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Generalized Hölder estimates via generalized Morrey norms for some ultraparabolic operators

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Abstract

We consider a class of hypoelliptic operators of the following type

$$\mathcal{L} = \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t,$$

where (a_{ij}), (b_{ij}) are constant matrices and (a_{ij}) is symmetric positive definite on $\mathbb{R}^{p_0}$ ($p_0 \le N$). We obtain generalized Hölder estimates for $\mathcal{L}$ on $\mathbb{R}^{N+1}$ by establishing several estimates of singular integrals in generalized Morrey spaces.

Keywords Ultraparabolic operators · Homogeneous type space · Singular integral operators · Generalized Morrey space · Generalized Hölder estimate

Mathematics Subject Classification Primary 35R03 · 35B45 · 42B20

1 Introduction and main results

Let us concern a class of ultraparabolic operators of Kolmogorov–Fokker–Planck type in $\mathbb{R}^{N+1}$:

$$\mathcal{L}_0 = div(A\nabla) + \langle x, B\nabla\rangle - \partial_t = \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t, \qquad (1.1)$$

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where $1 \leq p_0 \leq \mathbb{N}$, $A = (a_{ij})$ and $B = (b_{ij})$ are $N \times N$ matrices with constant real entries, $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N})$, div and $\langle \cdot, \cdot \rangle$ denote the gradient, the divergence and the inner product in $\mathbb{R}^N$, separately. The matrix A is supposed to be symmetric and positive semidefinite. We also assume that the following condition holds:

(H_0) Ker(A) does not contain nontrivial subspaces which are invariant for B.

Hormander in [1] pointed out that (H_0) implies (actually, is equivalent to) the hypoellipticity of (1.1). By introducing the matrix

$$C(t) = \int_0^t E(s) A E^T(s) ds, \tag{1.2}$$

where $E(s) = \exp(-sB^T)$, the authors in [2] showed that (H_0) is equivalent to the condition

$$C(t) > 0 \quad \text{for every} \quad t > 0. \tag{1.3}$$

It is interesting to remark that the condition (1.3) can also be expressed in geometric-differential terms. In fact, setting

$$X_i = \sum_{j=1}^{N} a_{ij} \partial_{x_j}, \ i = 1, \dots, N, \ Y = \langle x, B\nabla \rangle,$$

then (1.3) is equivalent to the following Hormander's condition

$$\text{rank}\, L(X_1, X_2, \dots, X_N, Y)(x) = N, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $L(X_1, X_2, \dots, X_N, Y)$ denotes the Lie algebra generated by $X_1, X_2, \dots, X_N, Y$. The proof of the equivalence between (H_0) and (1.4) is implicitly contained in the introduction of [1], and Kuptsov in [3] gave an explicit proof of the equivalence between (1.3) and (1.4).

The authors in [2] also proved that (1.4) implies that, for some basis on $\mathbb{R}^N$, the matrices A and B take the form:

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.5}$$

and

$$B = \begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{pmatrix} \tag{1.6}$$

respectively, where $A_0 = (a_{ij})_{i,j=1}^{p_0}$ is a $p_0 \times p_0$ constant matrix ($p_0 \leq N$) with rank p_0; B_j is a $p_{j-1} \times p_j$ block with rank p_j, $j = 1, 2, \ldots, r$. Moreover $p_0 \geq p_1 \geq \ldots > p_r \geq 1$ and $p_0 + p_1 + \ldots + p_r = N$.

Specially, if we denote by B_0 the matrix obtained by annihilating all the $*$ blocks of the matrix written as (1.6), then the operator $\mathcal{L}_0$ becomes

$$\mathcal{L} = div(A\nabla) + \langle x, B_0\nabla\rangle - \partial_t = \sum_{i,j=1}^{p_0} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t,$$

which is the principal part of $\mathcal{L}_0$. In this paper, we will consider the operator $\mathcal{L}$ and make the following assumption:

It is known that $\mathcal{L}$ is hypoelliptic (see [2]). On the other hand, $\mathcal{L}$ is a heat operator when $p_0 = N$, $B = 0$ and the degenerate operators (i.e., with $p_0 < N$) appear in many research fields. For instance, the Kolmogorov equation

$$\partial^2_{x_1x_1}u + x_1\partial_{x_2}u = \partial_t u, \quad (x, t) \in \mathbb{R}^3$$

occurs in the financial problem, in the kinetic theory as well as in the visual perception problem (see [4–6]).

The Kolmogorov equation was first introduced by Kolmogorov in 1934 to study the time evolution of the density of a Brownian test particle in the phase space. It is a linear strongly degenerate second order PDE whose diffusion part is governed by the Laplace operator in a subset of the variables (velocity variables) coupled with a transport term that contains the directions of missing ellipticity (position variables). Such a drift term makes the equation non-symmetric, but at the same time it is responsible for the hypoelliptic properties of the operator.

