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BÜLENT ALTUNKAYA

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The generalization of rectifying helices

Bülent ALTUNKAYA*
Department of Mathematics Education, Faculty of Education, Kırşehir Ahi Evran University, Kırşehir, Turkiye

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Abstract: This work establishes connections between different classes of arbitrary speed rectifying helices. We further analyze how the involute of a space curve can exhibit spherical helicity and how the evolute can inherit rectifying properties, thereby providing an alternative characterization of these helices. These relationships, along with their interplay with involutes and evolutes, are supported by illustrative examples that offer deeper insights into their geometric properties.

Key words: Rectifying curve, involute-evolute, spherical helix, slant helix, clad helix, g-clad helix

1. Introduction

In differential geometry, the study of curves is a fundamental area of research. It focuses on geometric properties such as curvature, torsion, and the associated Frenet frame. While unit-speed curves (curves parameterized by arc length) are often the primary focus due to their simplicity, arbitrary speed curves, parameterized by an arbitrary parameter are equally important in both theoretical and applied contexts. Arbitrary speed curves naturally arise in various physical and engineering problems, such as the motion of particles with variable speeds or the design of mechanical systems exhibiting nonuniform motion. Understanding the behavior of such curves and their associated Frenet frames is essential for extending the tools of differential geometry to more general and practical scenarios [13, 16, 17].

Helices, characterized by a constant ratio between their curvature and torsion, frequently appear in both natural and engineered structures. Notable examples include DNA strands [8, 21], helical springs [12], curve design in kinematics [6], and molecular modeling [15].

Slant helices were first introduced by Izumiya and Takeuchi [10] and have since been studied in various geometric settings. Their properties in Euclidean spaces have been explored in studies such as [5, 14], focusing on their structure and generalizations. The notion of generalized helices was further developed in [18], and the study of associated developable surfaces in [19] led to the introduction of clad helices and g-clad helices, which generalize the concept of slant helices.

Rectifying curves were first introduced by Chen [7]. A rectifying curve is characterized by the property that its position vector always lies in its rectifying plane. Further classification and characterization of these curves have been advanced, with studies exploring rectifying slant helices in Euclidean 3-space [2] and extending the theory to Minkowski 3-space for both spacelike and timelike scenarios [3, 4].

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*Correspondence: bulent.altunkaya@ahievran.edu.tr
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The involute–evolute pair relationship between curves offers a powerful tool for studying curve behavior and transformation properties. For instance, rectifying curves have been characterized through their involutes and evolutes in [11], while a geometric perspective on these classical constructs was provided in [9]. The mathematical groundwork laid by classical texts such as [13], [16], and [17] remains instrumental in understanding these foundational relationships.

In this work, we examine how the involute of a space curve can exhibit spherical helicity and how the evolute can possess rectifying properties. We establish new structural relationships between these curves and show that the involute–evolute framework offers an alternative way to characterize rectifying helices and their generalizations.

The paper is structured as follows: Section 2 lays the groundwork by introducing the necessary preliminaries for arbitrary speed curves, including the definitions of rectifying curves, involutes, evolutes, and various types of helices. Section 3 presents new characterizations for arbitrary speed rectifying curves and spatial involute-evolute pairs, along with simplified proofs for some previously known results. Section 4 extends the notion of rectifying helices, establishes connections between helices and involute-evolute curves, and presents several illustrative examples.

2. Preliminaries

The Euclidean 3-space $\mathbb{E}^3$ is the real vector space $\mathbb{R}^3$ with the standard inner product $\langle , \rangle$. Let $\beta : I \to \mathbb{E}^3$ be a curve in Euclidean 3-space $\mathbb{E}^3$ that has at least four continuous derivatives with a nonvanishing derivative $\beta'(t) \neq \mathbf{0}$ for all $t \in I$, where $\beta'(t) = \frac{d\beta}{dt}(t)$.

For a curve $\beta(t)$ in $\mathbb{R}^3$, parameterized by an arbitrary parameter t, the speed of the curve is given by:

$$\nu_\beta(t) = \|\beta'(t)\|,$$

where $\|\beta'(t)\| = \sqrt{\langle \beta'(t), \beta'(t)\rangle}$ represents the magnitude of the velocity vector. If $\nu_\beta(t) \neq 1$, the curve is said to be nonunit speed. In this case, the arc length s is related to the parameter t by:

$$s(t) = \int_{t_0}^{t} \nu_\beta(\theta)\, d\theta.$$

In this work, to ensure clarity and ease of use, we will index the elements of the Frenet frame according to the curve under consideration. For example, for a curve $\beta(t)$, the tangent vector field will be denoted as $\mathbf{T}_\beta(t)$, the normal vector field as $\mathbf{N}_\beta(t)$, and the binormal vector field as $\mathbf{B}_\beta(t)$. This indexing convention will help distinguish between the Frenet frames of different curves and simplify the presentation of formulas and derivations. The Frenet formulae for the curve $\beta(t)$ consist of the tangent, normal, and binormal vector fields are given by:

$$\begin{aligned}
\mathbf{T}'_\beta(t) &= \nu_\beta(t)\kappa_\beta(t)\mathbf{N}_\beta(t),\\
\mathbf{N}'_\beta(t) &= -\nu_\beta(t)\kappa_\beta(t)\mathbf{T}_\beta(t) + \nu_\beta(t)\tau_\beta(t)\mathbf{B}_\beta(t),\\
\mathbf{B}'_\beta(t) &= -\nu_\beta(t)\tau_\beta(t)\mathbf{N}_\beta(t),
\end{aligned}$$

where $\kappa_\beta(t)$ and $\tau_\beta(t)$ are the curvature and torsion of the curve β at t. In general, it is possible for $\mathbf{T}'_\beta(t)$ to vanish for certain values of t; however, we have assumed that this does not happen.

Rectifying curves are a relatively recent addition to the study of curves in differential geometry. They are defined by the condition that their position vector lies in the rectifying plane—the plane spanned by the tangent and binormal vectors [7]. This distinctive property results in a unique balance between curvature and torsion, making rectifying curves a subject of growing interest. For a curve $\beta(t)$, the condition for being a rectifying curve is:

$$\beta(t)=\mu_1(t)T_\beta(t)+\mu_2(t)B_\beta(t),$$

where $\mu_1(t)$ and $\mu_2(t)$ are scalar functions [7].

In the literature, the concept of involute-evolute for space curves is defined using the notion of tangent surfaces [13, 17]. The tangent surface of a curve $\beta(t)$ is formed by the family of tangent lines to the curve at each point, expressed as

$$\mathbf{x}(t,u)=\beta(t)+uT_\beta(t),$$

where u is a real parameter [13, 17].

