}{\lambda_0 - \lambda_1}.$$

Observe that

$$LM^{\lambda_0}_{p,p}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}) = L^p(\Omega, w_0, \mu), \quad LM^{\lambda_1}_{p,p}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}) = L^p(\Omega, w_1, \mu).$$

In words of the book [8], Burenkov and Nursultanov obtained the following interpolation result (9.8) in [18], which compliments the Stein-Weiss type interpolation (9.9):

Theorem 9.3 *Suppose that the parameters* $p, q, \lambda_0, \lambda_1, \theta \in (0, \infty]$ *satisfy*

$$\lambda_0, \lambda_1 < \infty, \quad \lambda_0 \neq \lambda_1, \quad \theta < 1,$$

and $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$. *Then*

$$(L^p(\Omega, w_0, \mu), L^p(\Omega, w_1, \mu))_{\theta,q} = LM^{\lambda}_{p,q}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}). \tag{9.8}$$

If $q = p$,

$$(L^p(\Omega, w_0, \mu), L^p(\Omega, w_1, \mu))_{\theta,p} = L^p(\Omega, {w_0}^{1-\theta}{w_1}^{\theta}, \mu). \tag{9.9}$$

Our method seems to be applicable to the anisotropic local Morrey-type spaces defined in [2]. Based on the definition above, Akbulut, Guliyev and Muradova discussed the boundedness property of the anisotropic Riesz potential in the anisotropic local Morrey-type spaces in [3]. We feel that the method employed in [29] seems to be applicable once we obtain a counterpart of Theorems 1.1 and 1.2. In [5], Aykol, Guliyev and Serbetci defined the local Lorentz Morrey spaces as the set of all measurable functions f for which the quasi-norm

$$\|f\|_{M^{\mathrm{loc}}_{p,q;1}} \equiv \sup_{t>0} t^{-\lambda/q} \|s^{1/p-1/q} f^*(s)\|_{L_q(0,t)} < \infty.$$

In [5, Theorem 4.1], Aykol, Guliyev and Serbetci obtained the boundedness of the Hardy–Littlewood maximal operator in the local Lorentz Morrey spaces. In [6, Theorem 3.1], Aykol, Guliyev, Kucukaslan and Serbetci obtained the boundedness of the Hilbert transform in the local Lorentz Morrey spaces. The modification of the argument to the anisotropic setting or to the local Lorentz Morrey spaces will be left as a future work.

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