We know that $\mathcal{L}$ is a class of Kolmogorov–Fokker–Planck ultraparabolic operator. Owing to its importance in physics and in mathematical finance, it has been extensively studied (see [2, 5, 6]). The authors in [2, 5, 6] proved an invariant Harnack inequality for the non-negative solutions of the equation $\mathcal{L}u = 0$. Based on the theory of singular integral, Polidoro and Ragusa [7] demonstrated Morrey-type imbedding results and gave a local Hölder continuity of the solution. In [8, 9] in particular was study pointwise regularity of solutions to problem (1.1) for Kolmogorov equations with right hand side in L^p. In this paper, we obtain generalized Hölder estimates for $\mathcal{L}$ on $\mathbb{R}^{N+1}$.

We note that a simple consequence of results contained in the following two recent articles [8, 9] is that if $\mathcal{L}u$ satisfied generalized Hölder continuous, then the second-order derivatives $\partial^2_{x_ix_j}u$, $i, j = 1, ..., m$, and the Lie derivative Yu are generalized Hölder continuous in $\mathbb{R}^{N+1}$. But consequence of results contained in this work is that if $\mathcal{L}u$ satisfied generalized Morrey continuous, then the function u and the first-order derivatives $\partial_{x_i}u$, $i = 1, ..., m$ are generalized Hölder continuous in $\mathbb{R}^{N+1}$.

Morrey spaces and their properties play an important role in the study of local behavior of solutions to elliptic partial differential equations, refer to [10, 11]. Moreover, various Morrey spaces are defined in the process of study. In [12–14] the authors introduced and studied the boundedness of the classical operators in generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [15–17]) and etc.

The aim of the paper is to prove global generalized Hölder estimates on the homogeneous group $\mathbb{G}$ for $\mathcal{L}$ by applying the properties of the fundamental solution for $\mathcal{L}$ and several estimates of singular integrals on the homogeneous space. The method here is inspired by that used in [7]. Our results reflect the relations between the generalized Morrey norms of $\mathcal{L}u$ and generalized Hölder exponents for u and $X_i u, i = 1, 2, \dots, N$. In order to state our main results, we first introduce the definition of generalized Morrey space.

Definition 1.1 (Generalized Morrey space). Let $1 \le p < \infty$ and $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^{N+1} \times (0, \infty)$. The generalized Morrey space $M^{p,\varphi}(\mathbb{R}^{N+1})$ is defined of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^{N+1})$ by the finite norm

$$\|f\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} = \sup_{z\in\mathbb{R}^{N+1}, r>0} \frac{r^{-\frac{Q+2}{p}}}{\varphi(z, r)} \|f\|_{L^p(B(z,r))}.$$

Also the weak generalized Morrey space $WM^{p,\varphi}(\mathbb{R}^{N+1})$ is defined of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^{N+1})$ by the finite norm

$$\|f\|_{WM^{p,\varphi}(\mathbb{R}^{N+1})} = \sup_{z\in\mathbb{R}^{N+1}, r>0} \frac{r^{-\frac{Q+2}{p}}}{\varphi(z, r)} \|f\|_{WL^p(B(z,r))}.$$

Remark 1.2 (1) If $\varphi(x, r) = r^{\frac{\lambda-(Q+2)}{p}}$ with $0 < \lambda < Q + 2$ (see reference to formula (2.2)), then $M^{p,\varphi}(\mathbb{R}^{N+1}) = L^{p,\lambda}(\mathbb{R}^{N+1})$ is the classical Morrey space and $WM^{p,\varphi}(\mathbb{R}^{N+1}) = WL^{p,\lambda}(\mathbb{R}^{N+1})$ is the weak Morrey space.

(2) If $\varphi(z, r) \equiv r^{-\frac{Q+2}{p}}$, then $M^{p,\varphi}(\mathbb{R}^{N+1}) = L^p(\mathbb{R}^{N+1})$ is the Lebesgue space and $WM^{p,\varphi}(\mathbb{R}^{N+1}) = WL^p(\mathbb{R}^{N+1})$ is the weak Lebesgue space.

Lemma 1.3 [18] *Let $\varphi(z, r)$ be a positive measurable function on $\mathbb{R}^{N+1} \times (0, \infty)$.*

(i) If

$$\sup_{t<r<\infty} \frac{r^{-\frac{Q+2}{p}}}{\varphi(z, r)} = \infty \quad \text{for some } t > 0 \text{ and for all } z \in \mathbb{R}^{N+1},$$

then $M^{p,\varphi}(\mathbb{R}^{N+1}) = \Theta$, where $\Theta = \Theta(\mathbb{R}^{N+1})$ is the set of all functions equivalent to 0 on $\mathbb{R}^{N+1}$..

(ii) If

$$\sup_{0<r<\tau} \varphi(z, r)^{-1} = \infty \quad \text{for some } \tau > 0 \text{ and for all } z \in \mathbb{R}^{N+1},$$

then $M^{p,\varphi}(\mathbb{R}^{N+1}) = \Theta$.