Involutes are curves that lie on this tangent surface and are orthogonal to the generating tangents. Thus, the involute of a curve can be defined in the following form [13, 16, 17]:

Definition 2.1 *Let $\beta : I \to \mathbb{R}^3$ be an arbitrary speed curve parameterized by t, and let $\gamma : I \to \mathbb{R}^3$ be another curve. For every $t \in I$, if the tangent to the curve β at the point $\beta(t)$ passes through the point $\gamma(t)$ and*

$$\langle T_\beta(t), T_\gamma(t)\rangle = 0,$$

then the curve γ is called an involute of the curve β.

Conversely, consider a curve α. We can determine another curve β such that α is an involute of β. In this case, the curve β is called the evolute of α. Thus, the evolute of a curve can be defined in the following form [13, 16, 17]:

Definition 2.2 *Let $\alpha : I \to \mathbb{R}^3$ be an arbitrary speed curve parameterized by t and $\beta : I \to \mathbb{R}^3$ be another curve defined on the same interval. For every $t \in I$, if the tangent line to the curve β at the point $\beta(t)$ passes through the point $\alpha(t)$ and*

$$\langle T_\beta(t), T_\alpha(t)\rangle = 0,$$

then the curve β is called an evolute of the curve α.

A general helix is a space curve characterized by the property that its tangent vector makes a constant angle with a fixed direction in space. Mathematically, this condition is equivalent to the ratio of the torsion (τ_β) to the curvature (κ_β) being constant for the curve β:

$$\frac{\tau_\beta}{\kappa_\beta}=\text{constant}. \tag{2.1}$$

A slant helix is defined by the property that its principal normal vector field makes a constant angle with a fixed direction in space [1]. This generalizes the concept of a helix. The characterizations of a unit speed slant helix is given in [5, 10, 14]. Based on these studies, we obtain the following proposition:

Proposition 2.3 *Let* $\beta(t)$ *be a space curve with* $\kappa_\beta(t) \neq 0$*. Then,* β *is a slant helix if and only if*

$$\sigma_\beta(t) = \left(\frac{\kappa_\beta^2}{\nu_\beta \left(\kappa_\beta^2 + \tau_\beta^2\right)^{3/2}} \left(\frac{\tau_\beta}{\kappa_\beta}\right)' \right)(t)$$

is a constant function.

The notion of a clad helix, introduced in [18, 19], generalizes slant helices. A curve is called a clad helix if the spherical image of its principal normal vector field is a part of a cylindrical helix. Based on these studies, we obtain the following proposition:

Proposition 2.4 *Let* $\beta(t)$ *be a space curve with* $\kappa_\beta(t) \neq 0$*. Then,* β *is a clad helix if and only if*

$$\rho_\beta(t) = \left(\frac{\sigma_\beta'}{\nu_\beta \left(\kappa_\beta^2 + \tau_\beta^2\right)^{1/2} \left(1 + \sigma_\beta^2\right)^{3/2}} \right)(t)$$

is a constant function.

The notion of a g-clad helix, which is also introduced in [18, 19], generalizes clad helices. A curve is called a g-clad helix if the spherical image of its principal normal vector field is a part of a slant helix. From these studies, we derive the following proposition:

Proposition 2.5 *Let* $\beta(t)$ *be a space curve with* $\kappa_\beta(t) \neq 0$*. Then,* β *is a g-clad helix if and only if*

$$\psi_\beta = \left(\frac{\rho_\beta'}{\nu_\beta(\kappa_\beta^2 + \tau_\beta^2)^{1/2}(1 + \sigma_\beta^2)^{1/2}(1 + \rho_\beta^2)^{3/2}} \right)(t)$$

is a constant function.

In this work, for the sake of ease of computation, we will work with the spheres centered at the origin. The results we obtain can easily be applied to spheres with other centers through translation. Now, we present a well-known proposition that we will use frequently throughout the paper.

Proposition 2.6 *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an arbitrary speed space curve. If*

$$\langle \alpha(t), T_\alpha(t) \rangle = 0$$

for all $t \in I$*, then the curve* α *lies on a sphere with center the origin.*

3. Rectifying curves and spatial involute-evolute pairs: extensions to arbitrary speed

In this section, we will present new characterizations for arbitrary speed rectifying curves and spatial involute-evolute pairs, and extend some characterizations to arbitrary speed curves. In addition, we provide new simplified proofs for some previously known results, particularly stated in [11, 16, 18–20].

3.1. Rectifying curves

Let $\beta : I \longrightarrow \mathbb{R}^3$ be an arbitrary speed rectifying curve. Then, the curve $\beta(t)$ is of the form

$$\beta(t) = \mu_1(t)T_\beta(t) + \mu_2(t)B_\beta(t)$$

for the functions μ_1 and μ_2. Differentiating β and using Frenet formulas, we have

$$\nu_\beta T_\beta = \mu_1' T_\beta + \nu_\beta \left[\mu_1\kappa_\beta - \mu_2\tau_\beta\right] N_\beta + \mu_2' B_\beta.$$

It implies that

$$\mu_1' = \nu_\beta, \qquad \mu_1\kappa_\beta - \mu_2\tau_\beta = 0, \qquad \mu_2' = 0, \tag{3.1}$$

so that

$$\mu_1 = \int_0^t \nu_\beta \, d\theta + k_1 \text{ and } \mu_2 = k_2,$$

where k_1 and $k_2 \neq 0$ constants. By means of these equations, we have the following proposition:

Proposition 3.1 *Let $\beta = \beta(t)$ be an arbitrary speed rectifying space curve. Then,*

$$\beta(t) = \left(\int_0^t \nu_\beta(\theta)\, d\theta + k_1\right) T_\beta(t) + k_2\, B_\beta(t), \tag{3.2}$$

where $k_1, k_2 \in \mathbb{R}$ with $k_2 \neq 0$.

Lemma 3.2 *Let $\beta = \beta(t)$ be an arbitrary speed space curve. Then, β is congruent to a rectifying curve if and only if*

$$\frac{\tau_\beta(t)}{\kappa_\beta(t)} = c_1 \int_0^t \nu_\beta(\theta)\, d\theta + c_2,$$

where $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq 0$.

Proof Let β be an arbitrary speed space curve.

⇒: Suppose β is a rectifying curve. Then, from equations (3.1) and (3.2), we have

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\int_0^t \nu_\beta\, d\theta + k_1}{k_2} = c_1 \int_0^t \nu_\beta\, d\theta + c_2,$$

where $c_1 = \dfrac{1}{k_2}$ and $c_2 = \dfrac{k_1}{k_2}$.

