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Some Remarks on a Classification of Nilpotent Compatible Leibniz Algebras

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Abstract

In this note, we describe compatible Leibniz algebras and present several of their properties. Our aim is to present a comprehensive classification of non-Lie nilpotent compatible Leibniz algebras in low dimensions. By using the full classification of non-Lie nilpotent Leibniz algebras over the complex field of dimensions two, three, and four, in this note, our aim is to provide the classification of non-Lie nilpotent compatible Leibniz algebras over the complex field of dimensions two, three, and four. Consequently, we show that there is no isomorphism class in dimension two, there are 5 isomorphism classes in dimension three, and there are 134 isomorphism classes in dimension four.

Keywords: Leibniz algebra; nilpotent algebra; compatible Leibniz algebra

MSC: 17A30; 17B30; 17A32

1. Introduction

J.L. Loday in [1] described some original sorts of algebras in conjunction with their (co)homologies and focused on the associated operads. Some instances of such algebras are Leibniz algebras, Zinbiel algebras, and their Koszul duals. A vector space $\mathcal{C}$ with a bilinear mapping $[,]$ is referred to as a Leibniz algebra if it holds the Leibniz identity

$$[[\kappa, \vartheta], r] = [\kappa, [\vartheta, r]] - [\vartheta, [\kappa, r]]$$

for each $\kappa, \vartheta, r \in \mathcal{C}$. In the existence of anti-commutativity, the Leibniz identity reduces to the Jacobi identity, indicating that every Lie algebra qualifies as a Leibniz algebra. Consequently, Leibniz algebras introduce a non-anti-commutative counterpart to Lie algebras. Actually, Bloh [2] in 1965 first presented such algebras, which were called $\mathcal{D}$-algebras. Then, Loday [3] examined Leibniz algebras, which are also known as Loday algebras in the literature.

In this note, a concept of compatible Leibniz algebras is presented and studied. Let $(\mathcal{C}, [,]_1)$ and $(\mathcal{C}, [,]_2)$ be two Leibniz algebras over a field $\mathcal{K}$. If $\mathcal{C}$ with the bilinear mapping $[,]$ defined by $[\kappa, \vartheta] = \lambda_1[\kappa, \vartheta]_1 + \lambda_2[\kappa, \vartheta]_2$ for each $\kappa, \vartheta \in \mathcal{C}$ and for any $\lambda_1, \lambda_2 \in \mathcal{K}$ becomes a Leibniz algebra structure, $(\mathcal{C}, [,]_1, [,]_2)$ is called compatible. Compatible Lie algebras were investigated in [4–6] by Golubchik and Sokolov. In [7], the authors classified the nilpotent compatible Lie algebras. Compatible Leibniz algebras were discussed in [8], by Makhlouf and Saha, and right Leibniz algebras were used. In our paper, by using left Leibniz algebras, we study compatible Leibniz algebras. One of the very difficult problems is to classify nilpotent compatible Leibniz algebras. On account of the absence of anti-commutativity

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in Leibniz algebras, classifying non-Lie nilpotent compatible Leibniz algebras is more challenging. By using the full classification of non-Lie nilpotent Leibniz algebras over the complex field of dimensions two, three, and four (refer to [9,10]), in this note, our aim is to provide the classification of non-Lie nilpotent compatible Leibniz algebras over the complex field of dimensions two, three, and four.

2. Preliminaries

We let $\mathcal{K}$ be an arbitrary field of characteristic zero.

A vector space $\mathcal{C}$ over $\mathcal{K}$ by a bilinear mapping $[,]$ is referred to as a (left) Leibniz algebra if it holds the Leibniz identity

$$[[\kappa, \vartheta], r] = [\kappa, [\vartheta, r]] - [\vartheta, [\kappa, r]]$$

for all $\kappa, \vartheta, r \in \mathcal{C}$.

Since, in the existence of anti-commutativity, the Leibniz identity reduces to the Jacobi identity, as an immediate result, each Lie algebra qualifies as a Leibniz algebra. Conversely, let $\mathcal{C}$ be a Leibniz algebra with $[\kappa, \kappa] = 0$ for any $\kappa \in \mathcal{C}$. Then, $\mathcal{C}$ becomes a Lie algebra. If $[\kappa, \vartheta] = 0$ for every $\kappa, \vartheta \in \mathcal{C}$, a Leibniz algebra $\mathcal{C}$ is abelian (see [11,12]). A Leibniz algebra $\mathcal{C}$ has a distinguished ideal which is denoted by $Leib(\mathcal{C})$. This ideal is spanned with such elements $[\kappa, \kappa]$, $\kappa \in \mathcal{C}$. It is obvious to show that $Leib(\mathcal{C})$ is one of ideals in $\mathcal{C}$. This ideal is said to be the Leibniz kernel of $\mathcal{C}$. Moreover, $\mathcal{C}/Leib(\mathcal{C})$ is a Lie algebra. We state a significant characteristic of the Leibniz kernel below:

$$[[\kappa, \kappa], \vartheta] = [\kappa, [\kappa, \vartheta]] - [\kappa, [\kappa, \vartheta]] = 0. \tag{1}$$

Therefore, $Leib(\mathcal{C})$ is an abelian subalgebra of $\mathcal{C}$.

Let $\mathcal{H}$ and $\mathcal{S}$ be two Leibniz algebras. A linear mapping $\varphi : \mathcal{H} \to \mathcal{S}$ is called a Leibniz algebra homomorphism if $\varphi([\kappa, r]) = [\varphi(\kappa), \varphi(r)]$ for all $\kappa, r \in \mathcal{H}$. If φ is bijective, we say that φ is an isomorphism.

The left and right centres $Cent^l(\mathcal{C})$ and $Cent^r(\mathcal{C})$ of $\mathcal{C}$ are described by the rules

$$Cent^l(\mathcal{C}) = \{\kappa \in \mathcal{C} : [\kappa, \vartheta] = 0 \text{ for each element } \vartheta \in \mathcal{C}\}$$
$$Cent^r(\mathcal{C}) = \{\kappa \in \mathcal{C} : [\vartheta, \kappa] = 0 \text{ for each element } \vartheta \in \mathcal{C}\}.$$

In general, $Cent^l(\mathcal{C}) \neq Cent^r(\mathcal{C})$. It is possible to prove that the left centre of $\mathcal{C}$ is an ideal; however, the right centre of $\mathcal{C}$ is only a subalgebra (see corresponding examples in [13]). When

$$Cent(\mathcal{C}) = \{\kappa \in \mathcal{C} : [\kappa, \vartheta] = 0 = [\vartheta, \kappa] \text{ for any element } \vartheta \in \mathcal{C}\},$$

the set $Cent(\mathcal{C})$ is said to be the centre of $\mathcal{C}$. It is easy to show that the centre is an ideal of $\mathcal{C}$, so we are able to mention the factor algebra $\mathcal{C}/Cent(\mathcal{C})$. Clearly, $Cent^l(\mathcal{C})$ and $Cent^r(\mathcal{C})$ are abelian.

The property (1) shows that the Leibniz kernel of $\mathcal{C}$ is an ideal of the left centre of $\mathcal{C}$. This is $Leib(\mathcal{C}) \leq Cent^l(\mathcal{C})$. Arising from the centre, we are able to determine the upper central series

$$\xi_1(\mathcal{C}) \leq \xi_2(\mathcal{C}) \leq \ldots \leq \xi_n(\mathcal{C}) \leq \xi_{n+1}(\mathcal{C}) \leq \ldots$$

of a Leibniz algebra $\mathcal{C}$ by the following regulation: $\xi_1(\mathcal{C}) = Cent(\mathcal{C})$ is the centre of $\mathcal{C}$, and, recursively, $\xi_{i+1}(\mathcal{C})/\xi_i(\mathcal{C}) = Cent(\mathcal{C}/\xi_i(\mathcal{C}))$ for all positive integers i.