Remark 1.4 [18] We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{R}^{N+1} \times (0,\infty)$ such that for all $t > 0$,

$$\sup_{z\in\mathbb{R}^{N+1}} \left\| \frac{r^{-\frac{Q+2}{p}}}{\varphi(z,r)} \right\|_{L^\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{z\in\mathbb{R}^{N+1}} \left\| \varphi(z,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 1.3, we always assume that $\varphi \in \Omega_p$.

Define

$$[u]_{C^\omega(\mathbb{R}^{N+1})} = \sup_{x,z\in\mathbb{R}^{N+1}, x\neq z} \frac{|u(x)-u(z)|}{\omega\big(\|x^{-1}\circ z\|\big)},$$

and set $C^{0,\omega}(\mathbb{R}^{N+1})$ for the space of all functions $u : \mathbb{R}^{N+1} \to \mathbb{R}$ of finite norm

$$\|u\|_{C^\omega(\mathbb{R}^{N+1})} = \|u\|_{L_\infty(\mathbb{R}^{N+1})} + [u]_{C^\omega(\mathbb{R}^{N+1})}.$$

In the case $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$ we get the Hölder spaces $C^\alpha(\mathbb{R}^{N+1})$.

The main results in this paper are as follows.

Theorem 1.5 *Let $1 < p < \infty$ and $\varphi = \varphi(z,t) \in \Omega_p$ satisfy the condition*

$$\int_0^1 \varphi(z,t)\, t\, dt + \int_1^\infty \varphi(z,t)\, dt < \infty$$

iniformly in $z \in \mathbb{R}^{N+1}$, then there exists a positive constant C, depending only on p, λ and the operator $\mathcal{L}$, such that for every $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$|u(z)-u(w)| \leq C\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \times \left(\int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\, t\, dt + \|z^{-1}\circ w\| \int_{\|z^{-1}\circ w\|}^\infty \varphi(z,t)\, dt \right)$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$, where $\circ$ is the group law given in Section 2.

Let $1 < p < \infty$ and $\varphi = \varphi(z,t) \in \Omega_p$ satisfy the condition

$$\int_0^1 \varphi(z,t)\, dt + \int_1^\infty \varphi(z,t)\, \frac{dt}{t} < \infty$$

iniformly in $z \in \mathbb{R}^{N+1}$, then there exists a positive constant C, depending only on p, λ and the operator $\mathcal{L}$, such that for every $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$|\partial_{x_j} u(z) - \partial_{x_j} u(w)| \leq C\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \times \left(\int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\, dt + \|z^{-1}\circ w\| \int_{\|z^{-1}\circ w\|}^\infty \varphi(z,t)\, \frac{dt}{t} \right)$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$ *and* $j = 1, 2, \ldots, p_0$.

Corollary 1.6 *Let* $1 < p < \infty$ *and* $\varphi = \varphi(z,t) \in \Omega_p$ *satisfy the condition*

$$\int_0^\delta \varphi(z,t)\, t\, dt + \delta \int_\delta^\infty \varphi(z,t)\, dt \lesssim \varphi(z,\delta)\, \delta^2$$

for all z *and* $\delta > 0$, *then there exists a positive constant* C, *depending only on* p, λ *and the operator* $\mathcal{L}$, *such that for every* $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$|u(z) - u(w)| \le C \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})}\, \varphi(z, \|z^{-1} \circ w\|)\, \|z^{-1} \circ w\|^2$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$, *where* $\circ$ *is the group law given in Section* 2. *Moreover,*

$$\|u\|_{C^{\varphi(\cdot,r)\,r^2}(\mathbb{R}^{N+1})} \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})}.$$

Let $1 < p < \infty$ *and* $\varphi = \varphi(z,t) \in \Omega_p$ *satisfy the condition*

$$\int_0^\delta \varphi(z,t)\, dt + \delta \int_\delta^\infty \varphi(z,t)\, \frac{dt}{t} \lesssim \varphi(z,\delta)\, \delta$$

for all z *and* $\delta > 0$, *then there exists a positive constant* C, *depending only on* p, λ *and the operator* $\mathcal{L}$, *such that for every* $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$|\partial_{x_j} u(z) - \partial_{x_j} u(w)| \le C \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})}\, \varphi(z, \|z^{-1} \circ w\|)\, \|z^{-1} \circ w\|$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$ *and* $j = 1, 2, \ldots, p_0$. *Moreover,*

$$\| |\partial_{x_j} u\|_{C^{\varphi(\cdot,r)\,r}} \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})}.$$

Note that for $\varphi(z,r) = |B(z,r)|^{\frac{\lambda-1}{p}}$, from Theorem 1.5 we get the following result, which proven in [4].