⇐: Suppose

$$\frac{\tau_\beta}{\kappa_\beta} = c_1 \int_0^t \nu_\beta\, d\theta + c_2$$

for the curve β. Then, if we substitute c_1 with $\dfrac{1}{k_2}$ and c_2 with $\dfrac{k_1}{k_2}$, we obtain

$$\left(\int_0^t \nu_\beta\, d\theta + k_1\right)\kappa_\beta - k_2\,\tau_\beta = 0.$$

By means of Frenet equations, we have

$$\frac{d}{dt}\left(\beta-\left(\int_0^t \nu_\beta\, d\theta+k_1\right)T_\beta+k_2\, B_\beta\right)=\nu_\beta\left(\left(\int_0^t \nu_\beta\, d\theta+k_1\right)\kappa_\beta\ -k_2\,\tau_\beta\right)N_\beta=0.$$

Therefore, β is congruent to a rectifying curve. □

3.2. Involute of a space curve

If α is an involute of β, from Definition 2.1, the curve α is of the form

$$\gamma(t)=\beta(t)+\lambda(t)T_\beta(t) \qquad \text{and} \qquad T_\gamma(t)\perp T_\beta(t).$$

By differentiating $\gamma=\beta+\lambda T_\beta$, we get

$$\nu_\gamma T_\gamma=\nu_\beta T_\beta+\lambda' T_\beta+\nu_\beta\kappa_\beta N_\beta.$$

Taking inner product of both sides with T_β and using Frenet-Serret equations we have,

$$0=\nu_\beta+\lambda'$$

and

$$\lambda=c-\int_0^t \nu_\beta\, d\theta.$$

Therefore, we can give the proposition below.

Proposition 3.3 *Let $\beta=\beta(t)$ be an arbitrary speed space curve and $\gamma=\gamma(t)$ be its involute. Then, γ is of the form*

$$\gamma(t)=\beta(t)+\left(c-\int_0^t \nu_\beta(\theta)\, d\theta\right)T_\beta(t), \tag{3.3}$$

where $c\in\mathbb{R}$.

The proposition demonstrates that each real value of c corresponds to a unique involute among the infinitely many possible involutes of the given curve. We now present a new lemma that establishes the relationship between the curvatures and torsions of the given curve and its involute.

Lemma 3.4 *Let $\beta=\beta(t)$ be an arbitrary speed space curve and $\gamma=\gamma(t)$ be its involute. Then,*

$$\kappa_\gamma(t)=\frac{\sqrt{\kappa_\beta^2(t)+\tau_\beta^2(t)}}{\left|c-\int_0^t \nu_\beta(\theta)\, d\theta\right|\kappa_\beta(t)},$$

$$\tau_\gamma(t)=\frac{\kappa_\beta(t)}{\left|c-\int_0^t \nu_\beta(\theta)\, d\theta\right|\nu_\beta(t)\left(\kappa_\beta^2(t)+\tau_\beta^2(t)\right)}\left(\frac{\tau_\beta(t)}{\kappa_\beta(t)}\right)'.$$

Proof Since γ is the involute of the curve β, from (3.3), we have

$$\gamma' = \nu_\gamma T_\gamma = \left(c - \int_0^t \nu_\beta \, d\theta\right) \nu_\beta \kappa_\beta N_\beta,$$

and by choosing $T_\gamma = -N_\beta$, we have

$$\gamma' = -\varphi N_\beta \qquad \text{and} \qquad \nu_\gamma = \varphi \nu_\beta \, \kappa_\beta, \tag{3.4}$$

where $\varphi = \left| c - \displaystyle\int_0^t \nu_\beta \, d\theta \right|$. Thus, we have

$$\begin{aligned}
\gamma'' &= \varphi \nu_\beta \kappa_\beta T_\beta - \varphi' N_\beta - \varphi \nu_\beta \tau_\beta B_\beta, \\
B_\gamma &= \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|} = \frac{1}{\sqrt{\kappa_\beta^2 + \tau_\beta^2}} \left(-\tau_\beta T_\beta + \kappa_\beta B_\beta\right), \\
N_\gamma &= B_\gamma \times T_\gamma = -B_\gamma \times N_\beta = \frac{1}{\sqrt{\kappa_\beta^2 + \tau_\beta^2}} \left(\kappa_\beta T_\beta + \tau_\beta B_\beta\right).
\end{aligned}$$

Thus, with straightforward calculations, we get

$$\kappa_\gamma = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \frac{\sqrt{\kappa_\beta^2 + \tau_\beta^2}}{\left| c - \int_0^t \nu_\beta \, d\theta \right| \kappa_\beta}$$

and

$$\tau_\gamma = \frac{1}{\nu_\gamma} \left\langle N_\gamma', B_\gamma \right\rangle = \frac{\kappa_\beta}{\left| c - \int_0^t \nu_\beta \, d\theta \right| \nu_\beta \left(\kappa_\beta^2 + \tau_\beta^2\right)} \left(\frac{\tau_\beta}{\kappa_\beta}\right)'.$$

□

We can now state an important theorem that will be frequently used later.

Theorem 3.5 *Let* $\beta = \beta(t)$ *be an arbitrary speed space curve and* $\gamma = \gamma(t)$ *be its involute. Then,* γ *is a spherical curve if and only if* β *is a rectifying curve.*

Proof Let $\beta = \beta(t)$ be an arbitrary speed space curve and $\gamma = \gamma(t)$ be its involute.

⇒: Suppose γ is a spherical curve. By differentiating (3.3), we get

$$\gamma' = \beta' - \nu_\beta T_\beta + \left(c - \int_0^t \nu_\beta \, dt\right) T_\beta'.$$

Taking the inner product of both sides with γ and using Frenet-Serret equations, we have

$$\langle N_\beta, \gamma \rangle = 0.$$

By using equation (3.3) again, we obtain

$$\begin{aligned}0 &= \langle N_\beta, \gamma\rangle \\ &= \left\langle N_\beta, \beta + \left(c - \int_0^t \nu_\beta \, dt\right) T_\beta \right\rangle \\ &= \langle N_\beta, \beta\rangle .\end{aligned}$$

Therefore, β is a rectifying curve.

