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ABSTRACT
In this paper, a new uncertainty modelling concept called strait fuzzy set is introduced, which brings new perspectives to both theoretical and practical advances in fuzzy mathematics. This set type allows objects/points to be graded with fuzzy membership intervals that are partitions of [0,1] (so that the union of these partitions is [0,1] and their intersection is empty) instead of fuzzy membership degrees represented as exact values in [0,1]. Moreover, some basic operations and properties of strait fuzzy sets are studied in detail. The concept of strait fuzzy rough set is put forward and its theoretical aspects are discussed. In addition, two different similarity approaches are proposed for strait fuzzy sets and strait fuzzy rough sets, and applied to measure the similarity rates of vaccines against influenza viruses.

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1. Introduction

In information science, information is sometimes imprecise or vaguely organised and has a level of granularity. In the twentieth century, fuzzy set theory (Zadeh, 1965), rough set theory (Pawlak, 1982), soft set theory (Molodtsov, 1999) were developed to address these issues of the knowledge system. Fuzzy logic is a form of multi-valued logic in which the true value of variables can be any real number (exact value) between 0 and 1, both being inclusive. Fuzzy systems as a subject was developed for modelling the uncertainty, ambiguity and vagueness present in the human thought process. Fuzzy set is a set in which each element is associated with value, which is between 0 to 1 based on the certainty. Basically it allows partial membership which means that it contain elements that have varying degrees of membership in the set. The use of fuzzy set theory, which extends the original set theory and gradually expands its application areas, to solve a certain complex problem has attracted the attention of many researchers in worldwide. Relatedly, the fuzzy set operations (Dubois & Prade, 1978; Mizumoto & Tanaka, 1981; Wu, 2020), fuzzy aggregation operators (Mardani et al., 2018; Sasikala & Petrou, 2001), fuzzy graph structures (Gani et al., 2016; Sitara et al., 2019) and fuzzy matrix forms (Kamacı et al., 2018a, 2018b; Pal, 2016; Petchimuthu et al., 2020; Petchimuthu & Kamacı, 2022) are studied in detail. But there are situations when assigning an exact number to an expert/decision maker opinion is too restrictive, and the assignment of an interval of values is more realistic. Therefore, there is an extensive theory of fuzzy sets, called an interval valued fuzzy set (Sambuc, 1975). In many real-life scenarios, these sets can model imperfect information better than fuzzy sets. Akram and Dudek (2011), Gehrke et al. (2001), Kamacı (2019), Peng and Garg (2016) developed some interval valued fuzzy structures and then studied some desirable properties of these structures. On the other hand, the rough set theory, introduced by Pawlak in the year 1982, is used as a mathematical tool for a perusal of uncertain, unsettled, and vagueness of imprecise data in information systems (Alisantoso & Khoo, 2009; Pawlak, 2002; Pawlak & Skowron, 2007). The rough set theory was applied in several fields including medical informatics, image processing, pattern recognition, data mining, expert systems and knowledge discovery. In the current literature, several research works were combined the rough set theory with other artificial intelligence methods

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such as neural networks (Ji et al., 2021; Pourpanah et al., 2021), fuzzy logic (Riaz & Hashmi, 2020; Zeng et al., 2019), additionally to other methods resulting in some good results (Kamacı, 2021a, 2021b; Karaaslan & Çagman, 2018). Mafarja and Abdullah (2015) and Shidpour et al. (2016) focussed on the combined form of rough set and fuzzy set. Mardani et al. (2017) and Yuan et al. (2021) systematically reviewed many studies that proposed or developed fuzzy rough sets theory. Besides, there exists quite a body of research work on applications of these set theories.

Similarity measure is a significant mean for measuring the uncertain information. Similarity measures are very useful in some areas, such as market prediction, pattern recognition, decision making, machine learning and medical diagnosis. Therefore, they have gained attention from researchers for their wide applications in real world. The fuzzy similarity measure is a measure that depicts the closeness among fuzzy sets. Different similarity measures in the literature were elucidated, and a comparative study between different pairs of fuzzy sets was presented. Some authors systematically gave the axiom definitions of similarity measures of fuzzy sets and discussed their basic properties (Liu, 1992; Wang, 1997; Wang et al., 1995). Beg and Ashraf (2009) proposed a set of axioms that a value in the range [0, 1] shall satisfy to be the rate or measure of similarity between two fuzzy subsets of a given universe. Omran and Hassaballah (2007) studied the summation and multiplication on the similarity measures of fuzzy sets. Kang et al. (2018) proposed new hesitation-based similarity and distance measures on fuzzy sets. Ye and Wu (2009) focussed on the similarity measures of fuzzy rough sets determined by a triangular norm. Qi and Chengyi (2008) suggested some rules for determining the degree of similarity between two fuzzy rough sets. Li et al. (2015) proposed three different approaches to measure similarity between two interval-valued fuzzy sets. Deng et al. (2017) described monotonic similarity measures on interval-valued fuzzy sets and investigated their monotonicity properties and applications. Presently, the studies on the similarity measures of fuzzy set theory, interval valued fuzzy set theory and fuzzy rough set theory are progressing rapidly.

Letter grade	Percentage
A	80 – 100
B	65 – 79
C	55 – 64
D	50 – 54
F	0 – 49

Figure 1. A figuration of academic grading (letter grade-percentage).

The prevailing concept of fuzzy set has numerous applications in various fields from real life. In some situations, this set theory is insufficient to represent the membership degrees of objects/points into a set. For example, in general, the percentile grading system is used in universities, but the letter grade is taken into account when calculating the student's GPA. In other words, students may have different fuzzy membership degrees when considering the percentile grading system, while they may have the same fuzzy membership degree when considering the letter grade (see Subsection 3.2 and Figure 1 for more details). The similar situations are encountered in many scenes of the real world. It would not be clear to express these situations by fuzzy variable. The main objective of this paper is to fill this research gap with the novel concept of strait fuzzy set. In this set theory, the objects/points are graded by fuzzy membership intervals that are partitions of [0,1] instead of fuzzy membership degrees represented as exact values in [0,1]. The strait fuzzy set is different from interval valued fuzzy set and is a new type of fuzzy set developed with a different perspective. Further, this study is an excellent introduction to the fundamentals of strait fuzzy rough sets. Also, it introduces the similarity measures of strait fuzzy sets and strait fuzzy rough sets. These measures are illustrated with the behaviour of vaccines against influenza viruses.

The rest of this paper has been split into the sections such as: Section 2 presents several basic explanations and operations on fuzzy sets, which are very useful for developing the main results of this paper. Section 3 introduces the concept of strait fuzzy set and discusses its real-life motivation and operations. Section 4 procures the idea of similarity measure for strait fuzzy sets. Section 5 presents an algorithm based on the proposed similarity measure of strait fuzzy sets and its application to determine the similarity of vaccines against influenza viruses. Section 6 gives the definition of strait fuzzy rough set and formulates several interesting results. It also proposes a similarity approach for strait fuzzy rough sets. Section 7 concludes the work done in this paper.