Corollary 1.7 [4, Theorem 1.2] *If* $2p + \lambda > Q + 2$, $p + \lambda < Q + 2$ *and* $\theta = \frac{2p+\lambda-(Q+2)}{p}$, *then there exists a positive constant* C, *depending only on* p, λ *and the operator* $\mathcal{L}$, *such that for every* $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$|u(z) - u(w)| \le C \|\mathcal{L}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\, \|z^{-1} \circ w\|^\theta,$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$, *where* $\circ$ *is the group law given in Section* 2;

If $p + \lambda > Q + 2$ *and* $\delta = \frac{p+\lambda-(Q+2)}{p}$, *then there exists a positive constant* C, *depending only on* p, λ *and the operator* $\mathcal{L}$, *such that for every* $u \in C_0^\infty(\mathbb{R}^{N+1})$,

$$\left|\partial_{x_j} u(z) - \partial_{x_j} u(w)\right| \le C \|\mathcal{L}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}\, \|z^{-1} \circ w\|^\delta$$

for every $z, w \in \mathbb{R}^{N+1}$, $z \neq w$ *and* $j = 1, 2, \ldots, p_0$.

Remark 1.8 Note that in the case of $N = 2$ Theorem 1.5 was proven in [19].

The paper is organized as follows: In Sect. 2, we introduce some preliminary and known results which will be used later. The proof of Theorem 1.5 is given in Sect. 3.

2 Preliminary

It is proved in [2] that the operator $\mathcal{L}$ is left-invariant with respect to the Lie group $\mathcal{K}$ whose underlying manifold is $\mathbb{R}^{N+1}$, endowed with the composition law

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau),$$

where $E(\tau) = \exp(-\tau B^T)$ and B^T denotes the transpose of B. Note that

$$(\xi,\tau)^{-1} = (-E(-\tau)\xi, -\tau).$$

There exists a group of dilations on $\mathbb{R}^{N+1}$, which we denote by $(D(\lambda))_{\lambda>0}$. More precisely, $D(\lambda)$ is defined by

$$D(\lambda) = \operatorname{diag}(\lambda^{\alpha_1}, \lambda^{\alpha_2}, \ldots, \lambda^{\alpha_N}, \lambda^2), \tag{2.1}$$

where

$$\alpha_1 = \ldots = \alpha_{p_0} = 1,\ \alpha_{p_0+1} = \ldots = \alpha_{p_0+p_1} = 3, \ldots,$$
$$\alpha_{p_0+\ldots+p_{r-1}+1} = \ldots = \alpha_N = 2r+1.$$

Therefore, we can write

$$D(\lambda) = \operatorname{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \ldots, \lambda^{2r+1} I_{p_r}, \lambda^2),$$

where I_{p_j}, $D(\lambda)$ denote the $p_j \times p_j$ identity matrix and the matrix of dilations on $\mathbb{R}^{N+1}$, respectively. Note that

$$\det(D(\lambda)) = \lambda^{Q+2},$$

where

$$Q + 2 = p_0 + 3p_1 + \ldots + (2r+1)p_r + 2 \tag{2.2}$$

is called the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$.

Definition 2.1 We say that a differential operator Y on $\mathbb{R}^{N+1}$ is homogeneous of degree $\beta > 0$, if

$$Y(\varphi(D(\lambda)z)) = \lambda^\beta (Y\varphi)(D(\lambda)z),\ \ z \in \mathbb{R}^{N+1},\ \lambda > 0$$

for every test function φ. Also, we say that a function f is homogeneous of degree α if

$$f((D(\lambda)z)) = \lambda^\alpha f(z),\ \lambda > 0,\ z \in \mathbb{R}^{N+1}.$$

Clearly, if Y is a homogeneous differential operator of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$. By Definition 2.1, it is easy to show that the operator $\mathcal{L}$ is homogeneous of degree two with respect to the dilations $D(\lambda)$, i.e.,

$$\mathcal{L}(u(D(\lambda)z)) = \lambda^2 (\mathcal{L}u)(D(\lambda)z),\ z \in \mathbb{R}^{N+1},\ \lambda > 0$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$.

Let us consider the norm and a quasidistance in $\mathbb{R}^{N+1}$, related to the groups of translations and dilations defined above.

Definition 2.2 Let $z = (x_1, x_2, \dots, x_N, t) \in \mathbb{R}^{N+1}$, if $z = 0$ we set $\|z\| = 0$, while if $z \in \mathbb{R}^{N+1} \setminus \{0\}$ we define $\|z\| = \varrho$, where ϱ is the unique positive solution to the equation

$$\frac{x_1^2}{\varrho^{2\alpha_1}} + \frac{x_2^2}{\varrho^{2\alpha_2}} + \dots + \frac{x_N^2}{\varrho^{2\alpha_N}} + \frac{t^2}{\varrho^4} = 1,$$

where $\alpha_1, \alpha_2, \dots, \alpha_N$ are the positive integers in (2.1).