The series of $\mathcal{C}$

$$\mathcal{C} = \gamma_1(\mathcal{C}) \geq \gamma_2(\mathcal{C}) \geq \ldots \geq \gamma_i(\mathcal{C}) \geq \gamma_{i+1}(\mathcal{C}) \geq \ldots$$

which is named the lower central series is described by the following regulation: $\gamma_1(\mathcal{C}) = \mathcal{C}, \gamma_{i+1}(\mathcal{C}) = [\mathcal{C}, \gamma_i(\mathcal{C})]$ for all positive integers i.

If there exists an element s in $\mathbb{Z}^+$ in such a way that $\gamma_s(\mathcal{C}) = \{0\}$, the algebra $\mathcal{C}$ is called nilpotent. Moreover, we say that n is named the nilpotency class of $\mathcal{C}$ if $\gamma_{n+1}(\mathcal{C}) = \{0\}$ but $\gamma_n(\mathcal{C}) \neq \{0\}$. This is denoted by $ncl(\mathcal{C})$.

3. Results

3.1. Compatible Leibniz Algebras

In this part, we begin by determining compatible Leibniz algebras and discussing their fundamentals.

Definition 1. *Let $\bar{\mathcal{C}} = (\mathcal{C}, [,]_1)$ and $\tilde{\mathcal{C}} = (\mathcal{C}, [,]_2)$ be two Leibniz algebras over the field $\mathcal{K}$. If $\mathcal{C}$ equipped with the bilinear mapping*

$$[\kappa, \vartheta] = \lambda_1[\kappa, \vartheta]_1 + \lambda_2[\kappa, \vartheta]_2, \tag{2}$$

for any $\lambda_1, \lambda_2 \in \mathcal{K}$ for all $\kappa, \vartheta \in \mathcal{C}$ forms a Leibniz algebra, then the triplet $(\mathcal{C}, [,]_1, [,]_2)$ is said to be a compatible Leibniz algebra.

Proposition 1. *Let $\bar{\mathcal{C}} = (\mathcal{C}, [,]_1)$ and $\tilde{\mathcal{C}} = (\mathcal{C}, [,]_2)$ be two Leibniz algebras over the field $\mathcal{K}$. Then, $(\mathcal{C}, [,]_1, [,]_2)$ is a compatible Leibniz algebra if and only if the following identity satisfies for each elements $\kappa, \vartheta, r \in \mathcal{C}$:*

$$[\kappa, [\vartheta, r]_1]_2 + [\kappa, [\vartheta, r]_2]_1 - [[\kappa, \vartheta]_1, r]_2 - [[\kappa, \vartheta]_2, r]_1 + [[\kappa, r]_1, \vartheta]_2 + [[\kappa, r]_2, \vartheta]_1 = 0. \tag{3}$$

The identity (3) is called the mixed Leibniz identity.

Proof. Since $(\mathcal{C}, [,]_1)$ and $(\mathcal{C}, [,]_2)$ are two Leibniz algebras, we have

$$[[\kappa, \vartheta]_1, r]_1 = [\kappa, [\vartheta, r]_1]_1 - [\vartheta, [\kappa, r]_1]_1 \tag{4}$$

and

$$[[\kappa, \vartheta]_2, r]_2 = [\kappa, [\vartheta, r]_2]_2 - [\vartheta, [\kappa, r]_2]_2. \tag{5}$$

Suppose that $(\mathcal{C}, [,]_1, [,]_2)$ is a compatible Leibniz algebra. Therefore, by Definition 1, $(\mathcal{C}, [,])$ with $[\kappa, \vartheta] = \lambda_1[\kappa, \vartheta]_1 + \lambda_2[\kappa, \vartheta]_2$, for any $\lambda_1, \lambda_2 \in \mathcal{K}$ and for all $\kappa, \vartheta \in \mathcal{C}$ forms a Leibniz algebra, namely, it satisfies

$$[[\kappa, \vartheta], r] = [\kappa, [\vartheta, r]] - [\vartheta, [\kappa, r]]. \tag{6}$$

By using (2), we acquire

$$\begin{aligned} [\kappa, [\vartheta, r]] &= [\kappa, [\vartheta, r]_1 + [\vartheta, r]_2]_1 + [\kappa, [\vartheta, r]_1 + [\vartheta, r]_2]_2 \\ &= [\kappa, [\vartheta, r]_1]_1 + [\kappa, [\vartheta, r]_2]_1 + [\kappa, [\vartheta, r]_1]_2 + [\kappa, [\vartheta, r]_2]_2 \\ &= [[\kappa, \vartheta]_1, r]_1 - [[\kappa, r]_1, \vartheta]_1 + [[\kappa, \vartheta]_2, r]_2 - [[\kappa, r]_2, \vartheta]_2 \\ &+ [\kappa, [\vartheta, r]_2]_1 + [\kappa, [\vartheta, r]_1]_2, \end{aligned} \tag{7}$$

$$\begin{aligned}
[\vartheta,[\kappa,r]] &= [\vartheta,[\kappa,r]_1+[\kappa,r]_2]_1+[\vartheta,[\kappa,r]_1+[\kappa,r]_2]_2 \\
&= [\vartheta,[\kappa,r]_1]_1+[\vartheta,[\kappa,r]_2]_1+[\vartheta,[\kappa,r]_1]_2+[\kappa,[\vartheta,r]_2]_2 \\
&= [[\kappa,\vartheta]_1,r]_1-[[\vartheta,r]_1,\kappa]_1+[[\vartheta,\kappa]_2,r]_2-[[\vartheta,r]_2,\kappa]_2 \\
&+ [\vartheta,[\kappa,r]_2]_1+[\vartheta,[\kappa,r]_1]_2
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
[[\kappa,\vartheta],r] &= [[\kappa,\vartheta]_1+[\kappa,\vartheta]_2,r]_1+[[\kappa,\vartheta]_1+[\kappa,\vartheta]_2,r]_2 \\
&= [[\kappa,\vartheta]_1,r]+[[\kappa,\vartheta]_2,r]_1+[[\kappa,\vartheta]_1,r]+[[\kappa,\vartheta]_2,r]_2 \\
&= [[\kappa,\vartheta]_1,r]_1-[[\kappa,r]_1,\vartheta]_1+[[\kappa,\vartheta]_2,r]_2-[[\kappa,r]_2,\vartheta]_2.
\end{aligned} \tag{9}$$

By substituting (7)–(9) into Equation (6), we obtain

$$[\kappa,[\vartheta,r]_1]_2+[\kappa,[\vartheta,r]_2]_1-[[\kappa,\vartheta]_1,r]_2-[[\kappa,\vartheta]_2,r]_1+[[\kappa,r]_1,\vartheta]_2+[[\kappa,r]_2,\vartheta]_1=0, \tag{10}$$

as required. Now, we assume that $\mathcal{C}$ holds the mixed Leibniz identity. By doing all processes conversely, the bilinear operation $[,]$ becomes a Leibniz bracket. □

It can be possible to restate that Definition 1 as a compatible Leibniz algebra is a triplet $(\mathcal{C},[,]_1,[,]_2)$ in such way that

(i) $(\mathcal{C},[,]_1)$ is a Leibniz algebra;
(ii) $(\mathcal{C},[,]_2)$ is a Leibniz algebra;
(iii) $[\kappa,[\vartheta,r]_1]_2+[\kappa,[\vartheta,r]_2]_1-[[\kappa,\vartheta]_1,r]_2-[[\kappa,\vartheta]_2,r]_1+[[\kappa,r]_1,\vartheta]_2+[[\kappa,r]_2,\vartheta]_1=0$, for every $\kappa,\vartheta,r\in\mathcal{C}$.