$\Leftarrow$: Suppose β is a rectifying curve. Then, by using (3.1) and (3.3), we have

$$\begin{aligned}\gamma &= \beta + \left(c - \int_0^t \nu_\beta \, d\theta\right) T_\beta \\ &= \left(\int_0^t \nu_\beta \, d\theta + k_1\right) T_\beta + k_2 \, B_\beta + \left(c - \int_0^t \nu_\beta \, d\theta\right) T_\beta \\ &= (k_1 + c) \, T_\beta + k_2 \, B_\beta,\end{aligned}$$

so that

$$\|\gamma\| = \sqrt{(k_1 + c)^2 + k_2^2} \, .$$

Therefore, γ is a spherical curve. □

3.3. Evolute of a space curve

If β is an evolute of α, from Definition 2.2, the curve β may be represented in the form

$$\beta(t) = \alpha(t) + \eta(t) \, u(t),$$

where $u(t) = \sin\varphi_\alpha(t) N_\alpha(t) + \cos\varphi_\alpha(t) B_\alpha(t)$, $\|\eta(t)\| = \|\beta(t) - \alpha(t)\|$, and $\varphi_\alpha(t)$ is the angle between the unit vectors $u(t)$ and $B_\alpha(t)$. As a consequence of the evolute definition, $T_\beta(t)$ must be parallel to $u(t)$. Thus, there must be a scalar $\omega_\alpha(t)$ where

$$\beta' = \omega_\alpha \, u. \tag{3.5}$$

Furthermore,

$$\beta' = \alpha' + \eta' u + \eta u'$$

so that

$$\alpha' + \eta u' = 0 \quad \text{and} \quad \nu_\beta = \eta' = \omega_\alpha. \tag{3.6}$$

By using Frenet equations, we have

$$\begin{aligned}\alpha' &= -\eta u', \\ \nu_\alpha T_\alpha &= -\eta \left[-\nu_\alpha \kappa_\alpha \sin\varphi_\alpha \, T_\alpha + (\varphi_\alpha' - \nu_\alpha \tau_\alpha) \cos\varphi_\alpha \, N_\alpha + (\nu_\alpha \tau_\alpha - \varphi_\alpha') \sin\varphi_\alpha \, B_\alpha \right].\end{aligned}$$

Therefore, we get

$$\eta \nu_\alpha \kappa_\alpha \sin\varphi_\alpha = \nu_\alpha \quad \Rightarrow \quad \eta = \frac{1}{\kappa_\alpha \sin\varphi_\alpha}, \tag{3.7}$$

and

$$\varphi'_\alpha = \nu_\alpha \tau_\alpha \quad \Rightarrow \quad \varphi_\alpha = \int_0^t \nu_\alpha \tau_\alpha \, d\theta + k,$$

where $k \in \mathbb{R}$. Therefore, we can give the proposition below.

Proposition 3.6 *Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. Then, β is of the form*

$$\beta(t) = \alpha(t) + \frac{1}{\kappa_\alpha(t)} N_\alpha(t) + \frac{1}{\kappa_\alpha(t)} \cot \varphi_\alpha(t) B_\alpha(t), \tag{3.8}$$

where $\varphi_\alpha(t) = \int_0^t \nu_\alpha(\theta)\tau_\alpha(\theta)\, d\theta + k$.

The proposition demonstrates that each real value of k corresponds to a unique evolute among the infinitely many possible evolutes of the given curve. We now introduce a new lemma that establishes the relationship between the curvatures and torsions of the given curve and its evolute.

Lemma 3.7 *Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. Then,*

$$\kappa_\beta(t) = \frac{1}{\omega_\alpha(t)} \nu_\alpha(t)\kappa_\alpha(t) \left|\sin \varphi_\alpha(t)\right|,$$

$$\tau_\beta(t) = \frac{1}{\omega_\alpha(t)} \nu_\alpha(t)\kappa_\alpha(t) \cos \varphi_\alpha(t),$$

where $\varphi_\alpha(t) = \int_0^t \nu_\alpha(\theta)\tau_\alpha(\theta)\, d\theta + k$ and $\omega_\alpha(t) = \frac{d}{dt}\left(\frac{1}{\kappa_\alpha(t)\sin \varphi_\alpha(t)}\right)$.

Proof From (3.5)–(3.8), we have

$$\nu_\beta T_\beta = \omega_\alpha \left[\sin \varphi_\alpha N_\alpha + \cos \varphi_\alpha B_\alpha\right].$$

Consequently, we obtain

$$T_\beta = \sin \varphi_\alpha N_\alpha + \cos \varphi_\alpha B_\alpha$$

and

$$\nu_\beta = \omega_\alpha = \left| \frac{d}{dt}\left(\frac{1}{\kappa_\alpha \sin \varphi_\alpha}\right)\right|. \tag{3.9}$$

Differentiating T_β and using Frenet equations give

$$T'_\beta = \nu_\beta \kappa_\beta N_\beta = -\nu_\alpha \kappa_\alpha \sin \varphi_\alpha T_\alpha,$$

from which we obtain

$$T_\alpha = \pm N_\beta.$$

We can choose $T_\alpha = -N_\beta$, which implies that

$$\nu_\beta \kappa_\beta = \nu_\alpha \kappa_\alpha \sin \varphi_\alpha$$

and from (3.9), we get

$$\kappa_\beta = \frac{1}{\omega_\alpha}\nu_\alpha \kappa_\alpha \left|\sin\varphi_\alpha\right|.$$

With straightforward calculations, we also have

$$\begin{aligned} B_\beta &= T_\beta \times N_\beta \\ &= [\sin\varphi_\alpha N_\alpha - \cos\varphi_\alpha B_\alpha] \times [-T_\alpha] \\ &= -\cos\varphi_\alpha N_\alpha + \sin\varphi_\alpha B_\alpha. \end{aligned}$$

Therefore, the Frenet frame field for the curve β can be written as

$$\begin{aligned} T_\beta &= \sin\varphi_\alpha N_\alpha + \cos\varphi_\alpha B_\alpha, \\ N_\beta &= -T_\alpha, \\ B_\beta &= -\cos\varphi_\alpha N_\alpha + \sin\varphi_\alpha B_\alpha. \end{aligned}$$

Differentiating B_β and using Frenet equations give

$$B'_\beta = -\nu_\beta \tau_\beta N_\beta = \nu_\alpha \kappa_\alpha \cos\varphi_\alpha T_\alpha,$$

and again from (3.9), it follows that

$$\tau_\beta = \frac{1}{\omega_\alpha}\nu_\alpha \kappa_\alpha \cos\varphi_\alpha.$$

□

We can now formulate an important theorem that will be frequently used in subsequent sections.

Theorem 3.8 *Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. Then, α is a spherical curve if and only if β is a rectifying curve.*

Proof Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute.

⇒: Suppose α is a spherical curve. Then, from using (3.8) and $T_\alpha = -N_\beta$, we get,

$$\begin{aligned} 0 = \langle \alpha, T_\alpha \rangle &= \langle \alpha, T_\alpha \rangle + \frac{1}{\kappa_\alpha}\langle N_\alpha + \cot\varphi_\alpha B_\alpha, T_\alpha \rangle \\ &= -\left\langle \alpha + \frac{1}{\kappa_\alpha}\left(N_\alpha + \cot\varphi_\alpha B_\alpha\right), N_\beta \right\rangle \\ &= -\langle \beta, N_\beta \rangle. \end{aligned}$$

Therefore, β is a rectifying curve.