Bramanti and Cerutti in [20] showed that the norm $\| \cdot \|$ satisfies

$$\|z^{-1}\| \le c_1 \|z\|, \quad z \in \mathbb{R}^{N+1} \tag{2.3}$$

and

$$\|z \circ \zeta\| \le c_2(\|z\| + \|\zeta\|), \quad z, \zeta \in \mathbb{R}^{N+1}, \tag{2.4}$$

where the positive constants c_1 and c_2 depend only on the matrix B. Clearly, we have

$$\|D(\lambda)z\| = \lambda \|z\|,\ \lambda > 0,\ z \in \mathbb{R}^{N+1}.$$

Definition 2.3 For every $z, w \in \mathbb{R}^{N+1}$, define a quasidistance by

$$d(z, w) = \|w^{-1} \circ z\|.$$

The ball with respect to d is denoted by

$$B(z, r) = B_r(z) = \{w \in \mathbb{R}^{N+1} : d(z, w) < r\}. \tag{2.5}$$

Since $B(0,r) = D(r)B(0,1)$ and $\det(D(\lambda)) = \lambda^{Q+2}$, we also have

$$|B_r(0)| = r^{Q+2}|B_1(0)|,$$

where $|B_1(0)| = w_{N+1}$ is the Lebesgue measure of the Euclidean unit ball of $\mathbb{R}^{N+1}$. This implies that the Lebesgue measure dz is a doubling measure with respect to d, since

$$|B(z,2r)| = 2^{Q+2}|B(z,r)|, \quad z \in \mathbb{R}^{N+1}, \quad r > 0.$$

Therefore, the space $(\mathbb{R}^{N+1}, dz, d)$ is a space of homogenous type. Recall that if f and g are functions on $\mathbb{R}^{N+1}$, their convolution $f * g$ is defined by

$$f * g(z) = \int_{\mathbb{R}^{N+1}} f(z \circ \zeta^{-1}) g(\zeta) d\zeta = \int_{\mathbb{R}^{N+1}} g(\zeta^{-1} \circ z) f(\zeta) d\zeta.$$

Lemma 2.4 ([2]). *The operator $\mathcal{L}$ possesses a fundamental solution*

$$\Gamma(z) = \begin{cases} 0, & t \le 0, \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\Big(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle \Big), & t > 0, \end{cases} \tag{2.6}$$

where $z = (x,t)$ and $C(t)$ is as in (1.2). Moreover, $\Gamma \in C^\infty(\mathbb{R}^{N+1} \setminus \{0\})$.

The authors in [21, 22] proved a representation formula:

$$u(z) = -(\mathcal{L}u * \Gamma)(z) = -\int_{\mathbb{R}^{N+1}} \Gamma(\zeta^{-1} \circ z)\mathcal{L}u(\zeta) d\zeta. \tag{2.7}$$

The following formula was given by Bramanti in [23]:

$$\partial^2_{x_i x_j} u(z) = -P.V.(\mathcal{L}u * \partial^2_{x_i x_j}\Gamma)(z) + c_{ij}\mathcal{L}u(z) \tag{2.8}$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$ and some constants c_{ij}, $i, j = 1, 2, \ldots, p_0$. The principal value in (2.8) is understood as

$$P.V.(\mathcal{L}u * \partial^2_{x_i x_j}\Gamma)(z) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{N+1} \setminus B(z,\varepsilon)} (\partial^2_{x_i x_j}\Gamma)(\zeta^{-1} \circ z)\mathcal{L}u(\zeta) d\zeta.$$

Set

$$\Gamma_i(z) = \partial_{x_i}\Gamma(z), \ \Gamma_{ij}(z) = \partial_{x_i}\partial_{x_j}\Gamma(z), \ i, j = 1, 2, \ldots, p_0.$$

We also observe that $\Gamma(z)$ is homogeneous of degree $-Q$ with respect to the group $(D(\lambda))_{\lambda>0}$ and $\Gamma_i(z)$ $(i, j = 1, 2, \ldots, p_0)$ are homogeneous of degree $-Q-1$. Recall that $\Gamma_{ij}(\cdot)$ has the following properties.

Lemma 2.5 [20] *For* $i, j = 1, 2, \ldots, p_0$, *one has*
(*a*) $\Gamma_{ij}(\cdot) \in C^\infty(\mathbb{R}^{N+1}\backslash\{0\})$;
(*b*) $\Gamma_{ij}(\cdot)$ *is homogeneous of degree* $-Q-2$;
(*c*) *for every* $R > r > 0$,

$$\int_{B(0,R)\backslash B(0,r)} \Gamma_{ij}(z)dz = \int_{\|z\|=1} \Gamma_{ij}(z)d\sigma(z) = 0.$$

Let us define a singular integral operator

$$T_{ij}g(z) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{N+1}\backslash B(\zeta^{-1}\circ z,\varepsilon)} (\partial_{x_i x_j}\Gamma)(\zeta^{-1}\circ z)g(\zeta)d\zeta, \quad i, j = 1, \ldots, p_0 \tag{2.9}$$

for every measurable function g.

3 Generalized Hölder continuity

In this section, by demonstrating generalized Hölder estimates of two integral operators, we prove Theorem 1.5.

