Analysis Math., **42 (2)** (2016), 165–172
DOI: 10.1007/s10476-016-0205-y

INCLUSION RESULTS ON STATISTICAL CLUSTER POINTS

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(Received June 29, 2015; revised February 2, 2016; accepted February 6, 2016)

Abstract. We study the concepts of statistical cluster points and statistical core of a sequence for A_λ methods defined by deleting some rows from a nonnegative regular matrix A. We also relate A_λ-statistical convergence to A_μ-statistical convergence. Finally we give a consistency theorem for A-statistical convergence and deduce a core equality result.

1. Introduction

In [7] Fridy introduced the concepts of statistical limit points and statistical cluster points. Also Fridy and Orhan [9] gave the definitions of statistical limit superior and inferior along with the statistical core of a real sequence. Then in [10] they studied the statistical core for complex number sequences. In [4] Demirci extended these concepts by taking a nonnegative regular matrix A instead of Cesàro matrix. On the other hand Goffman and Petersen [11] introduced the submethod by deleting some rows from a matrix method. Then using this idea Armitage and Maddox [1] studied relationship between C_1, the Cesàro matrix of order one, and C_λ methods which are defined by deleting some rows from the Cesàro matrix C_1. Osikiewicz and Khan [16] extended this concept to a more general matrix method A. Later Demirci [5] studied the concepts of statistical cluster point, statistical core of a sequence for C_λ methods.

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Key words and phrases: A-density, A-statistical cluster point, A-statical core of a sequence.
Mathematics Subject Classification: 40A05, 26A03, 11B05.

In this note, by taking a nonnegative regular matrix A instead of the Cesàro matrix, we give extentions of the results given in [5] under weaker conditions. In Section 2 of this paper we refine [5, Theorem 2.1] by omitting a condition. Hence we obtain more general theorems for submethods. The theory about the notion of A-statistical convergence, which is a generalization of the usual notion of convergence, follows parallel lines of development to that of the usual theory of convergence. We also study the A-statistical variant of the result given for matrix summability methods by Petersen in [17]. Finally we relate A_λ-statistical convergence to A_μ-statistical convergence and give a consistency theorem for A-statistical convergence along with a core equality result.

First we recall some definitions. Let A be a nonnegative regular matrix and $K \subseteq \mathbb{N}$. If the limit

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk}$$

exists, then we say that the set K has A-density $\delta_A(K)$.

Definition 1. A real or complex number sequence $x = (x_k)$ is A-statistically convergent to the number L provided that for every $\varepsilon > 0$

$$\lim_n (A\chi_{K_\varepsilon}(k))_n = \lim_n \sum_{k\in K_\varepsilon} a_{nk} = 0$$

i.e., $\delta_A(K_\varepsilon) = 0$ where $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ ([2], [6], [13], [15]). If we take $A = C_1$ the Cesàro matrix, then the A-density δ_A is just the asymptotic density δ and A-statistical convergence is the statistical convergence. By st_A we denote the set of all A-statistically convergent sequences.

Definition 2. If for every $\varepsilon > 0$, $\delta_A(K_\varepsilon) \neq 0$, γ is called the A-statistical cluster point of x, where $K_\varepsilon = \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$. We denote the set of all A-statistical cluster points of x by $\Gamma_A(x)$, ([4]).

Definition 3. If there is some $M > 0$ such that

$$\delta_A(\{k \in \mathbb{N} : |x_k| > M\}) = 0,$$

then, following [4] we say that the sequence x is A-statistically bounded.

On the other hand $\mathrm{st}_A\text{-}\limsup x$ and $\mathrm{st}_A\text{-}\liminf x$ are the greatest and least A-statistical cluster points of x, respectively. Also A-statistical boundedness implies that $\mathrm{st}_A\text{-}\limsup x$ and $\mathrm{st}_A\text{-}\liminf x$ are finite, ([4]).

For any complex sequence $x = (x_k)$ the A-statistical core of x is defined to be the set

$$\mathrm{st}_A\text{-}\operatorname{core}\{x\} = \bigcap_{H\in H(x)} H,$$

where $H(x)$ is the collection of all closed half-planes H that satisfy

$$\delta_A\big(\{k \in \mathbb{N} : x_k \in H\}\big) = 1$$

(see [4], [10]).

Remark 1. For every A-statistically bounded complex number sequence $x = (x_k)$,

$$\mathrm{st}_A\text{-}\operatorname{core}\{x\} = \bigcap_{z\in\mathbb{C}} B_x(z),$$

where

$$B_x(z) = \Big\{\omega \in \mathbb{C} : |\omega - z| \le \mathrm{st}_A\text{-}\limsup_k |x_k - z|\Big\}$$

(see [3]). If we take $A = C_1$, we obtain the results presented in [9], [10].