2. Preliminaries

In this section, we briefly review the fundamental concepts related to the fuzzy set and interval-valued fuzzy set.

Definition 2.1 (Zadeh, 1965): Let $\mathcal{A} \neq \emptyset$ be a universal set, then a fuzzy set Ψ on $\mathcal{A}$ is the set of pairs $\Psi = \{(x, \psi(x)) |\ x \in \mathcal{A}\}$, where $\psi : \mathcal{A} \to [0, 1]$ is the membership function of the fuzzy set.

The set of all fuzzy sets on $\mathcal{A}$ will be denoted by $\widetilde{F}(\mathcal{A})$. Some of the fuzzy set operations are given by the following:

Definition 2.2 (Zadeh, 1965): Let $\Psi, \Theta \in \widetilde{F}(\mathcal{A})$.

(a) If $\psi(x) \leq \theta(x)$ for all $x \in \mathcal{A}$, then Ψ is said to be contained in Θ, and denoted by $\Psi \subseteq \Theta$.
(b) The intersection of Ψ and Θ, is defined as $(\psi \cap \theta)(x) = \psi(x) \wedge \theta(x) = \min\{\psi(x), \theta(x)\}$, for all $x \in \mathcal{A}$.
(c) The union of Ψ and Θ, is defined as $(\psi \cup \theta)(x) = \psi(x) \vee \theta(x) = \max\{\psi(x), \theta(x)\}$, for all $x \in \mathcal{A}$.
(d) The complement of Ψ, denoted by Ψ^c, is defined as $\psi^c(x) = 1 - \psi(x)$ for all $x \in \mathcal{A}$.

Definition 2.3 (Luo, 1989): Let $\Psi \in \widetilde{F}(\mathcal{A})$. Given a number $t \in [0, 1]$, a t-level set (or t-cut) of a fuzzy set Ψ is a crisp subset of $\mathcal{A}$ defined by

$$\Psi_t = \{x \in \mathcal{A} \mid \psi(x) \geq t\}.$$

Definition 2.4 (Sambuc, 1975): Let $\mathcal{A} \neq \emptyset$ be a universal set, then an interval-valued fuzzy set (IVFS) Ψ_I on $\mathcal{A}$ is defined as $\Psi_I = \{[\psi^-(x), \psi^+(x)] | x \in \mathcal{A}\}$, in which $[\psi^-(x), \psi^+(x)] \subseteq [0, 1]$ represents the membership degree such that $0 \leq \sup[\psi^-(x), \psi^+(x)] \leq 1$.

3. Strait fuzzy set

3.1. The construction of strait fuzzy set

From now on, $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$ is the set of all finite partitions of the interval $[0, 1]$. First, let's give the following ordering definition for comparison between intervals within the interval [0,1].

Definition 3.1: Let $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$ and $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$. If $sup Y_i = inf Y_j$ for $i, j \in \{1, 2, \ldots, r\}$, then Y_i and Y_j are called *consecutive intervals* in the partition α_k and we show this by $Y_i < Y_j$.

From now on, the intervals Y_i in α_k will be taken as ordered by $<$, i.e. $Y_1 < Y_2 < \cdots < Y_r$. Since $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\} \in \mathfrak{P}[0, 1]$, then $Y_i \cap Y_j = \emptyset$ for all $i, j \in \{1, 2, \ldots, r\}$ such that $i \neq j$ and $\bigcup_{i\in\{1,2,\ldots,r\}} Y_i = [0, 1]$.

Definition 3.2: Let $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$, $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$ and let Ψ be a fuzzy set on $\mathcal{A}$. If for each $Y_a \in \alpha_k$ $(a = 1, 2, \ldots, r)$, there exists at least one $x \in \mathcal{A}$ such that $\psi(x) \in Y_a$, then a strait fuzzy set generated from Ψ using α_k is denoted by $\Psi^{\mathfrak{s}}(\alpha_k)$ and defined as

$$\Psi^{\mathfrak{s}}(\alpha_k) = \{Y_a(x) \mid a \in \{1, 2, \ldots, r\}, \\ x \in \mathcal{A}, \psi(x) \in Y_a\}$$

where (x) represents the sequence/set of x's providing $\psi(x) \in Y_a$.

If it doesn't cause any confusion $\Psi^{\mathfrak{s}}(\alpha_k)$ will be denoted by $\Psi^{\mathfrak{s}}$.

As can be understood from Definition 3.2, many strait fuzzy sets can be produced for one fuzzy set. Already, one of the facilitating features of strait fuzzy sets is that the intervals can be adjusted for the situations required by the problem. As is known, a partition of a set U corresponds to an equivalence relation on U. Classical fuzzy sets require a direct evaluation with a value. In strait fuzzy sets, it is possible to evaluate using values that are equivalent to the value obtained through classical fuzzy sets.

Definition 3.2 may remind the concept of IVFS, but in an IVFS, if there exists only one interval as [0,1], then such an IVFS is also a strait fuzzy set, in all other cases strait fuzzy set and IVFS are different concepts, as the intervals in an IVFS will never provide a partition of [0,1]. In strait fuzzy sets, a value can belong to only one equivalence class of the range $[0, 1]$, allowing the use of values equivalent to this value. In addition, the fact that the partitions of [0,1] can be selected according to the problem provides an advantage over IVFS. Since the equivalence classes that make up the partitions of a set are discrete, strait fuzzy sets allow us to precisely determine which class a value belongs to. For example, in IVFS assume that Jone's grade in the range of [0.6,0.7], when the instructor's grade range is $[0.4, 0.65) : CD$ and $[0.65, 0.75) : CC$, Jone's grade in the range of [0.6,0.7] creates a problem. Is Jone's grade CC or CD? Strait fuzzy set is an approach to deal with such problems.

From now on, $SF(\Psi)$ denotes the set of all strait fuzzy sets corresponding a fuzzy set Ψ and $\widetilde{SF\mathcal{A}}$ denotes the set of all strait fuzzy sets on the set $\mathcal{A} = \{x_1, x_2, \dots, x_n\}$.

For a given fuzzy set $\Psi \in \widetilde{F}(\mathcal{A})$, there are some special partitions can be constructed:

Definition 3.3: Let $\mathcal{A} = \{x_1, x_2, \dots, x_n\}$ and $\Psi \in \widetilde{F}(\mathcal{A})$.

(a) If $\alpha_0 = \{Y_1\} \in \mathfrak{P}[0, 1]$, where $Y_1 = [0, 1]$, then the strait fuzzy set $\Psi^{\mathfrak{s}} \in SF(\Psi)$ such that

$$\Psi^{\mathfrak{s}} = \{[0, 1](x) \mid x \in \mathcal{A}, \psi(x) \in [0, 1]\}$$

is called a **trivial strait fuzzy set** of Ψ and denoted by $\Psi^{\mathfrak{s}_o}$. Hence, $\Psi^{\mathfrak{s}_o} = \{[0, 1](x_1, x_2, \dots, x_n)\}$.