⇐: Suppose β is a rectifying curve. Taking the inner product of both sides of (3.8) with N_β and using $T_\alpha = -N_\beta$, we have

$$\begin{aligned} 0 = \langle \beta, N_\beta \rangle &= \langle \alpha, N_\beta \rangle + \langle N_\alpha + \cot\varphi_\alpha B_\alpha, N_\beta \rangle \\ &= \langle \alpha, -T_\alpha \rangle + \langle N_\alpha + \cot\varphi_\alpha B_\alpha, -T_\alpha \rangle \\ &= -\langle \alpha, T_\alpha \rangle. \end{aligned}$$

Therefore, from the Proposition 2.6, α is a spherical curve. □

4. Generalization of rectifying helices

In this section, we generalize the concept of rectifying helices, exploring their extended forms and examining connections among spherical helices, rectifying slant helices, spherical slant helices, rectifying clad helices, spherical clad helices, and rectifying g-clad helices using involute-evolute notions.

In order to address this, we present new theorems for the above cases, respectively. Since a curve has infinitely many involutes and evolutes, we believe it is more appropriate to present them separately: one for the involute of the curve and another for its evolute.

Theorem 4.1 *Let $\beta = \beta(t)$ be an arbitrary speed space curve and $\gamma = \gamma(t)$ be its involute. Then, γ is a spherical helix if and only if β is a rectifying slant helix.*

Proof Let $\beta = \beta(t)$ be an arbitrary speed space curve and $\gamma = \gamma(t)$ be its involute. From Theorem 3.5, we know that γ is a spherical curve if and only if β is a rectifying curve. Furthermore, by applying Lemma 3.4, we have

$$\begin{aligned}\frac{\tau_\gamma}{\kappa_\gamma} &= \frac{\dfrac{\kappa_\beta}{\left|c-\int_0^t \nu_\beta\, d\theta\right| \nu_\beta \left(\kappa_\beta^2+\tau_\beta^2\right)}\left(\dfrac{\tau_\beta}{\kappa_\beta}\right)'}{\dfrac{\sqrt{\kappa_\beta^2+\tau_\beta^2}}{\left|c-\int_0^t \nu_\beta\, d\theta\right| \kappa_\beta}} \\ &= \frac{\kappa_\beta^2}{\nu_\beta\left(\kappa_\beta^2+\tau_\beta^2\right)^{3/2}}\left(\frac{\tau_\beta}{\kappa_\beta}\right)' \\ &= \sigma_\beta.\end{aligned} \tag{4.1}$$

Therefore, under these assumptions, if $\dfrac{\tau_\gamma}{\kappa_\gamma} = c \in \mathbb{R}\setminus\{0\}$, then $\sigma_\beta = c$, and vice versa. □

Example 4.2 *The curve*

$$\beta(t) = \left(-\frac{1}{3}\sqrt{t^2+1}\cos\left(3\arctan(t)\right), -\frac{1}{3}\sqrt{t^2+1}\sin\left(3\arctan(t)\right), \frac{2}{3}\sqrt{2}\sqrt{t^2+1}\right)$$

is a rectifying slant helix (Figure 1) with

$$T_\beta(t) = \left(\frac{8t}{3\left(t^2+1\right)^2}, \frac{t^4+6t^2-3}{3\left(t^2+1\right)^2}, \frac{2\sqrt{2}t}{3\sqrt{t^2+1}}\right),$$

$$N_\beta(t) = \left(\frac{2\sqrt{2}\left(1-3t^2\right)}{3\left(t^2+1\right)^{3/2}}, -\frac{2\sqrt{2}t\left(t^2-3\right)}{3\left(t^2+1\right)^{3/2}}, \frac{1}{3}\right),$$

and

$$\sigma_\alpha(t) = \frac{1}{2\sqrt{2}}.$$

If we take $c = 0$ *in Proposition 3.3, we obtain the spherical helix* γ *(Figure 2), which lies on the unit sphere centered at the origin (Figure 3), and is also the involute of the curve* β *(Figure 4), where*

$$\gamma(t) = \left(\frac{3t^4 - 6t^2 - 1}{3\left(t^2+1\right)^2}, -\frac{8t^3}{3\left(t^2+1\right)^2}, \frac{2\sqrt{2}}{3\sqrt{t^2+1}}\right)$$

with

$$T_\gamma(t) = \left(\frac{2\sqrt{2}\left(3t^2-1\right)}{3\left(t^2+1\right)^{3/2}}, \frac{2\sqrt{2}t\left(t^2-3\right)}{3\left(t^2+1\right)^{3/2}}, -\frac{1}{3}\right) = -N_\beta(t),$$

$$N_\gamma(t) = \left(-\frac{t\left(t^2-3\right)}{\left(t^2+1\right)^{3/2}}, \frac{3t^2-1}{\left(t^2+1\right)^{3/2}}, 0\right),$$

and

$$\frac{\tau_\gamma(t)}{\kappa_\gamma(t)} = \frac{1}{2\sqrt{2}}.$$

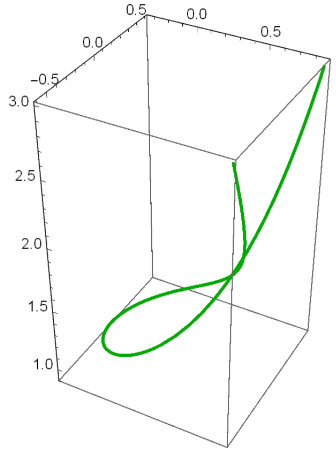

Figure 1. Rectifying slant helix β.

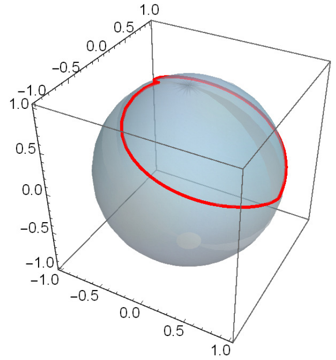

Figure 2. Spherical helix γ.