2. A_λ-statistical cluster points

Goffman and Petersen [11] introduced the submethod by deleting some rows from a matrix method. In [1] and [5] especially C_λ has been studied. In this section we consider the matrix A_λ which is obtained by deleting some rows from a nonnegative regular matrix A. The results presented here are extensions of those given in [5].

By A_λ we denote the method which is defined by deleting some rows from a nonnegative regular matrix A.

Let $\{\lambda(n)\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. The A_λ method is defined as follows:

$$(A_\lambda x)_n = \sum_{k=1}^{\infty} a_{\lambda(n),k}\, x_k.$$

For every λ, it can be easily seen that A_λ is regular. If we take $A = A_\lambda$, in Definition 2 we can obtain the A_λ-statistical cluster points.

We now present the main theorem in this section which gives a relationship between δ_{A_λ} and δ_{A_μ}.

Theorem 1. *Let A be a nonnegative regular summability matrix, and let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$. If $\delta_A(E \setminus F) = 0$ then for any subset $K \subseteq \mathbb{N}$*

$$\delta_{A_\lambda}(K) = 0 \quad \textit{implies} \quad \delta_{A_\mu}(K) = 0.$$

Proof. Let $H = E \cup F$ and $E \setminus F = \{\sigma(n) : n \in \mathbb{N}\}$. Observe that $H = (E \setminus F) \cup F$. Since $\delta_A(E \setminus F) = 0$, it follows that

$$\delta_{A_\lambda}(K) = \lim_n \sum_{k \in K} a_{\lambda(n),k} = \lim_n \left(\sum_{k \in K} a_{\sigma(n),k} + \sum_{k \in K} a_{\lambda(n),k} \right) \geq \lim_n \sum_{k \in K} a_{\mu(n),k}.$$

Then we immediately get that $\delta_{A_\mu}(K) = \lim_n (A\chi_K)_{\mu(n)} = 0$. □

Corollary 1. *Let A be a nonnegative regular summability matrix, and let $F = \{\lambda(n)\}$ be infinite subset of $\mathbb{N}$. If $\delta_A(F) = 1$, then for any subset $K \subseteq \mathbb{N}$*

$$\delta_{A_\lambda}(K) = 0 \quad \textit{implies} \quad \delta_A(K) = 0.$$

In Theorem 1 the case in which $A = C_1$, the Cesàro matrix of order one, yields the following

Corollary 2. *Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$ and $\delta(E \setminus F) = 0$ then*

$$\delta_{C_\lambda}(K) = 0 \quad \textit{implies} \quad \delta_{C_\mu}(K) = 0$$

for any subset $K \subseteq \mathbb{N}$.

Hence we obtain the main theorem of [5] as a special case of Corollary 2.

One may ask if the converse of Theorem 1 holds. Namely, if $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ are infinite subsets of $\mathbb{N}$ and

$$\delta_{A_\lambda}(K) = 0 \quad \text{implies} \quad \delta_{A_\mu}(K) = 0$$

then can one have $\delta_A(E \setminus F) = 0$? The answer is "no" which is shown by the following example.

Example 1. Define the matrix $A = (a_{nk})$ by $a_{2n-1,k} = a_{2n,k} = \frac{1}{n}$, if $1 \leq k \leq n$, and $a_{nk} = 0$ otherwise. Observe that A is a nonnegative regular matrix. Let $F = \{\lambda(n)\} = \{(2n+1)\}_{n \in \mathbb{N}}$ and $E = \{\mu(n)\} = \{(2n)\}_{n \in \mathbb{N}}$. Then $A_\lambda = C_1$ and $A_\mu = C_1$, the Cesàro matrix. If $K \subseteq \mathbb{N}$ then $\delta_{A_\lambda}(K) = 0$ if and only if $\delta_{A_\mu}(K) = 0$, yet $\delta_A(E \setminus F) = \frac{1}{2} \neq 0$.

Let us denote the symmetric difference by $E \triangle F = (E \setminus F) \cup (F \setminus E)$. Hence the next result follows immediately.

Theorem 2. *Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$.*
(i) *If $\delta_A(E \setminus F) = 0$, then $\Gamma_{A_\mu}(x) \subseteq \Gamma_{A_\lambda}(x)$,*
(ii) *If $\delta_A(E \triangle F) = 0$, then $\Gamma_{A_\lambda}(x) = \Gamma_{A_\mu}(x)$.*

We note that for an A-statistically bounded sequence x, $\Gamma_{A_\lambda}(x)$ is singleton if and only if x is A_λ-statistically convergent, i.e.,

$$\Gamma_{A_\lambda}(x) = \{L\} \quad \text{if and only if} \quad \text{st}_{A_\lambda}\text{-}\lim x = L$$

(see [14]). Eventhough Li and Fridy prove this proposition for the case $A = C_1$, the proof also works for a general nonnegative regular matrix A. Hence Theorem 2(ii) yields the following important result.