(b) If $\psi(x_1) < \psi(x_2) < \cdots < \psi(x_n)$, then $\alpha_d = \{Y_1, Y_2, \dots, Y_n\} \in \mathfrak{P}[0, 1]$, where $Y_1 = [0, \psi(x_1)]$, $Y_2 = (\psi(x_1), \psi(x_2)]$, $Y_3 = (\psi(x_2), \psi(x_3)], \dots$, $Y_{n-1} = (\psi(x_{n-2}), \psi(x_{n-1})]$ and $Y_n = (\psi(x_{n-1}), 1]$. The strait fuzzy set $\Psi^{\mathfrak{s}} \in SF(\Psi)$ such that

$$\Psi^{\mathfrak{s}} = \{Y_a(x_a) \mid a \in \{1, 2, \dots, n\}, x_a \in \mathcal{A}, \psi(x_a) \in Y_a\}$$

is called a **discrete strait fuzzy set** of Ψ and denoted by $\Psi^{\mathfrak{s}_d}$. Here (x_a) consists of element x_a. Hence, $\Psi^{\mathfrak{s}_d} = \{(Y_1(x_1)), (Y_2(x_2)), \dots, (Y_n(x_n))\}$.

3.2. The authentic life motivation about strait fuzzy sets

In this part, we focus on the real-life examples of strait fuzzy sets.

(1) It is up to the administration of each university to determine how letter grades will be converted to percentages in higher education institutions. For example, an A might be con verted as 80–100 percent at any university, while for another university it may be converted to 85–100 percent. However, a letter grade is taken into account rather than percentage whilst calculating the students' GPA. Now let us examine the matching of a letter grade and percentage given in Figure 1.

From Figure 1, we assume that John's success score/grade in a course is 66. In the same course, Mary's success score/grade is 73. Since these success scores/grades are taken out of 100, in fuzzy logic, we can think of them as 0.66 and 0.73. However, since both scores correspond to the letter grade B, they will have the same effect on the students' GPA. This is an outstanding real-life example of the strait fuzzy set.

(2) There are several different blood tests a health care practitioner can order to determine if you have diabetes. The three most common tests are described as 'Fasting Blood Glucose Test', 'HbA1c Test' and 'Oral Glucose Tolerance Test'. The HbA1c (or A1C) test is a common blood test used to diagnose type 1 and type 2 diabetes. If you are living with diabetes, the test is also used to monitor how well you are managing blood sugar levels. Measures the average level of sugar in your blood over the previous two-three months. Figure 2 is given for the HbA1c Test (Source: https://www.aarp.org/ppi/initiatives/learn-how-to-prevent-diabetes/). By Figure 2, we can say that two individuals with HbA1c levels of 5% and 5.2% are 'Normal' when diabetes membership is considered. That is, it is appropriate to say that their blood sugar rates are at the same level. Other HbA1c levels can be interpreted similarly. Consequently, it will be more convincing and efficient to use the strait fuzzy set form instead of classification using fuzzy values in the diagnosis of diabetes.

(3) In Turkey's electoral system, deputies are not allocated to political parties and independent candidates that receive less votes than the total number of valid votes cast in an electoral district, divided by the number of deputies to be elected from that circle. In other words, a party below this percentage cannot have a deputy in that region or

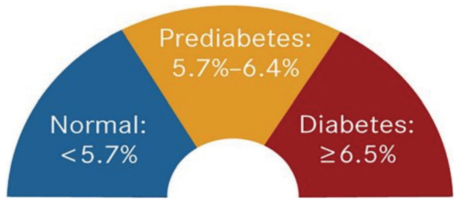

Figure 2. Diagnosis of diabetes using HbA1c test.

province. There are nearly 300,000 voters in Yozgat, a province in Turkey, and four deputies of the Grand National Assembly of Turkey are determined by the votes of these voters. In Figure 3, the 2015 (June) parliamentary election results for Yozgat are presented. In Figure 3, the names and logos/symbols of the political parties are not given, instead the political parties are represented by numbering. The following source for details of this figure can be checked. (Source: https://www.dailysabah.com/election/november-1-2015-general-elections/). In the 2015 parliamentary elections, for Yozgat, a party needs to receive about 20% of the total valid votes to have a deputy (i.e. this party needs to receive around 50,000 votes). By considering Figure 3, we can say that since Party 3, Party 4 and Others received less than 20% of the valid votes, they do not have any deputies and therefore it would be reasonable to consider them in the same class/standard regardless of their vote percentage. We can create a fuzzy set by considering the membership grades as the vote percentages (or numbers of votes) of the parties, but the most appropriate approach for the number of deputies based on the vote percentages is the strait fuzzy set.

The figures presented above are just a few examples that illustrate the authentic life motivation of the strait fuzzy sets. These examples can be reproduced as the election thresholds in national elections, the drug doses to be given to the patients, the irrigation and spraying intervals needed by the plants, the price range of a product and etc.

3.3. *Some basic operations of strait fuzzy sets*

In this part, we describe some basic operations on the strait fuzzy sets.

Definition 3.4: Let $\Psi^{\mathfrak{s}} \in SF(\Psi)$. The **complement** of $\Psi^{\mathfrak{s}}$ is defined as $(\Psi^{\mathfrak{s}})^c = (\Psi^c)^{\mathfrak{s}}$. That's mean if the strait fuzzy set

$$\Psi^{\mathfrak{s}} = \{Y_a(x) \mid a \in \{1,2,\ldots,r\},\\ x \in \mathcal{A}, \psi(x) \in Y_a\} \in SF(\Psi)$$

then the complement of $\Psi^{\mathfrak{s}}$ is defined as

$$(\Psi^{\mathfrak{s}})^c = \{Y_a(x) \mid a \in \{1,2,\ldots,r\},\\ x \in \mathcal{A}, \psi^c(x) \in Y_a\} \in SF(\Psi),$$

where $\mathfrak{P}[0,1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$, $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$.

Definition 3.5: Let $\Psi^{\mathfrak{s}} = \{Y_a(x) \mid a \in \{1,2,\ldots,r\}, x \in A, \psi(x) \in Y_a\}$ and $\Theta^{\mathfrak{s}} = \{W_b(x) \mid b \in \{1,2,\ldots,t\}, x \in A, \theta(x) \in W_b\}$, where $\mathfrak{P}[0,1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$, $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$, $\alpha_l = \{W_1, W_2, \ldots, W_t\}$. If

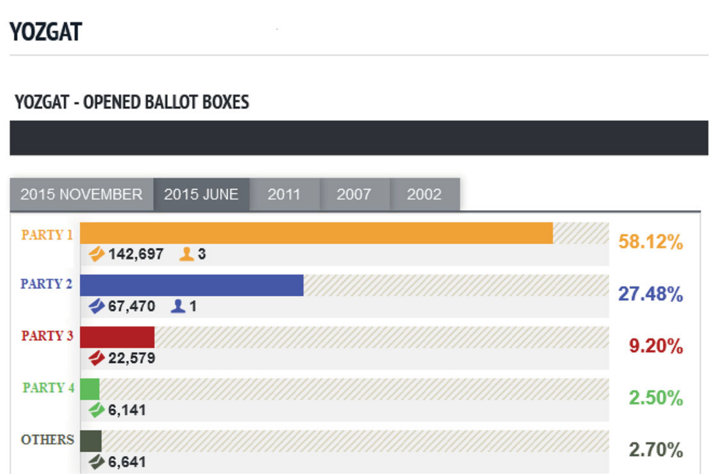

Figure 3. General Election Result for Yozgat in 2015 (June).