Example 1. *Let $\bar{\mathcal{C}}=(\mathcal{C},[,]_1)$ and $\tilde{\mathcal{C}}=(\mathcal{C},[,]_2)$ be two Leibniz algebras with $[\kappa,\kappa]_1=\vartheta$, $[\vartheta,r]_2=r$, and $[\kappa,\kappa]_2=r$. Then, $(\mathcal{C},[,]_1,[,]_2)$ is a compatible Leibniz algebra. To verify the compatibility of this pair of multiplications, we simply need to examine the mixed Leibniz identity on separate basis elements. We calculate*

$$\begin{aligned}
&[\kappa,[\vartheta,r]_1]_2+[\kappa,[\vartheta,r]_2]_1-[[\kappa,\vartheta]_1,r]_2-[[\kappa,\vartheta]_2,r]_1+[[\kappa,r]_1,\vartheta]_2+[[\kappa,r]_2,\vartheta]_1 \\
&\qquad = [\kappa,r]_1=0, \\
&[r,[\vartheta,r]_1]_2+[r,[\vartheta,r]_2]_1-[[r,\vartheta]_1,r]_2-[[r,\vartheta]_2,r]_1+[[r,r]_1,\vartheta]_2+[[r,r]_2,\vartheta]_1 \\
&\qquad = [r,r]_1=0, \\
&[\vartheta,[\vartheta,r]_1]_2+[\vartheta,[\vartheta,r]_2]_1-[[\vartheta,\vartheta]_1,r]_2-[[\vartheta,\vartheta]_2,r]_1+[[\vartheta,r]_1,\vartheta]_2+[[\vartheta,r]_2,\vartheta]_1 \\
&\qquad = [\vartheta,r]_1=0, \\
&[\vartheta,[\kappa,\kappa]_1]_2+[\vartheta,[\kappa,\kappa]_2]_1-[[\vartheta,\kappa]_1,\kappa]_2-[[\vartheta,\kappa]_2,\kappa]_1+[[\vartheta,\kappa]_1,\kappa]_2+[[\vartheta,\kappa]_2,\kappa]_1 \\
&\qquad = [\vartheta,\vartheta]_2+[\vartheta,r]_1=0, \\
&[r,[\kappa,\kappa]_1]_2+[r,[\kappa,\kappa]_2]_1-[[r,\kappa]_1,\kappa]_2-[[r,\kappa]_2,\kappa]_1+[[r,\kappa]_1,\kappa]_2+[[r,\kappa]_2,\kappa]_1 \\
&\qquad = [r,\vartheta]_2+[r,r]_1=0.
\end{aligned}$$

Definition 2. *A subalgebra of a compatible Leibniz algebra $\mathcal{C}$ is defined as a vector subspace of $\mathcal{C}$ that remains closed under both multiplications. An ideal $\mathcal{P}$ within a compatible Leibniz algebra $\mathcal{C}$ is characterised as a vector subspace that satisfies the following statements: $[\mathcal{P},\mathcal{C}]_1,[\mathcal{C},\mathcal{P}]_1,[\mathcal{P},\mathcal{C}]_2,[\mathcal{C},\mathcal{P}]_2\subseteq\mathcal{P}$. This is denoted by $\mathcal{P}\leq\mathcal{C}$.*

Definition 3. *The centre of a compatible Leibniz algebra $\mathcal{C}$, represented with $Cent(\mathcal{C})$, is an ideal described with*

$$Cent(\mathcal{C}) = \{\vartheta \in \mathcal{C} | [\vartheta, \mathcal{C}]_1 = [\mathcal{C}, \vartheta]_1 = 0 = [\vartheta, \mathcal{C}]_2 = [\mathcal{C}, \vartheta]_2\} = Cent(\bar{\mathcal{C}}) \cap Cent(\tilde{\mathcal{C}}).$$

Definition 4. *A compatible Leibniz algebra $\mathcal{C}$ is referred to as abelian if both of its multiplications are trivial, or, in other words, if it is identical to its centre, specifically, $[\kappa, \vartheta]_1 = [\kappa, \vartheta]_2 = 0$ for all $\kappa, \vartheta \in \mathcal{C}$.*

Let $\mathcal{H}$ and $\mathcal{S}$ be two subspaces of a compatible Leibniz algebra $\mathcal{C}$. The product of $\mathcal{H}$ and $\mathcal{S}$ is described as

$$[\mathcal{H}, \mathcal{S}] = [\mathcal{H}, \mathcal{S}]_1 + [\mathcal{H}, \mathcal{S}]_2 = \mathrm{span}\{[h, s]_1, [h, s]_2 | h \in \mathcal{H}, s \in \mathcal{S}\}.$$

The upper central series

$$\xi_1(\mathcal{C}) \leq \xi_2(\mathcal{C}) \leq \ldots \leq \xi_n(\mathcal{C}) \leq \xi_{n+1}(\mathcal{C}) \leq \ldots$$

of a compatible Leibniz algebra $\mathcal{C}$ is given with the following regulation: $\xi_1(\mathcal{C}) = Cent(\mathcal{C})$ is the centre of $\mathcal{C}$, and, recursively $\xi_{i+1}(\mathcal{C})/\xi_i(\mathcal{C}) = Cent(\mathcal{C}/\xi_i(\mathcal{C}))$ for all positive integers i.

The lower central series of $\mathcal{C}$

$$\mathcal{C} = \gamma_1(\mathcal{C}) \geq \gamma_2(\mathcal{C}) \geq \ldots \geq \gamma_i(\mathcal{C}) \geq \gamma_{i+1}(\mathcal{C}) \geq \ldots$$

of a compatible Leibniz algebra $\mathcal{C}$ is determined with the following regulation: $\mathcal{C} = \gamma_1(\mathcal{C}), \gamma_{i+1}(\mathcal{C}) = [\mathcal{C}, \gamma_i(\mathcal{C})]$ for all positive integers i.

Lemma 1. *Given a compatible Leibniz algebra $\mathcal{C}$, the subsequent circumstances are corresponding:*

(i) $\gamma_s(\mathcal{C}) = \{0\}$ *for an adequately large s;*
(ii) $\xi_s(\mathcal{C}) = \mathcal{C}$ *for an adequately large s;*
(iii) *There exists a decreasing chain of ideals of $\mathcal{C}$*

$$\mathcal{C}_1 \geq \mathcal{C}_2 \geq \ldots$$

where $\mathcal{C}_1 = \mathcal{C}, \mathcal{C}_s = \{0\}$, $[\mathcal{C}, \mathcal{C}_i] \subseteq \mathcal{C}_{i+1}$ for every $1 \leq i < s$;
(iv) *Each product*

$$[\kappa_1, [\kappa_2, \ldots, [\kappa_{s-1}, \kappa_s]_1]_1 \ldots]_1$$

including s elements is equal to zero or

$$[\kappa_1, [\kappa_2, \ldots, [\kappa_{s-1}, \kappa_s]_2]_2 \ldots]_2$$

including s elements is equal to zero.

The proof closely parallels the classical case and is therefore omitted.