Lemma 3.1 [24]. *Let* $b \in \mathbb{R}^1$ *and* $K \in C^1(\mathbb{R}^{N+1}\backslash\{0\})$ *be a homogeneous function with degree b with respect to the group* $(D(\lambda))_{\lambda>0}$ *and there exist two constants* $c > 0$ *and* $M > 1$ *such that if* $\|z\| > M\|z^{-1}\circ\zeta\|$. *Then*

$$|K(\zeta) - K(z)| \le c\|z^{-1}\circ\zeta\| \cdot \|z\|^{b-1}.$$

Lemma 3.2 [24] *For every* $z, w, \zeta \in \mathbb{R}^{N+1}$, *it holds*
(1) *there exists a constant* $c > 0$, *such that*

$$\Gamma(z^{-1}\circ w) \le \frac{c}{\|z^{-1}\circ w\|^Q}, \quad \Gamma_i(z^{-1}\circ w) \le \frac{c}{\|z^{-1}\circ w\|^{Q+1}}.$$

(2) *there exist two constant* $c > 0$ *and* $M > 1$, *such that if* $\|z^{-1}\circ w\| \ge M\|w^{-1}\circ\zeta\|$,

$$\left|\Gamma(z^{-1}\circ w) - \Gamma(z^{-1}\circ\zeta)\right| \le \frac{c\,\|w^{-1}\circ\zeta\|}{\|z^{-1}\circ w\|^{Q+1}},$$
$$\left|\Gamma_i(z^{-1}\circ w) - \Gamma_i(z^{-1}\circ\zeta)\right| \le \frac{c\,\|w^{-1}\circ\zeta\|}{\|z^{-1}\circ w\|^{Q+2}}.$$

Lemma 3.3 *Let* $1 < p < \infty$ *and* $\varphi = \varphi(z,t) \in \Omega_p$. *Fixed* $w \in \mathbb{R}^{N+1}$, $\alpha \in [0, Q+2)$, $\beta \in (0, Q+2)$ *and* $\sigma > 0$, *for every* $g \in M^{p,\varphi}(\mathbb{R}^{N+1})$, *we set*

$$T'_\alpha g(z) = \int_{\|\zeta^{-1}\circ z\|\ge\sigma\,\|w^{-1}\circ z\|} \frac{g(\zeta)}{\|\zeta^{-1}\circ z\|^{Q+2-\alpha}}d\zeta$$

and

$$T''_\beta g(z) = \int_{\|\zeta^{-1}\circ z\|<\sigma\|w^{-1}\circ z\|} \frac{g(\zeta)}{\|\zeta^{-1}\circ z\|^{Q+2-\beta}} d\zeta.$$

Then, if $\int_1^\infty \varphi(z,t)\, t^{\alpha-1}\, dt < \infty$, *then there exists* $c = c(p, \varphi, \alpha, \sigma) > 0$ *such that*

$$|T'_\alpha g(z)| \le c\, \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_{\|w^{-1}\circ z\|}^{\infty} \varphi(z,t)\, t^{\alpha-1}\, dt. \tag{3.1}$$

Moreover, if $\int_0^1 \varphi(z,t)\, t^{\beta-1}\, dt < \infty$, *then there exists* $c = c(p, \varphi, \beta, \sigma) > 0$ *such that*

$$|T''_\beta g(z)| \le c\, \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_0^{\|w^{-1}\circ z\|} \varphi(z,t)\, t^{\beta-1}\, dt. \tag{3.2}$$

Proof Observing that

$$\begin{aligned}
|T'_\alpha g(z)| &\le \sum_{k=1}^{\infty} \int_{2^{k-1}\sigma\|w^{-1}\circ z\|\le\|\zeta^{-1}\circ z\|<2^k\sigma\|w^{-1}\circ z\|} \frac{|g(\zeta)|}{\|\zeta^{-1}\circ z\|^{Q+2-\alpha}}\, d\zeta \\
&\le \sum_{k=1}^{\infty} \left(\frac{2}{2^k\sigma\|w^{-1}\circ z\|}\right)^{Q+2-\alpha} \int_{B_{2^k c_1\sigma\|w^{-1}\circ z\|}(z)} |g(\zeta)|\, d\zeta \\
&\le \sum_{k=1}^{\infty} \left(\frac{2}{2^k\sigma\|w^{-1}\circ z\|}\right)^{Q+2-\alpha} \|g\|_{L^p\left(B_{2^k c_1\sigma\|w^{-1}\circ z\|}(z)\right)} \left|B_{2^k c_1\sigma\|w^{-1}\circ z\|}(z)\right|^{\frac{1}{p'}} \\
&\le \sum_{k=1}^{\infty} \left(\frac{2}{2^k\sigma\|w^{-1}\circ z\|}\right)^{\frac{Q+2}{p}-\alpha} \|g\|_{L^p\left(B_{2^k c_1\sigma\|w^{-1}\circ z\|}(z)\right)} \lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \\
&\quad \times \sum_{k=1}^{\infty} \left(\frac{2}{2^k\sigma\|w^{-1}\circ z\|}\right)^{\frac{Q+2}{p}-\alpha} \left(2^k\sigma\|w^{-1}\circ z\|\right)^{\frac{Q+2}{p}} \varphi\left(z, 2^k c_1\sigma\|w^{-1}\circ z\|\right) \\
&\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \sum_{k=1}^{\infty} \left(2^k\sigma\|w^{-1}\circ z\|\right)^{\alpha} \varphi\left(z, 2^k\sigma\|w^{-1}\circ z\|\right) \\
&\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_{\|w^{-1}\circ z\|}^{\infty} \varphi(z,t)\, t^{\alpha-1}\, dt
\end{aligned}$$

we know that (3.1) is true, since the above series is convergent.