Table 1. The fuzzy sets Ψ, Θ and Ω (i.e. $\psi(x)$, $\theta(x)$ and $\omega(x)$) on the set $\mathcal{A}$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\psi(x)$	0.3	0.38	0.1	0.2	0.8	0.75	0.9
$\theta(x)$	0.46	0.4	0.2	0.36	0.85	0.68	0.8
$\omega(x)$	0.34	0.43	0.18	0.65	0.85	0.9	0.92
$\psi^c(x)$	0.7	0.62	0.9	0.8	0.2	0.25	0.1

for all $x \in A$, there exist a $b \in \{1, 2, \ldots, t\}$ such that $\psi(x) \in Y_a \subseteq W_b$, then $\Psi^{\mathfrak{s}}$ is said to be **strait fuzzy subset** of $\Theta^{\mathfrak{s}}$ and denoted by $\Psi^{\mathfrak{s}} \subseteq_s \Theta^{\mathfrak{s}}$. If $\Psi^{\mathfrak{s}} \subseteq_s \Theta^{\mathfrak{s}}$ and $\Theta^{\mathfrak{s}} \subseteq_s \Psi^{\mathfrak{s}}$, then $\Psi^{\mathfrak{s}}$ and $\Theta^{\mathfrak{s}}$ are called **strait equal fuzzy sets** and denoted by $\Psi^{\mathfrak{s}} =_s \Theta^{\mathfrak{s}}$.

Theorem 3.6: *Let $\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}} \in \widetilde{SF\mathcal{A}}$.*

(i) *Strait fuzzy equality $=_s$ is an equivalance relation on $\widetilde{SF\mathcal{A}}$.*
(ii) *$\Psi^{\mathfrak{s}} \subseteq_s \Psi^{\mathfrak{s}_o}$ and in particular $\Psi^{\mathfrak{s}_d} \subseteq_s \Psi^{\mathfrak{s}_o}$.*
(iii) *$\Psi^{\mathfrak{s}_o} =_s \Theta^{\mathfrak{s}_o}$. In particular $\Psi^{\mathfrak{s}_o} =_s (\Psi^c)^{\mathfrak{s}_o}$.*
(iv) *Two equality expressions $\Psi = \Theta$ and $\Psi^{\mathfrak{s}} =_s \Theta^{\mathfrak{s}}$ do not require each other.*

Proof: By Definitions 3.3, 3.4 and 3.5, the proofs of (i), (ii) and (iii) are straitforward, hence omitted. For the remain of the proof, we have Example 3.7. ■

There is generally no direct relationship between fuzzy subsets and strait fuzzy subsets. In general, even if Ψ is not a fuzzy subset of Θ, the corresponding strait fuzzy sets $\Psi^{\mathfrak{s}}$ can be a strait fuzzy subset of $\Theta^{\mathfrak{s}}$. On the other hand, even if Ψ is a fuzzy subset of Θ, the corresponding strait fuzzy sets $\Psi^{\mathfrak{s}}$ may not be a strait fuzzy subset of $\Theta^{\mathfrak{s}}$.

Example 3.7: Three fuzzy sets Ψ, Θ and Ω on the set $\mathcal{A} = \{x_1, x_2, \ldots, x_7\}$ are given as in Table 1.

It is seen that Ψ is a fuzzy subset of Ω, but Ψ is not a fuzzy subset of Θ, since $\psi(x_6) \nleq \theta(x_6)$.

Assume that

$$\alpha_1 = \{Y_1 = [0, 0.2), Y_2 = [0.2, 0.4),$$
$$Y_3 = [0.4, 0.8), Y_4 = [0.8, 1]\},$$
$$\alpha_2 = \{W_1 = [0, 0.8), W_2 = [0.8, 1]\} \text{ and}$$
$$\alpha_3 = \{T_1 = [0, 0.3), T_2 = [0.3, 0.5),$$
$$T_3 = [0.5, 0.9), T_4 = [0.9, 1]\}$$

are the corresponding partitions for the fuzzy sets Ψ, Θ and Ω, respectively. Then, we obtain the following strait fuzzy sets:

$$\Psi^{\mathfrak{s}} = \{Y_1 = [0, 0.2)(x_3),$$
$$Y_2 = [0.2, 0.4)(x_1, x_2, x_4), Y_3 = [0.4, 0.8)(x_6),$$
$$Y_4 = [0.8, 1](x_5, x_7)\},$$
$$\Theta^{\mathfrak{s}} = \{W_1 = [0, 0.8)(x_1, x_2, x_3, x_4, x_6),$$
$$W_2 = [0.8, 1](x_5, x_7)\} \text{ and}$$
$$\Omega^{\mathfrak{s}} = \{T_1 = [0, 0.3)(x_3), T_2 = [0.3, 0.5)(x_1, x_2),$$
$$T_3 = [0.5, 0.9)(x_4, x_5), T_4 = [0.9, 1](x_6, x_7)\}$$

It is seen that $\Psi^{\mathfrak{s}} \subseteq_s \Theta^{\mathfrak{s}}$, but $\Psi^{\mathfrak{s}}$ is not a strait fuzzy subset of $\Omega^{\mathfrak{s}}$, since $\psi(x_1) \in Y_2 \nsubseteq T_i$ for all $i \in \{1, 2, 3, 4\}$.

Furthermore, the complement of $\Psi^{\mathfrak{s}}$ is the set

$$(\Psi^{\mathfrak{s}})^c = \{Y_1 = [0, 0.2)(x_7), Y_2 = [0.2, 0.4)(x_5, x_6),$$
$$Y_3 = [0.4, 0.8)(x_1, x_2), Y_4 = [0.8, 1](x_3, x_4)\}.$$

For the proof of Theorem 3.6, firstly, consider the strait fuzzy set $\Psi^{\mathfrak{s}}(\alpha_3) = \{T_1(x_3, x_4), T_2(x_1, x_2), T_3(x_5, x_6), T_4(x_7)\}$. Then it is seen that $\Psi^{\mathfrak{s}}(\alpha_1) \neq_s \Psi^{\mathfrak{s}}(\alpha_3)$. Therefore, the fuzzy equality does not require strait fuzzy equality. Secondly, let $\alpha_4 = \{U_1 = [0, 0.7), U_2 = [0.7, 1]\}$. Then $\Psi^{\mathfrak{s}}(\alpha_4) = \{U_1(x_1, x_2, x_3, x_4), U_2(x_5, x_6, x_7)\}$ and $\Omega^{\mathfrak{s}}(\alpha_4) = \{U_1(x_1, x_2, x_3, x_4), U_2(x_5, x_6, x_7)\}$. It is seen that $\Psi^{\mathfrak{s}}(\alpha_4) =_s \Omega^{\mathfrak{s}}(\alpha_4)$, but $\Psi \neq \Omega$. Therefore the strait fuzzy equality does not require the fuzzy equality.