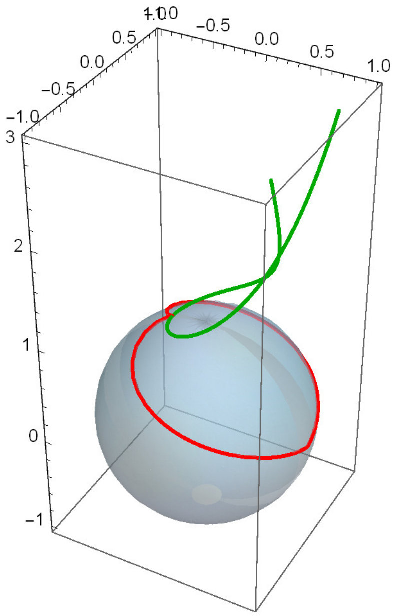

Figure 3. The curves γ and β.

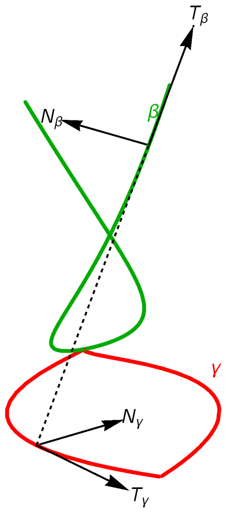

Figure 4. Tangent and normal vectors at $t = \frac{5}{2}$.

Theorem 4.3 *Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. Then, α is a spherical helix if and only if β is a rectifying slant helix.*

Proof Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. From Theorem 3.8, we know that α is a spherical curve if and only if β is a rectifying curve. Furthermore, by applying Lemma 3.7, we have

$$\begin{aligned}
\sigma_\beta &= \frac{\kappa_\beta^2}{\nu_\beta \left(\kappa_\beta^2+\tau_\beta^2\right)^{3/2}} \left(\frac{\tau_\beta}{\kappa_\beta}\right)' \\
&= \frac{\dfrac{1}{\omega_\alpha^2}\nu_\alpha^2\kappa_\alpha^2 \sin^2\varphi_\alpha}{\omega_\alpha \dfrac{1}{\omega_\alpha^3}\nu_\alpha^3\kappa_\alpha^3} (\cot\varphi_\alpha)' \\
&= \mp \frac{\sin^2\varphi_\alpha}{\nu_\alpha\kappa_\alpha}\varphi_\alpha' \left(\frac{-1}{\sin^2\varphi_\alpha}\right) \\
&= \pm \frac{\nu_\alpha\tau_\alpha}{\nu_\alpha\kappa_\alpha} \\
&= \pm \frac{\tau_\alpha}{\kappa_\alpha}.
\end{aligned} \tag{4.2}$$

Therefore, under these assumptions, if $\sigma_\beta = c \in \mathbb{R}\backslash\{0\}$ then $\dfrac{\tau_\gamma}{\kappa_\gamma} = \pm c$, and vice versa. $\square$

Theorem 4.4 *Let $\beta = \beta(t)$ be an arbitrary speed space curve and $\gamma = \gamma(t)$ be its involute. Then, γ is a spherical slant helix if and only if β is a rectifying clad helix.*

Proof Let $\beta = \beta(t)$ be an arbitrary speed space curve, and let $\gamma = \gamma(t)$ be its involute. From Theorem 3.5, we know that γ is a spherical curve if and only if β is a rectifying curve. Moreover, from equation (4.1), it follows that

$$\frac{\tau_\gamma}{\kappa_\gamma} = \sigma_\beta.$$

Let us denote $\left|c - \int_0^t \nu_\beta \, d\theta\right|$ with A and $\sqrt{\kappa_\beta^2+\tau_\beta^2}$ with B. From Lemma 3.4 and (3.4), we have

$$\begin{aligned}
\sigma_\gamma &= \frac{\kappa_\gamma^2}{\nu_\gamma \left(\kappa_\gamma^2+\tau_\gamma^2\right)^{3/2}} \left(\frac{\tau_\gamma}{\kappa_\gamma}\right)' \\
&= \frac{\dfrac{B^2}{A^2\kappa_\beta^2}}{A\kappa_\beta\nu_\beta \left(\dfrac{B^2}{A^2\kappa_\beta^2} + \dfrac{\kappa_\beta^2}{A^2\nu_\beta^2B^4}\left[\left(\dfrac{\tau_\beta}{\kappa_\beta}\right)'\right]^2\right)^{3/2}} \sigma_\beta' \\
&= \frac{\sigma_\beta'}{\nu_\beta B \left(1 + \dfrac{\kappa_\beta^4}{\nu_\beta^2 B^6}\left[\left(\dfrac{\tau_\beta}{\kappa_\beta}\right)'\right]^2\right)^{3/2}} \\
&= \frac{\sigma_\beta'}{\nu_\beta \left(1+\sigma_\beta^2\right)^{3/2} \left(\kappa_\beta^2+\tau_\beta^2\right)^{1/2}} \\
&= \rho_\beta.
\end{aligned} \tag{4.3}$$

Therefore, under these assumptions, if $\sigma_\gamma = c \in \mathbb{R}\backslash\{0\}$ then $\rho_\beta = c$, and vice versa. $\square$

Theorem 4.5 *Let* $\alpha = \alpha(t)$ *be an arbitrary speed space curve and* $\beta = \beta(t)$ *be its evolute. Then,* α *is a spherical slant helix if and only if* β *is a rectifying clad helix.*

Proof Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. From Theorem 3.8, we know that α is a spherical curve if and only if β is a rectifying curve. Additionally, from equation (4.2), it follows that

$$\sigma_\beta = \pm\frac{\tau_\alpha}{\kappa_\alpha}.$$

By means of Lemma 3.7, we have

$$\begin{aligned}\rho_\beta &= \frac{\sigma'_\beta}{\nu_\beta\left(1+\sigma_\beta^2\right)^{3/2}\left(\kappa_\beta^2+\tau_\beta^2\right)^{1/2}}\\ &= \frac{1}{\omega_\alpha\left(1+\dfrac{\tau_\alpha^2}{\kappa_\alpha^2}\right)^{3/2}\dfrac{1}{\omega_\alpha}\nu_\alpha\kappa_\alpha}\left(\pm\frac{\tau_\alpha}{\kappa_\alpha}\right)'\\ &= \frac{\kappa_\alpha^2}{\nu_\alpha\left(\kappa_\alpha^2+\tau_\alpha^2\right)^{3/2}}\left(\pm\frac{\tau_\alpha}{\kappa_\alpha}\right)'\\ &= \pm\sigma_\alpha.\end{aligned} \tag{4.4}$$

Therefore, under these assumptions, if $\rho_\beta = c \in \mathbb{R}\setminus\{0\}$ then $\sigma_\alpha = \pm c$, and vice versa. □