Definition 5. *Let $\mathcal{C}$ be a compatible Leibniz algebra. If there is a positive integer s in such a way that $\gamma_s(\mathcal{C}) = \{0\}$, the algebra $\mathcal{C}$ is called nilpotent. Moreover, we say that n is named the nilpotency class of $\mathcal{C}$ if $\gamma_{n+1}(\mathcal{C}) = \{0\}$ but $\gamma_n(\mathcal{C}) \neq \{0\}$. This is denoted by $ncl(\mathcal{C})$. Moreover, if $\mathcal{C}$ holds any of the equivalent statements of Lemma 1, $\mathcal{C}$ is a nilpotent compatible Leibniz algebra.*

This definition leads directly to the following observation:

Lemma 2. *Let $\mathcal{C}$ be a compatible Leibniz algebra. If $\mathcal{C} \neq \{0\}$, then $Cent(\mathcal{C}) \neq \{0\}$. Furthermore, $\mathcal{C}$ is nilpotent if and only if $\mathcal{C}/Cent(\mathcal{C})$ is nilpotent.*

Proof. Suppose that $\mathcal{C} \neq \{0\}$ so there exists a non-zero element ϑ in $\mathcal{C}$. Then, if $[\vartheta, \vartheta]_1 = 0$, we have $\vartheta \in Cent(\bar{\mathcal{C}}) \neq \{0\}$. If $[\vartheta, \vartheta]_1 \neq 0$, by (1), $[\vartheta, \vartheta]_1 \in Leib(\bar{\mathcal{C}})$. Therefore, we have $\{0\} \neq Leib(\bar{\mathcal{C}}) \leq Cent(\bar{\mathcal{C}})$. Similarly, we have $[\vartheta, \vartheta]_2 \in Leib(\tilde{\mathcal{C}})$ and $\{0\} \neq Leib(\tilde{\mathcal{C}}) \leq Cent(\tilde{\mathcal{C}})$. So, by the definition of centre, it is obvious to show that $Cent(\mathcal{C}) = Cent(\bar{\mathcal{C}}) \cap Cent(\tilde{\mathcal{C}} \neq \{0\}$.

Now suppose that $\mathcal{C}$ is nilpotent. Then, there is a positive integer m in such way that $\gamma_m(\mathcal{C}) \neq \{0\}$ and $\gamma_{m+1}(\mathcal{C}) = \{0\}$. By definition of factor algebra, we have $\gamma_{m+1}(\mathcal{C}/Cent(\mathcal{C})) = (\gamma_{m+1}(\mathcal{C}) + Cent(\mathcal{C}))/Cent(\mathcal{C}) = \{0\}$. Therefore, $\mathcal{C}/Cent(\mathcal{C})$ is nilpotent. □

It is obvious that, if $\mathcal{C}$ is a nilpotent compatible Leibniz algebra, then $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are nilpotent Leibniz algebras. Conversely, if $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are nilpotent Leibniz algebras, then $\mathcal{C}$ is not always a nilpotent compatible Leibniz algebra.

Example 2. *Let $\mathcal{C}$ be a three-dimensional compatible Leibniz algebra with a basis $\{\kappa, \vartheta, r\}$ by the following products $[\kappa, \vartheta]_1 = r$ and $[\kappa, r]_2 = \vartheta$. $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are nilpotent Leibniz algebras. Clearly, $\gamma_1(\bar{\mathcal{C}}) = \bar{\mathcal{C}}$, $\gamma_2(\bar{\mathcal{C}}) = span\{r\}$, and $\gamma_3(\bar{\mathcal{C}}) = \{0\}$, and so $ncl(\bar{\mathcal{C}}) = 2$. Similarly, $\gamma_1(\tilde{\mathcal{C}}) = \tilde{\mathcal{C}}$, $\gamma_2(\tilde{\mathcal{C}}) = span\{\vartheta\}$, and $\gamma_3(\tilde{\mathcal{C}}) = \{0\}$, and so $ncl(\tilde{\mathcal{C}}) = 2$. However, since $\gamma_1(\mathcal{C}) = \mathcal{C}$, $\gamma_2(\mathcal{C}) = span\{r, \vartheta\}$, and $\gamma_3(\bar{\mathcal{C}}) = span\{r, \vartheta\}$, $\mathcal{C}$ cannot be nilpotent.*

Definition 6. *A compatible Leibniz algebra homomorphism between two compatible Leibniz algebras $(\mathcal{H}, [,]_1, [,]_2)$ and $(\mathcal{S}, [,]_1, [,]_2)$ is a linear mapping $\theta : \mathcal{H} \longrightarrow \mathcal{S}$, which is a Leibniz algebra homomorphism between $\bar{\mathcal{H}}$ and $\bar{\mathcal{S}}$ and a Leibniz algebra homomorphism between $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{S}}$.*

An invertible homomorphism is an isomorphism. If $\mathcal{H}$ and $\mathcal{S}$ are isomorphic, they are denoted by $\mathcal{H} \cong \mathcal{S}$.

3.2. Classification of Nilpotent Compatible Leibniz Algebras

Throughout the present part, the field $\mathcal{K}$ is assumed to be the field of complex numbers, and we present the classification of non-Lie nilpotent compatible Leibniz algebras for dimensions two, three, and four. To present non-Lie nilpotent compatible pairs, we consider only the case which there are two non-Lie nilpotent Leibniz algebras in the pairs in this section.

3.2.1. Nilpotent Compatible Leibniz Algebras for Two Dimensions

There are three non-abelian non-isomorphic two-dimensional Leibniz algebras. They are given with respect to basis $\{\vartheta_1, \vartheta_2\}$ by

$$\begin{aligned} L_1 &: [\vartheta_1, \vartheta_2] = \vartheta_2, [\vartheta_2, \vartheta_1] = -\vartheta_2 \text{ (non-nilpotent)}; \\ L_2 &: [\vartheta_1, \vartheta_1] = \vartheta_2 \text{ (nilpotent)}; \\ L_3 &: [\vartheta_1, \vartheta_1] = \vartheta_2, [\vartheta_1, \vartheta_2] = \vartheta_2 \text{ (non-nilpotent)}. \end{aligned}$$

Proposition 2. *There is, up to isomorphism, only one pair of two-dimensional compatible Leibniz algebras. It is given by $\mathcal{C}_1 = (L_2, L_3)$, where $\bar{\mathcal{C}} = L_2$ and $\tilde{\mathcal{C}} = L_3$. Moreover, it is non-nilpotent.*

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_1$ is compatible. It is clear to see that $[\vartheta_1, \vartheta_1] = [\vartheta_1, \vartheta_1]_1 + [\vartheta_1, \vartheta_1]_2 = 2\vartheta_2$ and $[\vartheta_1, \vartheta_2] = \vartheta_2$. Then, we obtain that $\gamma_1(\mathcal{C}_1) = \mathcal{C}_1$, $\gamma_2(\mathcal{C}_1) = \text{span}\{\vartheta_2\}$ and $\gamma_3(\mathcal{C}_1) = \text{span}\{\vartheta_2\}$. This means that $\mathcal{C}_1$ is not nilpotent. □

3.2.2. Nilpotent Compatible Leibniz Algebras for Three Dimensions

Theorem 1 ([9], Theorem 6.4)**.** *Let L be a non-Lie nilpotent Leibniz algebra and $dim(L) = 3$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3\}$ with the non-zero products given by one of the following:*

$$\begin{aligned}
&L_1 : [\vartheta_1, \vartheta_1] = \vartheta_2, [\vartheta_1, \vartheta_2] = \vartheta_3.\\
&L_2 : [\vartheta_1, \vartheta_1] = \vartheta_3.\\
&L_3 : [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_3] = \vartheta_3.\\
&L_4 : [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = -\vartheta_3, [\vartheta_2, \vartheta_2] = \vartheta_3.\\
&L_5 : [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \alpha\vartheta_3, \alpha \in \mathbb{C}\backslash\{1, -1\}.
\end{aligned}$$