The theorem below shows the existence of strait fuzzy sets corresponding to the intersections and unions of fuzzy sets and how to obtain them.

Theorem 3.8 (A construction method of partitions): *Let $\Psi^{\mathfrak{s}} \in SF(\Psi)$ and $\Theta^{\mathfrak{s}} \in SF(\Theta)$ over $\mathcal{A}$. Then there exists at least a strait fuzzy set $\Omega^{\mathfrak{s}}$ (or $\Gamma^{\mathfrak{s}}$), where Ω (or Γ) is a fuzzy set over $\mathcal{A}$ such that $\Omega = \Psi \cap \Theta$ (or $\Gamma = \Psi \cup \Theta$).*

Proof: Let $\Psi^{\mathfrak{s}} = \{Y_a(x) |\ a \in \{1, 2, \ldots, r\}, x \in \mathcal{A}, \psi(x) \in Y_a\}$ and $\Theta^{\mathfrak{s}} = \{W_b(x) | b \in \{1, 2, \ldots, t\}, x \in \mathcal{A}, \theta(x) \in W_b\}$, where $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$, $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$, $\alpha_l = \{W_1, W_2, \ldots, W_t\}$ such that $Y_1 < Y_2 < \cdots < Y_r$ and $W_1 < W_2 < \cdots < W_t$.

Construction method of partitions:

Step 1. Determining the partition set for both intersection and union (or any other fuzzy set operations) is the same. Firstly, consider the intervals $K_{ij} = Y_i \cap W_j$, then the set

$$P(YW) = \{K_{ij} | \ i \in \{1, 2, \ldots, r\}, j \in \{1, 2, \ldots, t\}, \\ K_{ij} \neq \emptyset\}$$

is a partition of $[0, 1]$. To prove this, we have three situations:

(1) If $j \neq p$, then $K_{ij} \cap K_{ip} = (Y_i \cap W_j) \cap (Y_i \cap W_p) = Y_i \cap (W_j \cap W_p) = Y_i \cap \emptyset = \emptyset$.
(2) If $i \neq p$, then $K_{ij} \cap K_{pj} = (Y_i \cap W_j) \cap (Y_p \cap W_j) = (Y_i \cap Y_p) \cap W_j = \emptyset \cap W_j = \emptyset$.
(3) If $i \neq s, j \neq p$, then $K_{ij} \cap K_{sp} = (Y_i \cap W_j) \cap (Y_s \cap W_p) = (Y_i \cap Y_s) \cap (W_j \cap W_p) = \emptyset \cap \emptyset = \emptyset$.

Since $\bigcup_{i \in \{1,2,\ldots,r\}} Y_i = \bigcup_{j \in \{1,2,\ldots,t\}} W_j = [0, 1]$, we have

$$\begin{aligned} \bigcup P(YW) &= \bigcup_{i \in \{1,2,\ldots,r\}, j \in \{1,2,\ldots,t\}} K_{ij} \\ &= \bigcup_{i \in \{1,2,\ldots,r\}, j \in \{1,2,\ldots,t\}} (Y_i \cap W_j) \\ &= \left(\bigcup_{i \in \{1,2,\ldots,r\}} Y_i \right) \cap \left(\bigcup_{j \in \{1,2,\ldots,t\}} W_j \right) \\ &= [0, 1] \cap [0, 1] = [0, 1] \end{aligned}$$

and then we see that $P(YW) \in \mathfrak{P}[0, 1]$, i.e. we obtained $\alpha_p = P(YW) = \{P_1, P_2, \ldots, P_n\}$ where $P_1 < P_2 < \cdots < P_n$.

The fact that $P(YW)$ is a partition of $[0, 1]$ does not indicate that this partition corresponds to the fuzzy set $\Omega = \Psi \cap \Theta$ (or $\Gamma = \Psi \cup \Theta$).

Step 2. To find the partition corresponding to the fuzzy set Ω (or Γ), we have the following arrangement:

(i) If [there exists at least $x_q \in \mathcal{A}$ such that $\omega(x_q) \in P_j$ (or $\gamma(x_q) \in P_j$) but there is no $x \in \mathcal{A}$ such that $\omega(x) \in P_{j+1}$ (or $\gamma(x) \in P_{j+1}$)], then we take $\omega(x_q) \in (P_j \cup P_{j+1}) = P_{j'}$ (or $\gamma(x_q) \in (P_j \cup P_{j+1}) = P_{j'}$).
(ii) If [there exists at least $x_q \in \mathcal{A}$ such that $\omega(x_q) \in P_2$ (or $\gamma(x_i) \in P_2$) but there is no $x \in \mathcal{A}$ such that $\omega(x) \in P_1$ (or $\gamma(x) \in P_1$)], then we take $\omega(x_q) \in (P_1 \cup P_2) = P_{1'}$ (or $\gamma(x_q) \in (P_1 \cup P_2) = P_{1'}$).

By this arrangement we obtained a new partition set say $\alpha_{p'} = \{P'_1, P'_2, \ldots, P'_m\}$ using the partition $\alpha_p = P(YW) = \{P_1, P_2, \ldots, P_n\}$, obviously $m \leq n$.

Therefore the proof is completed, because

$$\Omega^{\mathfrak{s}} = \{P'_e(x) \ | e \in \{1, 2, \ldots, m\}, x \in \mathcal{A}, \omega(x) \in P'_e\}$$

or

$$\Gamma^{\mathfrak{s}} = \{P'_e(x) \ | e \in \{1, 2, \ldots, m\}, x \in \mathcal{A}, \gamma(x) \in P'_e\}$$

are strait fuzzy sets where $\Omega = \Psi \cap \Theta$ and $\Gamma = \Psi \cup \Theta$. ■

Then, we have the definitions of intersection and union of strait fuzzy sets:

Definition 3.9: Let $\Psi^{\mathfrak{s}} \in SF(\Psi)$ and $\Theta^{\mathfrak{s}} \in SF(\Theta)$ over $\mathcal{A}$. Then

$$\Psi^{\mathfrak{s}} \cap \Theta^{\mathfrak{s}} = (\Psi \cap \Theta)^{\mathfrak{s}} \quad \text{and} \quad \Psi^{\mathfrak{s}} \cup \Theta^{\mathfrak{s}} = (\Psi \cup \Theta)^{\mathfrak{s}},$$

here the partitions corresponding to $\Psi \cap \Theta$ and $\Psi \cup \Theta$ obtained by the contruction method given in Theorem 3.8.