Similarly, by integrating on the set

$$\{\zeta \in \mathbb{R}^{N+1} : 2^{-k}\sigma\|w^{-1}\circ z\| \le \|\zeta^{-1}\circ z\| < 2^{1-k}\sigma\|w^{-1}\circ z\|\},$$

it yields

$$
\begin{aligned}
|T''_\beta g(z)| &\le \sum_{k=1}^{\infty} \int_{2^{-k}\sigma\|w^{-1}\circ z\| \le \|\zeta^{-1}\circ z\| < 2^{1-k}\sigma\|w^{-1}\circ z\|} \frac{|g(\zeta)|}{\|\zeta^{-1}\circ z\|^{Q+2-\beta}}\, d\zeta \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^{1-k}\sigma\|w^{-1}\circ z\|}\Big)^{Q+2-\beta} \int_{B_{2^{1-k}c_1\sigma\|w^{-1}\circ z\|}(z)} |g(\zeta)|\, d\zeta \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^{1-k}\sigma\|w^{-1}\circ z\|}\Big)^{Q+2-\beta} \|g\|_{L^p\left(B_{2^{1-k}c_1\sigma\|w^{-1}\circ z\|}(z)\right)} \left|B_{2^{1-k}c_1\sigma\|w^{-1}\circ z\|}(z)\right|^{\frac{1}{p'}} \\
&\le \sum_{k=1}^{\infty} \Big(\frac{2}{2^{1-k}\sigma\|w^{-1}\circ z\|}\Big)^{\frac{Q+2}{p}-\beta} \|g\|_{L^p\left(B_{2^{1-k}c_1\sigma\|w^{-1}\circ z\|}(z)\right)} \lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \\
&\quad \times \sum_{k=1}^{\infty} \Big(\frac{1}{2^{-k}\sigma\|w^{-1}\circ z\|}\Big)^{\frac{Q+2}{p}-\beta} \big(2^{-k}\sigma\|w^{-1}\circ z\|\big)^{\frac{Q+2}{p}} \varphi\big(z, 2^{-k}\sigma\|w^{-1}\circ z\|\big) \\
&\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \sum_{k=1}^{\infty} \big(2^{-k}\sigma\|w^{-1}\circ z\|\big)^{\beta} \varphi\big(z, 2^{-k}\sigma\|w^{-1}\circ z\|\big) \\
&\lesssim \|g\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_0^{\|w^{-1}\circ z\|} \varphi(z,t)\, t^{\beta-1}\, dt.
\end{aligned}
$$

Noting that the above series is convergent, (3.2) is proved. □

Proof of Theorem 1.5. For $u \in C_0^\infty(\mathbb{R}^{N+1})$, by Lemmas 3.2 and 3.3, there exist $M, c > 0$ such that

$$
\begin{aligned}
|u(z) - u(w)| &\le \int_{\mathbb{R}^{N+1}} |\Gamma(\zeta^{-1}\circ z) - \Gamma(\zeta^{-1}\circ w)||\mathcal{L}(\zeta)| d\zeta \\
&\lesssim \int_{\|\zeta^{-1}\circ z\| \ge M\|z^{-1}\circ w\|} \frac{\|z^{-1}\circ w\|}{\|\zeta^{-1}\circ z\|^{Q+1}} |\mathcal{L}u(\zeta)| d\zeta \\
&\quad + \int_{\|\zeta^{-1}\circ z\| < M\|z^{-1}\circ w\|} \frac{1}{\|\zeta^{-1}\circ z\|^{Q}} |\mathcal{L}u(\zeta)| d\zeta \\
&\quad + \int_{\|\zeta^{-1}\circ z\| < M\|z^{-1}\circ w\|} \frac{1}{\|\zeta^{-1}\circ w\|^{Q}} |\mathcal{L}u(\zeta)| d\zeta \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
$$

By applying Lemma 3.3 and choosing $\alpha = 1$ and $\sigma = M/c_1$, there exists a positive constant c such that

$$
|I_1| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \|z^{-1}\circ w\| \int_{\|z^{-1}\circ w\|}^{\infty} \varphi(z,t)\, dt, \tag{3.3}
$$

choosing $\beta = 2$ and $\sigma = Mc_1$ in Lemma 3.3, there exists a positive constant c such that that

$$|I_2| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\, t\, dt, \tag{3.4}$$

choosing $\beta = 2$ and $\sigma = c_2(1+M)$ in Lemma 3.3, there exists a positive constant c such that

$$|I_3| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\, t\, dt. \tag{3.5}$$

Hence, by (3.3), (3.4) and (3.5), it is easy to obtain

$$\begin{aligned}|u(z)-u(w)| &\lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})}\\ &\quad\times\Big(\int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\, t\, dt + \|z^{-1}\circ w\| \int_{\|z^{-1}\circ w\|}^{\infty} \varphi(z,t)\, dt\Big).\end{aligned}$$