Theorem 2. *Let $\mathcal{C}$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension three. Then, $\mathcal{C}$ is isomorphic to a compatible Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3\}$ with the non-zero multiplications defined as follows:*

$$\begin{aligned}
\mathcal{C}_1 = (L_1, L_2) :&[\vartheta_1, \vartheta_1]_1 = \vartheta_2, [\vartheta_1, \vartheta_2]_1 = \vartheta_3.\\
&[\vartheta_1, \vartheta_1]_2 = \vartheta_3.\\
\mathcal{C}_2 = (L_1, L_5) :&[\vartheta_1, \vartheta_1]_1 = \vartheta_2, [\vartheta_1, \vartheta_2]_1 = \vartheta_3.\\
&[\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_3, \alpha \in \mathbb{C}\backslash\{1, -1\}.\\
\mathcal{C}_3 = (L_2, L_4) :&\vartheta_1, \vartheta_1]_1 = \vartheta_3.\\
&[\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_3.\\
\mathcal{C}_4 = (L_2, L_5) :&\vartheta_1, \vartheta_1]_1 = \vartheta_3.\\
&[\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_3, \alpha \in \mathbb{C}\backslash\{1, -1\}.\\
\mathcal{C}_5 = (L_4, L_5) :&\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = -\vartheta_3, [\vartheta_2, \vartheta_2]_1 = \vartheta_3.\\
&[\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_3, \alpha \in \mathbb{C}\backslash\{1, -1\}.
\end{aligned}$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 1, 2, \ldots, 5$ is compatible. Clearly, $\mathcal{C}_1$ is a nilpotent compatible Leibniz algebra with the multiplication $[\vartheta_1, \vartheta_1] = [\vartheta_1, \vartheta_1]_1 + [\vartheta_1, \vartheta_1]_2 = \vartheta_3 + \vartheta_2$ and $[\vartheta_1, \vartheta_2] = \vartheta_3$. By doing some calculations, we obtain $\gamma_1(\mathcal{C}_1) = \mathcal{C}_1$, $\gamma_2(\mathcal{C}_1) = \text{span}\{\vartheta_3, \vartheta_3 + \vartheta_2\}$, $\gamma_3(\mathcal{C}_1) = \text{span}\{\vartheta_z\}$ and $\gamma_4(\bar{\mathcal{C}}_1) = \{0\}$. Thus, $\mathcal{C}_1$ is nilpotent and $ncl(\mathcal{C}_1) = 3$.

$\mathcal{C}_2$ is a nilpotent compatible Leibniz algebra with the multiplication $[\vartheta_1, \vartheta_1] = [\vartheta_1, \vartheta_1]_1 + [\vartheta_1, \vartheta_1]_2 = \vartheta_2$, $[\vartheta_1, \vartheta_2] = 2\vartheta_3$ and $[\vartheta_2, \vartheta_1] = \alpha\vartheta_3$. After some calculations, we obtain $\gamma_1(\mathcal{C}_2) = \mathcal{C}_2$, $\gamma_2(\mathcal{C}_2) = \text{span}\{\vartheta_2, \vartheta_3\}$, $\gamma_3(\mathcal{C}_2) = \text{span}\{\vartheta_3\}$ and $\gamma_4(\bar{\mathcal{C}}_2) = \{0\}$. Thus, $\mathcal{C}_2$ is nilpotent and $ncl(\mathcal{C}_2) = 3$. $\mathcal{C}_3$ is a nilpotent compatible Leibniz algebra with the multiplication $[\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = -\vartheta_3$ and $[\vartheta_2, \vartheta_2] = \vartheta_3$. After some calculations, we obtain $\gamma_1(\mathcal{C}_3) = \mathcal{C}_3$, $\gamma_2(\mathcal{C}_3) = \text{span}\{\vartheta_3\}$, and $\gamma_3(\mathcal{C}_3) = \{0\}$. Thus, $\mathcal{C}_3$ is nilpotent and $ncl(\mathcal{C}_3) = 2$. Similarly, $\mathcal{C}_4$ and $\mathcal{C}_5$ are nilpotent of class two.

By Theorem 1, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}_n$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}_n$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 1, 2 \ldots, 5$. □

3.2.3. Nilpotent Compatible Leibniz Algebras for Four Dimensions

Let L be a four-dimensional non-Lie nilpotent Leibniz algebra. L is non-Lie, which implies that $Leib(L) \neq \{0\}$; hence, $\gamma_2(L) \neq \{0\}$. Then, $\dim(\gamma_2(L)) = 1, 2$, or 3. The classification of non-abelian non-Lie nilpotent Leibniz algebras when $\dim(\gamma_2(L)) = 1, 2$, or 3 is known [9,10,14].

Theorem 3 ([9], Theorem 2.2)**.** *Let L be a four-dimensional non-abelian non-Lie nilpotent Leibniz algebra with $dim(\gamma_2(L)) = 1$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero products given by one of the following:*

$$
\begin{aligned}
L_1 &: [\vartheta_1, \vartheta_3] = \vartheta_4, [\vartheta_3, \vartheta_2] = \vartheta_4.\\
L_2 &: [\vartheta_1, \vartheta_3] = \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_3] = \vartheta_4, [\vartheta_3, \vartheta_1] = \vartheta_4, [\vartheta_3, \vartheta_2] = -\vartheta_4.\\
L_3 &: [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = -\vartheta_4, [\vartheta_3, \vartheta_3] = \vartheta_4.\\
L_4 &: [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = -\vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4, [\vartheta_3, \vartheta_3] = \vartheta_4.\\
L_5 &: [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, [\vartheta_3, \vartheta_3] = \vartheta_4, \alpha \in \mathbb{C}\backslash\{1, -1\}.\\
L_6 &: [\vartheta_1, \vartheta_1] = \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4, [\vartheta_3, \vartheta_3] = \vartheta_4.
\end{aligned}
$$

Theorem 4 ([14], Corollary 3)**.** *Let L be a four-dimensional non-abelian non-Lie nilpotent Leibniz algebra with $dim(\gamma_2(L)) = 3$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero products given by the following:*

$$L_7 : [\vartheta_1, \vartheta_1] = \vartheta_2, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_1, \vartheta_3] = \vartheta_4.$$

Theorem 5 ([10], Theorem 2.3)**.** *Let L be a four-dimensional non-abelian non-Lie nilpotent Leibniz algebra with $dim(\gamma_2(L)) = 2$, $dim(\gamma_3(L)) = 0$, and $dim(Leib(L)) = 1$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero products given by one of the following:*

$$
\begin{aligned}
L_8 &: [\vartheta_1, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \vartheta_3 = -[\vartheta_2, \vartheta_1].\\
L_9 &: [\vartheta_1, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \vartheta_3 = -[\vartheta_2, \vartheta_1], [\vartheta_2, \vartheta_2] = \vartheta_4.
\end{aligned}
$$