Definition 3.10: Let $\Psi \in \widetilde{F}(\mathcal{A})$. Given a number $t \in [0, 1]$, the strait t-level fuzzy set of Ψ is defined as

$$\Psi^{\mathfrak{s}}_t = \{[0, t)(x_p), [t, 1](x_q)\}$$

and a t-level set (or t-cut) of $\Psi^{\mathfrak{s}}_t$ is a crisp subset of $\mathcal{A}$ defined by

$$(\Psi^{\mathfrak{s}}_t)_t = \{x \in \mathcal{A} | \ \psi(x) \in [t, 1]\}.$$

Clearly, if there exist $x_p, x_q \in \mathcal{A}$ such that $\psi(x_p) \in [0, t)$ and $\psi(x_q) \in [t, 1]$, then $\Psi^{\mathfrak{s}}_t$ is a strait fuzzy set of Ψ.

Theorem 3.11: *Let $\Psi \in \widetilde{F}(\mathcal{A})$, $t \in [0, 1]$ and $\Psi^{\mathfrak{s}}_t$ be the strait t-level fuzzy set of Ψ. Then $\Psi_t = (\Psi^{\mathfrak{s}}_t)_t$.*

Proof: It is easily seen by Definitions 2.3 and 3.10. ■

The following example illustrates all the new concepts and results given above:

Example 3.12: The math teacher Mr. X gives two performance homeworks of varying difficulty to his 10 students who take the course. He wants to enter a single grade by combining the results of these two assignments. The students set is $\mathcal{A} =$

Table 2. The fuzzy sets Ψ, Θ, Ω and Γ (i.e. $\psi(x), \theta(x), \omega(x)$ and $\gamma(x)$) on the set $\mathcal{A}$.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
$\psi(x)$	0.4	0.6	0.2	0.3	0.7	0.5	0.6	0.8	0.8	0.55
$\theta(x)$	0.6	0.3	0.5	0.68	0.82	0.9	0.38	0.56	0.18	0.76
$\omega(x)$	0.4	0.3	0.2	0.3	0.7	0.5	0.38	0.56	0.18	0.55
$\gamma(x)$	0.6	0.6	0.5	0.68	0.82	0.9	0.6	0.8	0.8	0.76

$\{s_1, s_2, \ldots, s_{10}\}$. Mr. X determines grade ranges $\alpha_1 = \{Y_1 = [0, 0.45), Y_2 = [0.45, 0.65), Y_3 = [0.65, 1]\}$ and $\alpha_2 = \{W_1 = [0, 0.3), W_2 = [0.3, 0.5), W_3 = [0.5, 0.8), W_4 = [0.8, 1]\}$ according to the difficulty level of the first and second homeworks, respectively. The grades of students are given with fuzzy sets Ψ for the first homework and Θ for the second homework. The fuzzy sets $\Psi, \Theta, \Omega = \Psi \cap \Theta$ and $\Gamma = \Psi \cup \Theta$ are given as in Table 2.

Then, we have $\Psi^{\mathfrak{s}} = \{Y_1 = [0, 0.45)(s_1, s_3, s_4), Y_2 = [0.45, 0.65)(s_2, s_6, s_7, s_{10}), Y_3 = [0.65, 1](s_5, s_8, s_9)\}$ and $\Theta^{\mathfrak{s}} = \{W_1 = [0, 0.3)(s_9), W_2 = [0.3, 0.5)(s_2, s_7), W_3 = [0.5, 0.8)(s_1, s_3, s_4, s_8, s_{10}), W_4 = [0.8, 1](s_5, s_6)\}$.

Firstly, consider the intervals $K_{ij} = Y_i \cap W_j$ for $i \in \{1, 2, 3\}, j \in \{1, 2, 3, 4\}$. Then $K_{11} = [0, 0.3)$, $K_{12} = [0.3, 0.45)$, $K_{13} = \emptyset$, $K_{14} = \emptyset$, $K_{21} = \emptyset$, $K_{22} = [0.45, 0.5)$, $K_{23} = [0.5, 0.65)$, $K_{24} = \emptyset$, $K_{31} = \emptyset$, $K_{32} = \emptyset$, $K_{33} = [0.65, 0.8)$, $K_{34} = [0.8, 1]$. Then the set

$$\begin{aligned} P(YW) &= \{K_{ij} | i \in \{1, 2, 3\}, j \in \{1, 2, 3, 4\}, K_{ij} \neq \emptyset\} \\ &= \{K_{11}, K_{12}, K_{22}, K_{23}, K_{33}, K_{34}\} \\ &= \{P_1, P_2, P_3, P_4, P_5, P_6\} = \alpha_p \end{aligned}$$

is a partition of $[0, 1]$, such that $P_1 < P_2 < P_3 < P_4 < P_5 < P_6$.

If Mr. X finds appropriate to use the intersection operation, he obtains the following:

He needs to do arrangements on α_p, since $s_1, s_2, s_4, s_7 \in P_2$ but there is no $x \in \mathcal{A}$ such that $\omega(x) \in P_3$ and $s_5 \in P_5$ but there is no $x \in \mathcal{A}$ such that $\omega(x) \in P_6$. So, he takes $P_{2'} = P_2 \cup P_3$ and $P_{5'} = P_5 \cup P_6$. In this case, the arranged partition set of α_p is the set

$$\begin{aligned} \alpha_{p'} &= \{P_1, P_{2'}, P_4, P_{5'}\} \\ &= \{P'_1 = [0, 0.3), P'_2 = [0.3, 0.5), \\ P'_3 &= [0.5, 0.65), P'_4 = [0.65, 1]\} \end{aligned}$$

is the corresponding partition set of Ω and $P'_1 < P'_2 < P'_3 < P'_4$. Finally the strait fuzzy set corresponding Ω is

$$\begin{aligned} \Omega^{\mathfrak{s}} &= \{P'_e(x_q) | \ e \in \{1, 2, 3, 4\}, x_q \in \mathcal{A}, \omega(x_q) \in P'_e\} \\ &= \{[0, 0.3)(s_3, s_9), [0.3, 0.5)(s_1, s_2, s_4, s_7), \\ &\quad [0.5, 0.65)(s_6, s_8, s_{10}), [0.65, 1](s_5)\} \end{aligned}$$

Then Mr.X can give the grades of the students by determining the letter grade equivalents of the intervals, for example $[0, 0.3) = F$, $[0.3, 0.5) = C$, $[0.5, 0.65) = B$, $[0.65, 1] = A$. That's mean the student s_5 received the grade A as the joint grade of these two exams.

If Mr. X finds appropriate to use the union operation, he obtains the following:

He needs to do arrangements on α_p, since $s_1, s_2, s_3, s_7 \in P_4$ but there is no $x \in \mathcal{A}$ such that $\gamma(x) \in P_1$(and $\in P_2$)(and $\in P_3$). So, he takes $P_{1'} = P_1 \cup P_2 \cup P_3 \cup P_4$. In this case, the arranged partition set of α_p is the set

$$\begin{aligned} \alpha_{p'} &= \{P_{1'}, P_5, P_6\} \\ &= \{P'_1 = [0, 0.65), P'_2 = [0.65, 0.8), P'_3 = [0.8, 1]\} \end{aligned}$$

is the corresponding partition set of γ and $P'_1 < P'_2 < P'_3$. Finally the strait fuzzy set corresponding γ is

$$\begin{aligned} \Gamma^{\mathfrak{s}} &= \{P'_e(x_q) | \ e \in \{1, 2, 3\}, x_i \in A, \gamma(x_q) \in P'_e\} \\ &= \{[0, 0.65)(s_1, s_2, s_3, s_7), [0.65, 0.8)(s_4, s_{10}), \\ &\quad [0.8, 1](s_5, s_6, s_8, s_9)\} \end{aligned}$$

Then Mr.X can give the grades of the students by determining the letter grade equivalents of the intervals, for example $[0, 0.65) = D$, $[0.65, 0.8) = B$, $[0.8, 1] = A$. That's mean the students s_5, s_6, s_8, s_9 received the grade A as the joint grade of these two exams.