Example 4.6 *The curve*

$$\begin{aligned}\alpha(t) =&\Bigg(\sin\left(\sqrt{2}t\right)\cos(t)\cos(\sin(t)) - \frac{\cos\left(\sqrt{2}t\right)\left(\sin(\sin(t))+\sin(t)\cos(\sin(t))\right)}{\sqrt{2}}\\ &, -\cos(t)\cos\left(\sqrt{2}t\right)\cos(\sin(t)) - \frac{\sin\left(\sqrt{2}t\right)\left(\sin(\sin(t))+\sin(t)\cos(\sin(t))\right)}{\sqrt{2}}\\ &, \frac{\sin(t)\cos(\sin(t))-\sin(\sin(t))}{\sqrt{2}}\Bigg)\end{aligned}$$

is a spherical slant helix which lies on the unit sphere centered at the origin (Figure 5) with

$$\begin{aligned}T_\alpha(t) &= \left(\sin(t)\sin\left(\sqrt{2}t\right) + \frac{\cos(t)\cos\left(\sqrt{2}t\right)}{\sqrt{2}}, \frac{\sin\left(\sqrt{2}t\right)\cos(t)}{\sqrt{2}} - \sin(t)\cos\left(\sqrt{2}t\right), -\frac{\cos(t)}{\sqrt{2}}\right),\\ N_\alpha(t) &= \left(\frac{\cos\left(\sqrt{2}t\right)}{\sqrt{2}}, \frac{\sin\left(\sqrt{2}t\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\end{aligned}$$

and $\sigma_\alpha(t) = -1$*. If we take* $k = \dfrac{\pi}{2}$ *in Proposition 3.6, we obtain the rectifying clad helix* β *(Figures 6 and 7), which is also the evolute of the curve* α *(Figure 8), where*

$$\begin{aligned}\beta(t) = \Bigg(&\frac{1}{2}\sec(\sin(t))\left(2\sin\left(\sqrt{2}t\right)\cos(t) - \sqrt{2}\sin(t)\cos\left(\sqrt{2}t\right)\right)\\ &, -\frac{1}{2}\sec(\sin(t))\left(\sqrt{2}\sin(t)\sin\left(\sqrt{2}t\right) + 2\cos(t)\cos\left(\sqrt{2}t\right)\right)\\ &, \frac{\sec(\sin(t))\sin(t)}{\sqrt{2}}\Bigg)\end{aligned}$$

with

$$\begin{aligned}T_\beta(t) =&\Bigg(\sin\left(\sqrt{2}t\right)\sin(\sin(t))\cos(t) + \frac{\cos\left(\sqrt{2}t\right)\left(\cos(\sin(t)) - \sin(t)\sin(\sin(t))\right)}{\sqrt{2}}\\ &, \frac{\sin\left(\sqrt{2}t\right)\left(\cos(\sin(t)) - \sin(t)\sin(\sin(t))\right)}{\sqrt{2}} - \sin(\sin(t))\cos(t)\cos\left(\sqrt{2}t\right)\\ &, \frac{\sin(t)\sin(\sin(t)) + \cos(\sin(t))}{\sqrt{2}}\Bigg),\\ N_\beta(t) =& \left(-\sin(t)\sin\left(\sqrt{2}t\right) - \frac{\cos(t)\cos\left(\sqrt{2}t\right)}{\sqrt{2}}, \sin(t)\cos\left(\sqrt{2}t\right) - \frac{\sin\left(\sqrt{2}t\right)\cos(t)}{\sqrt{2}}, \frac{\cos(t)}{\sqrt{2}}\right)\\ =& -T_\alpha(t),\end{aligned}$$

and $\rho_\beta(t) = -1$.

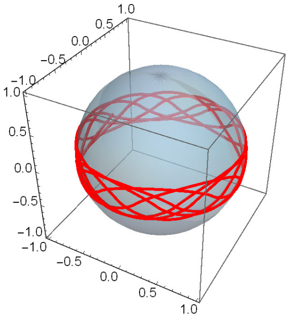

Figure 5. Spherical slant helix α.

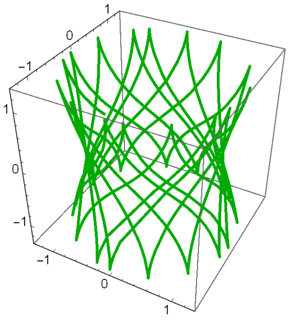

Figure 6. Rectifying clad helix β.

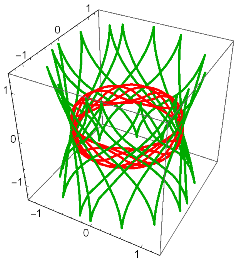

Figure 7. The curves α and β.

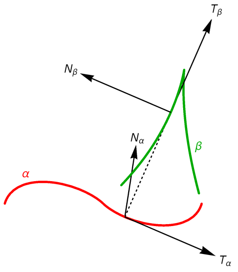

Figure 8. Tangent and normal vectors at $t = 1$.

Theorem 4.7 *Let $\beta = \beta(t)$ be an arbitrary speed space curve and $\gamma = \gamma(t)$ be its involute. Then, γ is a spherical clad helix if and only if β is a rectifying g-clad helix.*

Proof Let $\beta = \beta(t)$ be an arbitrary speed space curve, and let $\gamma = \gamma(t)$ be its involute. From Theorem 3.5, we know that γ is a spherical curve if and only if β is a rectifying curve. Furthermore, from equation (4.3), we have

$$\sigma_\gamma = \rho_\beta .$$

Let us denote $\left| c - \int_0^t \nu_\beta \, d\theta \right|$ with A and $\sqrt{\kappa_\beta^2 + \tau_\beta^2}$ with B. From Lemma 3.4 and (3.4), we obtain

$$\begin{aligned}
\rho_\gamma &= \frac{\sigma_\gamma'}{\nu_\gamma \left(\kappa_\gamma^2 + \tau_\gamma^2\right)^{1/2} (1 + \sigma_\gamma^2)^{3/2}} \\
&= \frac{\rho_\beta'}{A \kappa_\beta \nu_\beta \left(\dfrac{B^2}{A^2 \kappa_\beta^2} + \dfrac{\kappa_\beta^2}{A^2 \nu_\beta^2 B^4} \left[\left(\dfrac{\tau_\beta}{\kappa_\beta} \right)' \right]^2 \right)^{1/2} (1 + \rho_\beta^2)^{3/2}} \\
&= \frac{\rho_\beta'}{\nu_\beta B \left(1 + \dfrac{\kappa_\beta^4}{\nu_\beta^2 B^6} \left[\left(\dfrac{\tau_\beta}{\kappa_\beta} \right)' \right]^2 \right)^{1/2} (1 + \rho_\beta^2)^{3/2}} \\
&= \frac{\rho_\beta'}{\nu_\beta \left(\kappa_\beta^2 + \tau_\beta^2\right)^{1/2} \left(1 + \sigma_\beta^2\right)^{3/2} (1 + \rho_\beta^2)^{3/2}} \\
&= \psi_\beta .
\end{aligned}$$