Corollary 3. *Let* $F = \{\lambda(n)\}$ *and* $E = \{\mu(n)\}$ *be infinite subsets of* $\mathbb{N}$. *If* $\delta_A(E \triangle F) = 0$, *then for an* A*-statistically bounded sequence* $x = (x_n)$, *we have*

$$\mathrm{st}_{A_\lambda}\text{-}\lim x = L \quad \textit{if and only if} \quad \mathrm{st}_{A_\mu}\text{-}\lim x = L.$$

Note that Demirci [5] proved that if $E \triangle F$ is finite and

$$\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0 \tag{1}$$

then $\Gamma_{C_\lambda}(x) = \Gamma_{C_\mu}(x)$ where C is the Cesàro matrix.

However our Theorem 2 shows that condition (1) is superfluous and the condition "$E \triangle F$ is finite" is replaced by the weaker condition "$\delta_A(E \triangle F) = 0$".

It follows from Theorem 2(i) that every bounded sequence $x = (x_k)$ of complex numbers we have

$$\mathrm{st}_{A_\mu}\text{-}\limsup |x| \leq \mathrm{st}_{A_\lambda}\text{-}\limsup |x|.$$

This implies that

$$\bigcap_{z\in\mathbb{C}} \Big\{\omega \in \mathbb{C} : |\omega - z| \leq \mathrm{st}_{A_\mu}\text{-}\limsup_k |x_k - z|\Big\}$$
$$\subseteq \bigcap_{z\in\mathbb{C}} \Big\{\omega \in \mathbb{C} : |\omega - z| \leq \mathrm{st}_{A_\lambda}\text{-}\limsup_k |x_k - z|\Big\},$$

hence more precisely we have the following.

$$\mathrm{st}_{A_\mu}\text{-}\mathrm{core}\{x\} \subseteq \mathrm{st}_{A_\lambda}\text{-}\mathrm{core}\{x\}.$$

So we have proved the following

Theorem 3. *Let* $F = \{\lambda(n)\}$ *and* $E = \{\mu(n)\}$ *be infinite subsets of* $\mathbb{N}$.
(i) *If* $\delta_A(E \setminus F) = 0$ *then* $\mathrm{st}_{A_\mu}\text{-}\mathrm{core}\{x\} \subseteq \mathrm{st}_{A_\lambda}\text{-}\mathrm{core}\{x\}$,
(ii) *If* $\delta_A(E \triangle F) = 0$ *then* $\mathrm{st}_{A_\mu}\text{-}\mathrm{core}\{x\} = \mathrm{st}_{A_\lambda}\text{-}\mathrm{core}\{x\}$.

3. *A*-statistical inclusion

In [8] Fridy and Khan introduced the concept of statistical consistency. Later on Demirci [5] examined this concept.

The main object of this section is to give an A-statistical consistency theorem, which is the A-statistically convergent version of bounded consistency theorem, and study conditions for which the equality $\mathrm{st}_A\text{-}\operatorname{core}\{x\} = \mathrm{st}_B\text{-}\operatorname{core}\{x\}$ holds.

Now recall the following definitions.

Definition 4. Let A and B be nonnegative regular matrices. If $\mathrm{st}_A \supset \mathrm{st}_B$, it is said to be that A is stronger than B in the statistical convergence sense.

Definition 5. Let A and B be nonnegative regular matrices. If for every $x \in \mathrm{st}_A \cap \mathrm{st}_B$, $\mathrm{st}_A\text{-}\lim x = \mathrm{st}_B\text{-}\lim x$ then matrices A and B are called consistent in the statistical convergence sense. If A is stronger than B in the statistical convergence sense and consistent with B in the statistical convergence sense, then we write $A \overset{\mathrm{st}}{\supset} B$ (see [8]). If $A \overset{\mathrm{st}}{\supset} B$ and $B \overset{\mathrm{st}}{\supset} A$, then it is said that A and B are equivalent in the statistical convergence sense and denoted by $A \overset{\mathrm{st}}{\sim} B$.