This method, which is applied to intersection and union operations, can also be applied to specially defined operations. For example, if Mr. X wants to use a new operation such that $\lambda(x) = 0.4\psi(x) + 0.6\theta(x)$, then Λ is a fuzzy set over $\mathcal{A}$ (see Table 3).

He needs to do arrangements on α_p, since $s_2, s_3, s_9 \in P_2$ but there is no $x \in \mathcal{A}$ such that $\lambda(x) \in P_1$ and $s_5, s_6, s_8, s_{10} \in P_5$ but there is no $x \in \mathcal{A}$ such that $\lambda(x) \in P_6$. So, he takes $P_{1'} = P_1 \cup P_2$ and $P_{5'} = P_5 \cup P_6$. In this case, the arranged partition set of α_p is the set

$$\alpha_{p'} = \{P_{1'}, P_3, P_4, P_{5'}\}$$

Table 3. The fuzzy sets Ψ, Θ and Λ (i.e. $\psi(x)$, $\theta(x)$ and $\lambda(x)$) on the set $\mathcal{A}$.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
$\psi(x)$	0.4	0.6	0.2	0.3	0.7	0.5	0.6	0.8	0.8	0.55
$\theta(x)$	0.6	0.3	0.5	0.68	0.82	0.9	0.38	0.56	0.18	0.76
$\lambda(x)$	0.52	0.42	0.38	0.53	0.78	0.74	0.47	0.66	0.43	0.68

$$= \{P'_1 = [0, 0.45), P'_2 = [0.45, 0.5), \\ P'_3 = [0.5, 0.65), P'_4 = [0.65, 1]\}$$

is the corresponding partition set of Λ and $P'_1 < P'_2 < P'_3 < P'_4$. Finally the strait fuzzy set corresponding Λ is

$$\begin{aligned}\Lambda^{\mathfrak{s}} &= \{P'_e(x_q) \mid e \in \{1,2,3,4\}, x_q \in \mathcal{A}, \lambda(x_q) \in P'_e\} \\ &= \{[0, 0.45)(s_2, s_3, s_9), [0.45, 0.5)(s_7), [0.5, 0.65) \\ &\quad (s_1, s_4), [0.65, 1](s_5, s_6, s_8, s_{10})\}\end{aligned}$$

Then Mr. X can give the grades of the students by determining the letter grade equivalents of the intervals, for example $[0, 0.45) = F$, $[0.45, 0.5) = C$, $[0.5, 0.65) = B$, $[0.65, 1] = A$. That's mean the students s_5, s_6, s_8, s_{10} received the grade A as the joint grade of these two exams.

To determine who passed the course, Mr. X can use t-level set. For $t = 0.45$, $\Lambda^{\mathfrak{s}}_t = \{[0, 0.45)(s_2, s_3, s_9), [0.45, 1](s_1, s_4, s_5, s_6, s_7, s_8, s_{10})\}$ and then $(\Lambda^{\mathfrak{s}}_t)_{0.45} = \{s_1, s_4, s_5, s_6, s_7, s_8, s_{10}\}$ is the set of students who passed the course. It is seen that $(\Lambda^{\mathfrak{s}}_t)_{0.45} = \Lambda_{0.45}$.

4. Similarity measure of strait fuzzy sets

In this section, we propose a similarity measure approach to quantify the similarity between two strait fuzzy sets.

Definition 4.1: A real function $N : \widetilde{SFA} \times \widetilde{SFA} \to [0, 1]$ is called similarity measure of strait fuzzy sets, if N satisfies the following properties:

Sim1. $N(\Psi^{\mathfrak{s}}, (\Psi^{\mathfrak{s}})^c) = 0$ if $\mathcal{A}$ is a crisp set.
Sim2. $N(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}) = 1$ if and only if $\Psi^{\mathfrak{s}} =_s \Theta^{\mathfrak{s}}$.
Sim3. $N(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}) = N(\Theta^{\mathfrak{s}}, \Psi^{\mathfrak{s}})$.
Sim4. For all $\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}, \Omega^{\mathfrak{s}} \in \widetilde{SFA}$, if $\Psi^{\mathfrak{s}} \subseteq_s \Theta^{\mathfrak{s}} \subseteq_s \Omega^{\mathfrak{s}}$, then $N(\Psi^{\mathfrak{s}}, \Omega^{\mathfrak{s}}) \leq N(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}})$ and $N(\Psi^{\mathfrak{s}}, \Omega^{\mathfrak{s}}) \leq N(\Theta^{\mathfrak{s}}, \Omega^{\mathfrak{s}})$.

Let $\Psi^{\mathfrak{s}} = \{Y_a(x) \mid a \in \{1, 2, \dots, r\}, x \in \mathcal{A}, \psi(x) \in Y_a\}$ and $\Theta^{\mathfrak{s}} = \{W_b(x) \mid b \in \{1, 2, \dots, t\}, x \in \mathcal{A}, \theta(x) \in W_b\}$, where $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \dots, \alpha_k, \dots\}$, $\alpha_k = \{Y_1, Y_2, \dots, Y_r\}$, $\alpha_l = \{W_1, W_2, \dots, W_t\}$ such that $Y_1 < Y_2 < \dots < Y_r$ and $W_1 < W_2 < \dots < W_t$.

Consider the function $\Im : \widetilde{SFA} \times \widetilde{SFA} \to [0, 1]$ such that

$$\Im(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}) = 1 - \frac{1}{n} \sum_{q=1}^{n} \begin{cases} (|infY_r - infW_t| \\ +|supY_r - supW_t|) & \text{if } Y_r \nsubseteq W_t \text{ and } W_t \nsubseteq Y_r \\ |supY_r - supW_t| & \text{if } Y_r \subseteq W_t \text{ or } W_t \subseteq Y_r \end{cases} \tag{1}$$

where $\psi(x) \in Y_r$ and $\theta(x) \in W_t$.