Theorem 6 ([10], Theorem 2.4)**.** *Let L be a four-dimensional non-abelian non-Lie nilpotent Leibniz algebra with $dim(\gamma_2(L)) = 2$ and $dim(Leib(L)) = 1 = dim(\gamma_3(L))$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero products given by one of the following:*

$$
\begin{aligned}
L_{10} &: [\vartheta_1, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \vartheta_3 = -[\vartheta_2, \vartheta_1], [\vartheta_1, \vartheta_3] = \vartheta_4 = -[\vartheta_3, \vartheta_1].\\
L_{11} &: [\vartheta_1, \vartheta_2] = \vartheta_3 = -[\vartheta_2, \vartheta_1], [\vartheta_2, \vartheta_2] = \vartheta_4, [\vartheta_1, \vartheta_3] = \vartheta_4 = -[\vartheta_3, \vartheta_1].\\
L_{12} &: [\vartheta_1, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3] = \vartheta_4 = -[\vartheta_3, \vartheta_1].\\
L_{13} &: [\vartheta_2, \vartheta_2] = \vartheta_4, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3] = \vartheta_4 = -[\vartheta_3, \vartheta_1].
\end{aligned}
$$

Theorem 7 ([10], Theorem 2.5)**.** *Let L be a four-dimensional non-abelian non-Lie nilpotent Leibniz algebra with $dim(\gamma_2(L)) = 2 = dim(Leib(L))$ and $dim(\gamma_3(L)) = 0$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero products given by the following:*

$$
\begin{aligned}
L_{14} &: [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4.\\
L_{15} &: [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4.\\
L_{16} &: [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_3.\\
L_{17} &: [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\}.\\
L_{18} &: [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\}.\\
L_{19} &: [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4.
\end{aligned}
$$

Theorem 8 ([10], Theorem 2.6)**.** *Let L be a four-dimensional non-abelian non-Lie nilpotent Leibniz algebra with $dim(\gamma_2(L)) = 2 = dim(Leib(L))$ and $dim(\gamma_3(L)) = 1$. Then, L is isomorphic to a Leibniz algebra spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero products given by the following:*

$$
\begin{aligned}
&L_{20}: [\vartheta_1,\vartheta_2] = \vartheta_3, [\vartheta_1,\vartheta_3] = \vartheta_4.\\
&L_{21}: [\vartheta_1,\vartheta_2] = \vartheta_3, [\vartheta_2,\vartheta_2] = \vartheta_4, [\vartheta_1,\vartheta_3] = \vartheta_4.\\
&L_{22}: [\vartheta_1,\vartheta_2] = \vartheta_3, [\vartheta_2,\vartheta_1] = \vartheta_4, [\vartheta_1,\vartheta_3] = \vartheta_4.\\
&L_{23}: [\vartheta_1,\vartheta_2] = \vartheta_3, [\vartheta_2,\vartheta_1] = \vartheta_4, [\vartheta_2,\vartheta_2] = \vartheta_4, [\vartheta_1,\vartheta_3] = \vartheta_4.\\
&L_{24}: [\vartheta_1,\vartheta_1] = \vartheta_3, [\vartheta_2,\vartheta_1] = \vartheta_4, [\vartheta_1,\vartheta_3] = \vartheta_4.\\
&L_{25}: [\vartheta_1,\vartheta_1] = \vartheta_3, [\vartheta_2,\vartheta_2] = \vartheta_4, [\vartheta_1,\vartheta_3] = \vartheta_4.
\end{aligned}
$$

By using Theorems 3–8, we obtain the pairs of four-dimensional compatible Leibniz algebras as follows:

Theorem 9. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four with $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 1$. Then, $\mathcal{C}$ is isomorphic to a compatible Leibniz algebra which is generated with $\{\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_1 = (L_1, L_2):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_3]_2 = \vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_3]_2 = \vartheta_4, [\vartheta_3,\vartheta_1]_2 = \vartheta_4, [\vartheta_3,\vartheta_2]_2 = -\vartheta_4\\
\mathcal{C}_2 = (L_1, L_3):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = -\vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_3 = (L_1, L_4):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = -\vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_4 = (L_1, L_5):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = \alpha\vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4, \alpha \in \mathbb{C}\backslash\{1,-1\}\\
\mathcal{C}_5 = (L_1, L_6):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_1]_2 = \vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_6 = (L_2, L_3):&[\vartheta,\vartheta]_1 = \vartheta_4, [\vartheta_2,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_1]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = -\vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = -\vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_7 = (L_2, L_4):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_2,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_1]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = -\vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = -\vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_8 = (L_2, L_5):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_2,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_1]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = -\vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = \alpha\vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4, \alpha \in \mathbb{C}\backslash\{1,-1\}\\
\mathcal{C}_9 = (L_2, L_6):&[\vartheta_1,\vartheta_3]_1 = \vartheta_4, [\vartheta_2,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_3]_1 = \vartheta_4, [\vartheta_3,\vartheta_1]_1 = \vartheta_4, [\vartheta_3,\vartheta_2]_1 = -\vartheta_4\\
&[\vartheta_1,\vartheta_1]_2 = \vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_{10} = (L_3, L_4):&[\vartheta_1,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_1]_1 = -\vartheta_4, [\vartheta_3,\vartheta_3]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = -\vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_{11} = (L_3, L_5):&[\vartheta_1,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_1]_1 = -\vartheta_4, [\vartheta_3,\vartheta_3]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = \alpha\vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4, \alpha \in \mathbb{C}\backslash\{1,-1\}\\
\mathcal{C}_{12} = (L_3, L_6):&[\vartheta_1,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_1]_1 = -\vartheta_4, [\vartheta_3,\vartheta_3]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_1]_2 = \vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_{13} = (L_4, L_5):&[\vartheta_1,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_1]_1 = -\vartheta_4, [\vartheta_2,\vartheta_2]_1 = \vartheta_4, [\vartheta_3,\vartheta_3]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_2]_2 = \vartheta_4, [\vartheta_2,\vartheta_1]_2 = \alpha\vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4, \alpha \in \mathbb{C}\backslash\{1,-1\}\\
\mathcal{C}_{14} = (L_4, L_6):&[\vartheta_1,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_1]_1 = -\vartheta_4, [\vartheta_2,\vartheta_2]_1 = \vartheta_4, [\vartheta_3,\vartheta_3]_1 = \vartheta_4\\
&[\vartheta_1,\vartheta_1]_2 = \vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4\\
\mathcal{C}_{15} = (L_5, L_6):&[\vartheta_1,\vartheta_2]_1 = \vartheta_4, [\vartheta_2,\vartheta_1]_1 = \alpha\vartheta_4, [\vartheta_3,\vartheta_3]_1 = \vartheta_4, \alpha \in \mathbb{C}\backslash\{1,-1\}\\
&[\vartheta_1,\vartheta_1]_2 = \vartheta_4, [\vartheta_2,\vartheta_2]_2 = \vartheta_4, [\vartheta_3,\vartheta_3]_2 = \vartheta_4.
\end{aligned}
$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 1, 2, \ldots, 15$ is compatible. Clearly, $\mathcal{C}_1$ is a nilpotent compatible Leibniz algebra with the multiplication $[\vartheta_1,\vartheta_3] = 2\vartheta_4, [\vartheta_2,\vartheta_2] = \vartheta_4, [\vartheta_2,\vartheta_3] = \vartheta_4$ and $[\vartheta_3,\vartheta_1] = \vartheta_4$. After some calculations, we obtain $\gamma_1(\mathcal{C}_1) = \mathcal{C}_1$, $\gamma_2(\mathcal{C}_1) = \text{span}\{\vartheta_4\}$ and $\gamma_3(\mathcal{C}_1) = \{0\}$. Thus, $\mathcal{C}_1$ is nilpotent and $ncl(\mathcal{C}_1) = 2$. By using a similar method, we prove that the others are nilpotent compatible Leibniz algebras.