Lemma 1. *Let B be a nonnegative regular matrix. If $\delta_B(\mathbb{N} \setminus N) = 1$ then there is a sequence (z_n) defined for $n \in \mathbb{N} \setminus N$ such that any extension of (z_n) to all of $\mathbb{N}$ is not in st_B.*

Proof. Since B is nonnegative, regular and $\delta_B(S) = 1$ where $S = \mathbb{N} \setminus N$, there exist n_1 and m_1 such that

$$\sum \big[b_{n_1 k} : k \le m_1 \text{ and } k \in S \big] > \frac{9}{10}\,.$$

Define $z_k = 0$, for $k \le m_1$ and $k \in S$. There also exist $n_2 > n_1$ and $m_2 > m_1$ such that

$$\sum \big[b_{n_2 k} : m_1 < k \le m_2 \text{ and } k \in S \big] > \frac{9}{10}\,.$$

Define $z_k = 1$, for $m_1 < k \le m_2$ and $k \in S$. By continuing this process and alternating the sequence between 0 and 1, we obtain a partial sequence (z_k) defined on $I_1 \cup I_2 \cup \cdots$, where $I_1 = \{k : k \le m_1 \text{ and } k \in S\}$, $I_2 = \{k : m_1 < k \le m_2 \text{ and } k \in S\}$. Then any sequence extending this partial sequence to $\mathbb{N}$ will not be B-statistically convergent. □

The next result is a statistical version of Petersen [17, Theorem 1].

Theorem 4. *Let A and B be nonnegative regular matrices and $x = (x_k)$ be a sequence such that*

$$\mathrm{st}_A\text{-}\lim x \ne \mathrm{st}_B\text{-}\lim x. \tag{2}$$

Then there is a sequence $y = (y_k)$ so that $(y_k) \in \mathrm{st}_A$ and $(y_k) \notin \mathrm{st}_B$, (i.e., $(y_k) \in st_A \setminus st_B$).

PROOF. Let $\mathrm{st}_A\text{-}\lim x = a$ and $\mathrm{st}_B\text{-}\lim x = b$ such that $c = |b-a| \neq 0$. Since $\mathrm{st}_A\text{-}\lim x = a$ there exist increasing sequences (n_i) and (m_i) such that

$$\sum\Big[a_{nk} : k \le m_1 \text{ and } |x_k - a| < \frac{c}{10}\Big] > \frac{9}{10}, \quad \text{for } n_1 \le n < n_2,$$
$$\sum\Big[a_{nk} : k \le m_2 \text{ and } |x_k - a| < \frac{c}{100}\Big] > \frac{99}{100}, \quad \text{for } n_2 \le n < n_3,$$
$$\sum\Big[a_{nk} : k \le m_3 \text{ and } |x_k - a| < \frac{c}{1000}\Big] > \frac{999}{1000}, \quad \text{for } n_3 \le n < n_4,$$
$$\dots .$$

Let $N = N_1 \cup N_2 \cup \cdots$, where $N_1 = \{k \le m_1 : |x_k - a| < \frac{c}{10}\}$, $N_2 = \{k \le m_2 : |x_k - a| < \frac{c}{100}\}, \dots$. Suppose $y = (y_k)$ is any sequence satisfying

$$y_k = x_k, \quad k \in N.$$

This implies $\mathrm{st}_A\text{-}\lim y = a$. First note that $\delta_B(\mathbb{N} \setminus N) = 1$. Then by the above lemma define y_k on for $k \in \mathbb{N} \setminus N$ such that $(y_k) \notin \mathrm{st}_B$. ☐

THEOREM 5. *Let A and B be nonnegative regular matrices.*

(i) *If* $\mathrm{st}_A \subset \mathrm{st}_B$, *then* $A \overset{\mathrm{st}}{\supset} B$,

(ii) *If* $\mathrm{st}_A = \mathrm{st}_B$, *then* $A \overset{\mathrm{st}}{\sim} B$.

THEOREM 6. *Let A and B be nonnegative regular matrices. If* $\mathrm{st}_A \subset \mathrm{st}_B$, *then for every* $x \in \mathrm{st}_A \cap \mathrm{st}_B$ *we have*

(i) $\Gamma_A(x) = \Gamma_B(x)$,

(ii) $\mathrm{st}_A\text{-}\mathrm{core}\{x\} = \mathrm{st}_B\text{-}\mathrm{core}\{x\}$.

The next consistency result is easily deduced from Theorem 6.

COROLLARY 4. *If* $\mathrm{st}_A \subset \mathrm{st}_B$, *then for every* $x \in \mathrm{st}_A \cap \mathrm{st}_B$ *we have*

$$\mathrm{st}_A\text{-}\lim x = L \quad \textit{if and only if} \quad \mathrm{st}_B\text{-}\lim x = L.$$

A direct proof of Corollary 4 could also be obtained from [3, Proposition 4(2)].

We finally remark that some further results on statistical consistency may be found in [3] and [12].

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