Therefore, under these assumptions, if $\rho_\gamma = c \in \mathbb{R}\setminus\{0\}$ then $\psi_\beta = c$, and vice versa. $\square$

Theorem 4.8 *Let $\alpha = \alpha(t)$ be an arbitrary speed space curve and $\beta = \beta(t)$ be its evolute. Then, α is a spherical clad helix if and only if β is a rectifying g-clad helix.*

Proof Let $\alpha = \alpha(t)$ be an arbitrary speed space curve, and let $\beta = \beta(t)$ be its evolute. From Theorem 3.8, we know that α is a spherical curve if and only if β is a rectifying curve. Additionally, from equations (4.2) and (4.4), we have

$$\rho_\beta = \pm\sigma_\alpha \quad \text{and} \quad \sigma_\beta = \pm\frac{\tau_\alpha}{\kappa_\alpha}.$$

By means of Lemma 3.7, we obtain

$$\begin{aligned}
\psi_\beta &= \frac{\rho'_\beta}{\nu_\beta(\kappa_\beta^2+\tau_\beta^2)^{1/2}(1+\sigma_\beta^2)^{1/2}(1+\rho_\beta^2)^{3/2}} \\
&= \frac{\pm\sigma'_\alpha}{\omega_\alpha \dfrac{1}{\omega_\alpha}\nu_\alpha\kappa_\alpha\left(1+\dfrac{\tau_\alpha^2}{\kappa_\alpha^2}\right)^{1/2}(1+\sigma_\alpha^2)^{3/2}} \\
&= \pm\frac{\sigma'_\alpha}{\nu_\alpha\left(\kappa_\alpha^2+\tau_\alpha^2\right)^{1/2}(1+\sigma_\alpha^2)^{3/2}} \\
&= \pm\rho_\alpha.
\end{aligned}$$

Therefore, under these assumptions, if $\psi_\beta = c \in \mathbb{R}\setminus\{0\}$ then $\rho_\alpha = \pm c$, and vice versa. $\square$

5. Conclusion

This study generalizes rectifying helices and establishes new geometric relationships through the framework of involute–evolute curves. Various special helices—including slant, clad, and g-clad helices—are characterized via their involutes and evolutes, highlighting their connections with spherical and rectifying helices. These results provide a foundation for further research. They contribute to the existing literature and suggest future research directions such as extending these concepts to higher-dimensional and alternative geometric settings like Minkowski space, as well as developing computational approaches for curve classification based on the proposed invariants.

References

[1] Altınbaş H, Mak M, Altunkaya B, Kula L. Mappings that transform helices from Euclidean space to Minkowski space. Hacettepe Journal of Mathematics and Statistics 2023; 52 (4): 915-925. https://doi.org/10.15672/hujms.915138

[2] Altunkaya B, Aksoyak FK, Kula L, Aytekin C. On rectifying slant helices in Euclidean 3-space. Konuralp Journal of Mathematics 2016; 4 (2): 17-24.

[3] Altunkaya B, Kula L. On spacelike rectifying slant helices in Minkowski 3-space. Turkish Journal of Mathematics 2018; 41 (5): 1234–1245. https://doi.org/10.3906/mat-1705-109

[4] Altunkaya B, Kula L. On timelike rectifying slant helices in Minkowski 3-space. International Electronic Journal of Geometry 2018; 12 (2): 229–240. https://doi.org/10.36890/iejg.545073

[5] Altunkaya B. Slant helices that constructed from hyperspherical curves in the n-dimensional Euclidean space. International Electronic Journal of Geometry 2019; 12 (2): 1–10. https://doi.org/10.36890/iejg.585408

[6] Bottema O, Roth B. Theoretical Kinematics. Amsterdam, Netherlands: North-Holland, 1979.

[7] Chen BY. When does the position vector of a space curve always lie in its rectifying plane? The American Mathematical Monthly 2003; 110 (2): 147–152. https://doi.org/10.2307/3647775

[8] Dickerson RE. The DNA helix and how it is read. Scientific American 1983; 249 (6): 94–111. https://doi.org/10.1038/scientificamerican1283-94

[9] Fuchs D. Evolutes and involutes of spatial curves. The American Mathematical Monthly 2013; 120 (3): 217–231. https://doi.org/10.4169/amer.math.monthly.120.03.217

[10] Izumiya S, Takeuchi N. New special curves and developable surfaces. Turkish Journal of Mathematics 2004; 28 (2): Article 6.

[11] Jianu M, Achimescu S, Däug L, Mihai A, Roman OA et al. Characterization of rectifying curves by their involutes and evolutes. Mathematics 2021; 9 (23): 3077. https://doi.org/10.3390/math9233077

[12] Kessler DA, Rabin Y. Stretching instability of helical springs. Physical Review Letters 2003; 90 (2): 024301. https://doi.org/10.1103/PhysRevLett.90.024301

[13] Kreyszig E. Differential Geometry. New York, NY, USA: Dover Publications, 1991.

[14] Kula L, Yayli Y. On slant helix and its spherical indicatrix. Applied Mathematics and Computation 2005; 169 (1): 600–607. https://doi.org/10.1016/j.amc.2004.09.078

[15] Lu XJ, Olson WK. 3DNA: A versatile toolkit for the analysis of 3D nucleic acid structures. Nature Protocols 2008; 3 (7): 1213–1227. https://doi.org/10.1038/nprot.2008.104

[16] Sabuncuoğlu A. Diferensiyel Geometri. 4th ed. Ankara, Türkiye: Nobel Publications, 2014 (in Turkish).

[17] Struik DJ. Lectures on Classical Differential Geometry: Second Edition. New York, NY, USA: Dover Publications, 1988.

[18] Takahashi T. The generalization of helices. Caspian Journal of Mathematical Sciences 2019; 8 (2): 178–195. https://doi.org/10.22080/cjms.2019.14595.1356

[19] Takahashi T, Takeuchi N. Clad helices and developable surfaces. Bulletin of Tokyo Gakugei University, Division of Natural Sciences 2014; 66: 1–9. https://u-gakugei.repo.nii.ac.jp/records/32153

[20] Wang Y, Pei D, Gao R. Generic properties of framed rectifying curves. Mathematics 2019; 7 (1):37. https://doi.org/10.3390/math7010037

[21] Watson JD, Crick FH. Molecular structure of nucleic acids: A structure for deoxyribose nucleic acid. Nature 1953; 171 (4356): 737–738. https://doi.org/10.1038/171737a0