By Theorem 3, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 1, 2 \ldots, 15$. □

Theorem 10. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = dim(\gamma_3(\tilde{\mathcal{C}})) = 0$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 1$. Then, $\mathcal{C}$ is isomorphic to a compatible Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_{16} = (L_8, L_9) : & [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4.
\end{aligned}
$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_{16}$ is compatible. $\mathcal{C}_{16}$ is a nilpotent compatible Leibniz algebra with the multiplication $[\vartheta_1, \vartheta_1] = 2\vartheta_4$, $[\vartheta_1, \vartheta_2] = 2\vartheta_3$, $[\vartheta_2, \vartheta_1] = -2\vartheta_3$ and $[\vartheta_2, \vartheta_2] = \vartheta_4$. After some calculations, we obtain $\gamma_1(\mathcal{C}_{16}]) = \mathcal{C}_{16}$, $\gamma_2(\mathcal{C}_{16}) = \text{span}\{\vartheta_3, \vartheta_4\}$ and $\gamma_3(\mathcal{C}_{16}) = \{0\}$. Thus, $\mathcal{C}_{16}$ is nilpotent and $ncl(\mathcal{C}_{16}) = 2$.

By Theorem 5, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_{16}}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_{16}}$, and so $\mathcal{C} \cong \mathcal{C}_{16}$. □

Theorem 11. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = 0$, $dim(\gamma_3(\tilde{\mathcal{C}})) = 1$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 1$. Then, $\mathcal{C}$ is isomorphic to a compatible Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_{17} = (L_8, L_{10}) : & [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{18} = (L_8, L_{11}) : & [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{19} = (L_8, L_{12}) : & [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{20} = (L_8, L_{13}) : & [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{21} = (L_9, L_{10}) : & [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{22} = (L_9, L_{11}) : & [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{23} = (L_9, L_{12}) : & [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{24} = (L_9, L_{13}) : & [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
& [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2.
\end{aligned}
$$

Proof. By checking the varied Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 17, 18, \ldots, 24$ is compatible. By Theorems 5 and 6, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 17, 18, \ldots, 24$. □

Theorem 12. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four with $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = dim(\gamma_3(\tilde{\mathcal{C}})) = 1$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 1$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$\begin{aligned}
\mathcal{C}_{25} = (L_{10}, L_{11}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_2, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{26} = (L_{10}, L_{12}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{27} = (L_{10}, L_{13}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{28} = (L_{11}, L_{12}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{29} = (L_{11}, L_{13}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2 \\
\mathcal{C}_{30} = (L_{12}, L_{13}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = -\vartheta_3 + \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_2.
\end{aligned}$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 25, 26, \ldots, 30$ is compatible. By Theorem 6, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 25, 26, \ldots, 30$. □

Theorem 13. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = dim(\gamma_3(\tilde{\mathcal{C}})) = 0$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 2$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$\begin{aligned}
\mathcal{C}_{31} = (L_{14}, L_{15}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4 \\
\mathcal{C}_{32} = (L_{14}, L_{16}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_3 \\
\mathcal{C}_{33} = (L_{14}, L_{17}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
\mathcal{C}_{34} = (L_{14}, L_{18}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
\mathcal{C}_{35} = (L_{14}, L_{19}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{36} = (L_{15}, L_{16}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_3 \\
\mathcal{C}_{37} = (L_{15}, L_{17}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
\mathcal{C}_{38} = (L_{15}, L_{18}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
\mathcal{C}_{39} = (L_{15}, L_{19}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{40} = (L_{16}, L_{17}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
\mathcal{C}_{41} = (L_{16}, L_{18}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
\mathcal{C}_{42} = (L_{16}, L_{19}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{43} = (L_{17}, L_{18}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
\mathcal{C}_{44} = (L_{17}, L_{19}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \alpha\vartheta_4, \alpha \in \mathbb{C} - \{-1, 0\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{45} = (L_{18}, L_{19}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4.
\end{aligned}$$

Proof. By checking the varied Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 31, 32, \ldots, 45$ is compatible. By Theorem 7, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 31, 32, \ldots, 45$. □

Theorem 14. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a four-dimensional non-abelian non-Lie nilpotent compatible Leibniz algebra by $dim(\gamma_2(\bar{\mathcal{C}})) = 3, dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\tilde{\mathcal{C}})) = 0$ and $dim(Leib(\tilde{\mathcal{C}})) = 2$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$\begin{aligned} \mathcal{C}_{46} = (L_7, L_{14}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_2, [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\ & [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4. \end{aligned}$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_{46}$ is compatible. $\mathcal{C}_{46}$ is a nilpotent compatible Leibniz algebra with the multiplication $[\vartheta_1, \vartheta_1] = \vartheta_2 + \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3 + \vartheta_4$ and $[\vartheta_1, \vartheta_3] = \vartheta_4$. After some calculations, we obtain $\gamma_1(\mathcal{C}_{46}]) = \mathcal{C}_{46}$, $\gamma_2(\mathcal{C}_{46}) = \mathrm{span}\{\vartheta_2 + \vartheta_3, \vartheta_3 + \vartheta_4, \vartheta_4\}$, $\gamma_3(\mathcal{C}_{46}) = \mathrm{span}\{\vartheta_3 + \vartheta_4, \vartheta_4\}$, $\gamma_4(\mathcal{C}_{46}) = \mathrm{span}\{\vartheta_4\}$ and $\gamma_5(\mathcal{C}_{46}) = \{0\}$. Thus, $\mathcal{C}_{46}$ is nilpotent.

By Theorems 4 and 7, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_{46}}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_{46}}$, and so $\mathcal{C} \cong \mathcal{C}_{46}$. □

Theorem 15. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four with $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = dim(\gamma_3(\tilde{\mathcal{C}})) = 0$ and $dim(Leib(\bar{\mathcal{C}})) = 1$, $dim(Leib(\tilde{\mathcal{C}})) = 2$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$\begin{aligned}
\mathcal{C}_{47} = (L_8, L_{14}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{48} = (L_8, L_{15}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4 \\
\mathcal{C}_{49} = (L_8, L_{16}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_3 \\
\mathcal{C}_{50} = (L_8, L_{17}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
\mathcal{C}_{51} = (L_8, L_{18}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
\mathcal{C}_{52} = (L_8, L_{19}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{53} = (L_9, L_{14}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4 \\
\mathcal{C}_{54} = (L_9, L_{15}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4 \\
\mathcal{C}_{55} = (L_9, L_{16}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_2, \vartheta_2]_2 = \vartheta_1, [\vartheta_2, \vartheta_3]_2 = \vartheta_1, [\vartheta_3, \vartheta_2]_2 = \vartheta_1 \\
\mathcal{C}_{56} = (L_9, L_{17}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_4, [\vartheta_2, \vartheta_1]_2 = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
\mathcal{C}_{57} = (L_9, L_{18}) :& [\vartheta_1, \vartheta_1]_1 = \kappa_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_2]_2 = \alpha\vartheta_3, [\vartheta_2, \vartheta_2]_2 = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
\mathcal{C}_{58} = (L_9, L_{19}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4.
\end{aligned}$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 47, 48, \ldots, 58$ is compatible. By Theorems 5 and 7, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 47, 48, \ldots, 58$. □