By (2.7), we write

$$\partial_{x_j} u(z) = -\int_{\mathbb{R}^{N+1}} \Gamma_j(\zeta^{-1}\circ z)\mathcal{L}u(\zeta)d\zeta$$

for every $z \in \mathbb{R}^{N+1}$ and $j = 1, 2, \ldots, p_0$. Analogously, by Lemmas 3.2 and 3.3, we get that there exist $M, c > 0$ such that

$$\begin{aligned}|\partial_{x_j}u(z) - \partial_{x_j}u(w)| &\le \int_{\mathbb{R}^{N+1}} |\Gamma_j(\zeta^{-1}\circ z) - \Gamma_j(\zeta^{-1}\circ w)||\mathcal{L}u(\zeta)|d\zeta\\ &\le \int_{\|\zeta^{-1}\circ z\|\ge M\|z^{-1}\circ w\|} \frac{c\|z^{-1}\circ w\|}{\|\zeta^{-1}\circ z\|^{Q+2}}|\mathcal{L}u(\zeta)|d\zeta\\ &\quad+ \int_{\|\zeta^{-1}\circ z\|< M\|z^{-1}\circ w\|} \frac{c}{\|\zeta^{-1}\circ z\|^{Q+1}}|\mathcal{L}u(\zeta)|d\zeta\\ &\quad+ \int_{\|\zeta^{-1}\circ z\|< M\|z^{-1}\circ w\|} \frac{c}{\|\zeta^{-1}\circ w\|^{Q+1}}|\mathcal{L}u(\zeta)|d\zeta\\ &\equiv I_1' + I_2' + I_3'.\end{aligned}$$

By applying Lemma 3.3 and choosing $\alpha = 0$ and $\sigma = M/c_1$, there exists a positive constant c such that

$$|I_1'| \lesssim \|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \|z^{-1}\circ w\| \int_{\|z^{-1}\circ w\|}^{\infty} \varphi(z,t)\, \frac{dt}{t}, \tag{3.6}$$

choosing $\beta = 1$ and $\sigma = Mc_1$ in Lemma 3.3, there exists a positive constant c such that

$$|I_2'| \le c\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\, dt, \tag{3.7}$$

choosing $\beta = 1$ and $\sigma = c_2(1 + M)$ in Lemma 3.3, there exists a positive constant c such that

$$|I_3'| \le c\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\,dt. \tag{3.8}$$

Hence, by (3.6), (3.7) and (3.8), we derive

$$\begin{aligned} \left|\partial_{x_j} u(z) - \partial_{x_j} u(w)\right| &\le C\,\|\mathcal{L}u\|_{M^{p,\varphi}(\mathbb{R}^{N+1})} \\ &\times \left(\int_0^{\|z^{-1}\circ w\|} \varphi(z,t)\,dt + \|z^{-1}\circ w\| \int_{\|z^{-1}\circ w\|}^{\infty} \varphi(z,t)\,\frac{dt}{t} \right), \end{aligned}$$

where C is a positive constant, $z, w \in \mathbb{R}^{N+1}$, $z \ne w$. This ends the proof.

Proof of Corollary 1.7. In Theorem 1.5 if we take $\varphi(z,r) = |B(z,r)|^{\frac{\lambda-1}{p}}$, then we get the following

$$\int_{\|w^{-1}\circ z\|}^{\infty} \varphi(z,t)\,t^{\alpha-1}\,dt = \int_{\|w^{-1}\circ z\|}^{\infty} t^{\frac{\lambda-Q-2}{p}+\alpha-1}\,dt = \|w^{-1}\circ z\|^{\frac{\lambda-Q-2}{p}+\alpha}$$

and

$$\begin{aligned} \int_1^{\infty} \varphi(z,t)\,t^{\alpha-1}\,dt = \int_1^{\infty} t^{\frac{\lambda-Q-2}{p}+\alpha-1}\,dt < \infty \Leftrightarrow \frac{\lambda-Q-2}{p} + \alpha > 0 \\ \Leftrightarrow \lambda + p\alpha < Q + 2. \end{aligned}$$

Also

$$\int_0^{\|w^{-1}\circ z\|} \varphi(z,t)\,t^{\beta-1}\,dt = \int_0^{\|w^{-1}\circ z\|} t^{\frac{\lambda-Q-2}{p}+\beta-1}\,dt = \|w^{-1}\circ z\|^{\frac{\lambda-Q-2}{p}+\beta}$$

and

$$\begin{aligned} \int_0^{1} \varphi(z,t)\,t^{\beta-1}\,dt = \int_0^{1} t^{\frac{\lambda-Q-2}{p}+\beta-1}\,dt < \infty \Leftrightarrow \frac{\lambda-Q-2}{p} + \beta > 0 \\ \Leftrightarrow \lambda + p\beta > Q + 2. \end{aligned}$$

This ends the proof.

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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