Theorem 16. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}}^{)}) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = 0$, $dim(\gamma_3(\tilde{\mathcal{C}})) = 1$ and $dim(Leib(\bar{\mathcal{C}})) = 1$, $dim(Leib(\tilde{\mathcal{C}})) = 2$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_{59} = (L_8, L_{20}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{60} = (L_8, L_{21}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{61} = (L_8, L_{22}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{62} = (L_8, L_{23}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{63} = (L_8, L_{24}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{64} = (L_8, L_{25}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{65} = (L_9, L_{20}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{66} = (L_9, L_{21}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{67} = (L_9, L_{22}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{68} = (L_9, L_{23}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{69} = (L_9, L_{24}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{70} = (L_9, L_{25}):& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4.
\end{aligned}
$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 59, 60, \ldots, 70$ is compatible. By Theorems 5 and 8, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 59, 60, \ldots, 70$. □

Theorem 17. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = dim(\gamma_3(\tilde{\mathcal{C}})) = 0$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 1$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_{71} = (L_{10}, L_{20}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{72} = (L_{10}, L_{21}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{73} = (L_{10}, L_{22}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{74} = (L_{10}, L_{23}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{75} = (L_{10}, L_{24}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{76} = (L_{10}, L_{25}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{77} = (L_{11}, L_{20}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{78} = (L_{11}, L_{21}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{79} = (L_{11}, L_{22}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{80} = (L_{11}, L_{23}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{81} = (L_{11}, L_{24}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{82} = (L_{11}, L_{25}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3 = -[\vartheta_2, \vartheta_1]_1, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 = -[\vartheta_3, \vartheta_1]_1 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4.
\end{aligned}
$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 71, 72, \ldots, 82$ is compatible. By Theorems 6 and 8, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 71, 72, \ldots, 82$. □

Theorem 18. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = dim(\gamma_3(\tilde{\mathcal{C}})) = 0$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 1$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_{83} = (L_{20}, L_{21}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{84} = (L_{20}, L_{22}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{85} = (L_{20}, L_{23}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{86} = (L_{20}, L_{24}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{87} = (L_{20}, L_{25}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{88} = (L_{21}, L_{22}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{89} = (L_{21}, L_{23}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{90} = (L_{21}, L_{24}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{91} = (L_{21}, L_{25}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{92} = (L_{22}, L_{23}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [x_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{93} = (L_{22}, L_{24}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{94} = (L_{22}, L_{25}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{95} = (L_{23}, L_{24}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{96} = (L_{23}, L_{25}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_2, \vartheta_2]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{97} = (L_{24}, L_{25}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4.
\end{aligned}
$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 83, 84, \ldots, 97$ is compatible. By Theorem 8, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 83, 84, \ldots, 97$. □

Theorem 19. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four by $dim(\gamma_2(\bar{\mathcal{C}})) = 3, dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\tilde{\mathcal{C}})) = 1$ and $dim(Leib(\tilde{\mathcal{C}})) = 2$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is generated with $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$
\begin{aligned}
\mathcal{C}_{98} = (L_7, L_{20}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_2, [\vartheta_1, \vartheta_2]_1 = \vartheta_3, [\vartheta_1, \vartheta_3]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4.
\end{aligned}
$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_{92}$ is compatible. By Theorems 4 and 8, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_{98}}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_{98}}$, and so $\mathcal{C} \cong \mathcal{C}_{98}$. $\square$

Theorem 20. *Let $(\mathcal{C}, [,]_1, [,]_2)$ be a non-abelian non-Lie nilpotent compatible Leibniz algebra in dimension four with $dim(\gamma_2(\bar{\mathcal{C}})) = dim(\gamma_2(\tilde{\mathcal{C}})) = 2$, $dim(\gamma_3(\bar{\mathcal{C}})) = 0, dim(\gamma_3(\tilde{\mathcal{C}})) = 1$ and $dim(Leib(\bar{\mathcal{C}})) = dim(Leib(\tilde{\mathcal{C}})) = 2$. Then, $\mathcal{C}$ is isomorphic to a Leibniz algebra which is spanned by $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with the non-zero multiplications given as follows:*

$$\begin{aligned}
\mathcal{C}_{99} = (L_{14}, L_{20}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{100} = (L_{14}, L_{21}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{101} = (L_{14}, L_{22}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{102} = (L_{14}, L_{23}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{103} = (L_{14}, L_{24}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{104} = (L_{14}, L_{25}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{105} = (L_{15}, L_{20}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{106} = (L_{15}, L_{21}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{107} = (L_{15}, L_{22}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{108} = (L_{15}, L_{23}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_4, [\vartheta_1, \vartheta_2]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{109} = (L_{15}, L_{24}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{110} = (L_{15}, L_{25}) :& [\vartheta_1, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_1]_1 = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{111} = (L_{16}, L_{20}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{112} = (L_{16}, L_{21}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_{113} = (L_{16}, L_{22}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{114} = (L_{16}, L_{23}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{115} = (L_{16}, L_{24}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{116} = (L_{16}, L_{25}) :& [\vartheta_1, \vartheta_2]_1 = \vartheta_4, [\vartheta_2, \vartheta_1]_1 = \vartheta_3, [\vartheta_2, \vartheta_2]_1 = -\vartheta_3 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{117} = (L_{17}, L_{20}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{118} = (L_{17}, L_{21}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{119} = (L_{17}, L_{22}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{120} = (L_{17}, L_{23}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{121} = (L_{17}, L_{24}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{122} = (L_{17}, L_{25}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_4, [\vartheta_2, \vartheta_1] = \alpha\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1, 0\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{123} = (L_{18}, L_{20}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{124} = (L_{18}, L_{21}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{125} = (L_{18}, L_{22}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{126} = (L_{18}, L_{23}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{127} = (L_{18}, L_{24}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{128} = (L_{18}, L_{25}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_4, [\vartheta_1, \vartheta_2] = \alpha\vartheta_3, [\vartheta_2, \vartheta_2] = -\vartheta_4, \alpha \in \mathbb{C}\backslash\{-1\} \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{129} = (L_{19}, L_{20}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{130} = (L_{19}, L_{21}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{131} = (L_{19}, L_{22}) :& [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_{132} = (L_{19}, L_{23}) : & [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4 \\
& [\vartheta_1, \vartheta_2]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{133} = (L_{19}, L_{24}) : & [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_1]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4 \\
\mathcal{C}_{134} = (L_{19}, L_{25}) : & [\vartheta_1, \vartheta_1] = \vartheta_3, [\vartheta_1, \vartheta_2] = \vartheta_3, [\vartheta_2, \vartheta_1] = \vartheta_3 + \vartheta_4, [\vartheta_2, \vartheta_2] = \vartheta_4 \\
& [\vartheta_1, \vartheta_1]_2 = \vartheta_3, [\vartheta_2, \vartheta_2]_2 = \vartheta_4, [\vartheta_1, \vartheta_3]_2 = \vartheta_4.
\end{aligned}$$

Proof. By checking the mixed Leibniz identity on distinct basis elements, we show that $\mathcal{C}_n$ for $n = 99, 100, \ldots, 134$ is compatible. By Theorems 7 and 8, there are Leibniz algebra isomorphisms between $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}_n}$ and between $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}_n}$, and so $\mathcal{C} \cong \mathcal{C}_n$ for $n = 99, 100, \ldots, 134$. □

4. Discussion

This paper is focused on one of the central questions in studying the comprehensive classification of non-Lie nilpotent compatible Leibniz algebras in low dimensions by using two nilpotent non-Lie Leibniz algebras over the complex field. We show that there is no isomorphism class in dimension two, there are 5 isomorphism classes in dimension three, and there are 134 isomorphism classes in dimension four.

Building upon these findings, our future work will aim to generalize these results by showing for at least five-dimensional Leibniz algebras over a field with characteristic zero. To this end, we will investigate the structure of such algebras using the similar approach used in this paper.

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