Theorem 4.2: *$\Im$ is a similarity measure on $\widetilde{SFA}^*$, where $\widetilde{SFA}^*$ is the set of all strait fuzzy sets over $\mathcal{A}$ except trivial strait fuzzy sets.*

Proof: Let $\Psi^{\mathfrak{s}} = \{Y_a(x) \mid a \in \{1, 2, \dots, r\}, x \in \mathcal{A}, \psi(x) \in Y_a\}$, $\Theta^{\mathfrak{s}} = \{W_b(x) \mid b \in \{1, 2, \dots, t\}, x \in \mathcal{A}, \theta(x) \in W_b\}$ and $\Omega^{\mathfrak{s}} = \{T_c(x) \mid c \in \{1, 2, \dots, m\}, x \in \mathcal{A}, \omega(x) \in T_c\}$, where $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \dots, \alpha_k, \dots\}$, $\alpha_k = \{Y_1, Y_2, \dots, Y_r\}$, $\alpha_l = \{W_1, W_2, \dots, W_t\}$ and $\alpha_z = \{T_1, T_2, \dots, T_m\}$ such that $Y_1 < Y_2 < \dots < Y_r$, $W_1 < W_2 < \dots < W_t$ and $T_1 < T_2 < \dots < T_m$.

Sim1. By Definition 3.4, $(\Psi^{\mathfrak{s}})^c = \{Y_a(x) \mid a \in \{1, 2, \dots, r\}, x \in \mathcal{A}, \psi^c(x) \in Y_a\} \in SF(\Psi)$. Let $\mathcal{A}$ be a crisp set. Since $\psi(x) = 0$ implies $\psi^c(x) = 1$ and $\psi(x) = 1$ implies $\psi^c(x) = 0$, then $|infY_r - infY_t| + |supY_r - supY_t| = 1$ where $\psi(x) \in Y_r$ and $\psi^c(x) \in Y_t$ such that $r \neq t$. Therefore $\Im(\Psi^{\mathfrak{s}}, (\Psi^{\mathfrak{s}})^c) = 0$.
Sim2. Let $\Im(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}) = 1$. If $Y_r \nsubseteq W_t$ and $W_t \nsubseteq Y_r$, then $\sum_{q=1}^{n}(|infY_r - infW_t| + |supY_r - supW_t|) = 0$ which implies $infY_r = infW_t$ and $supY_r = supW_t$ for all $\psi(x) \in Y_r$ and $\theta(x) \in W_t$. Then $Y_r \subseteq W_t$ or $W_t \subseteq Y_r$, this is a contradiction. Since $Y_r \subseteq W_t$ or $W_t \subseteq Y_r$ where $\psi(x_q) \in Y_r$ and $\theta(x_q) \in W_t$ for all $x_q \in \mathcal{A}$, then $\Psi^{\mathfrak{s}} =_s \Theta^{\mathfrak{s}}$ by Definition 3.5. Now let $\Psi^{\mathfrak{s}} =_s \Theta^{\mathfrak{s}}$. Then $Y_r \subseteq W_t$ or $W_t \subseteq Y_r$ where $\psi(x) \in Y_r$ and $\theta(x_q) \in W_t$ for all $x_q \in \mathcal{A}$ which implies $supY_r = supW_t$. Therefore $\Im(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}) = 1 - \frac{1}{n} \sum_{q=1}^{n} |supY_r - supW_t| = 1$.

Sim3. Symmetry is clearly seen by definition of the function $\Im$.

Sim4. Let $\Psi^{\mathfrak{s}} \subseteq_s \Theta^{\mathfrak{s}} \subseteq_s \Omega^{\mathfrak{s}}$. Then for all $x \in \mathcal{A}$, there exist a $b \in \{1, 2, \ldots, t\}$ such that $\psi(x) \in Y_a \subseteq W_b$ and there exist a $c \in \{1, 2, \ldots, m\}$ such that $\theta(x) \in W_b \subseteq T_c$. Then $supY_a \leq supW_b \leq supT_c$ which implies $|supY_a - supW_b| \leq |supY_a - supT_c|$ and $|supW_b - supT_c| \leq |supY_a - supT_c|$. Therefore $\Im(\Psi^{\mathfrak{s}}, \Omega^{\mathfrak{s}}) \leq \Im(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}})$ and $\Im(\Psi^{\mathfrak{s}}, \Omega^{\mathfrak{s}}) \leq \Im(\Theta^{\mathfrak{s}}, \Omega^{\mathfrak{s}})$.

■

Example 4.3: Consider the three strait fuzzy sets $\Psi^{\mathfrak{s}}$, $\Theta^{\mathfrak{s}}$ and $\Omega^{\mathfrak{s}}$ of fuzzy sets Ψ, Θ and Ω on the set $\mathcal{A} = \{x_1, x_2, \ldots, x_7\}$ given in Example 3.7. Then, we calculate similarity measures for strait fuzzy sets $\Psi^{\mathfrak{s}}$, $\Theta^{\mathfrak{s}}$ and $\Omega^{\mathfrak{s}}$ as

$$\begin{aligned}\Im(\Psi^{\mathfrak{s}}, \Theta^{\mathfrak{s}}) &= 1 - \frac{1}{7}[(|0.4 - 0.8| + |0.4 - 0.8| \\ &\quad + |0.2 - 0.8| + |0.4 - 0.8| + |1 - 1| \\ &\quad + |0.8 - 0.8| + |1 - 1|)] \\ &= 1 - 0.257 = 0.743 \quad \text{and} \\ \Im(\Psi^{\mathfrak{s}}, \Omega^{\mathfrak{s}}) &= 1 - \frac{1}{7}[(|0.2 - 0.3| + |0.4 - 0.5|) \\ &\quad + (|0.2 - 0.3| + |0.4 - 0.5|) + |0.3 - 0.2| \\ &\quad + (|0.2 - 0.5| + |0.4 - 0.9|) \\ &\quad + (|0.8 - 0.5| + |1 - 0.9|) \\ &\qquad + (|0.4 - 0.9| + |0.8 - 1|) \\ &\quad + (|0.4 - 0.9| + |0.8 - 1|) + |1 - 1|] \\ &= 1 - 0.442 = 0.558.\end{aligned}$$

5. An application regarding the similarity rates of vaccines against influenza viruses

In this section, we construct a method using the proposed similarity measure of strait fuzzy sets, and then apply it to calculate the similarity rates of vaccines against influenza viruses. The reason for choosing such a problem is that the companies producing vaccines have statistical data and thus they are already doing the comparative tests with different methods. Therefore, the necessary data to obtain Algorithm 1 are available to companies. In addition, with this method, the same problem allows companies that will produce in various fields to make comparisons before and after production.

The similarity measure is an interesting tool applied to many practical situations such as multi-criteria decision making, medical diagnosis, pattern recognition, and therefore has attracted the attention of researchers interested in fuzzy mathematics. So far, many similarity measure approaches based on the fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, q-rung orthopair fuzzy set, picture fuzzy set, spherical fuzzy set and q-rung picture fuzzy set have been proposed. Moreover, their advantages and efficiency in dealing with real-world problems have been demonstrated. In short, the idea of similarity measure in fuzzy set theory is important and valuable.

Now, we present an algorithm based on the proposed similarity measure of strait fuzzy sets and follow its steps to compare the similarity rates of vaccines produced by a pharmaceutical company or to determine the similarity rates of vaccines produced by different companies. (Of course, this comparison method can also be used in different fields. In particular, companies that produce the same type of product compare their own production capacities, sales rates of products, production amounts, etc. with rival companies. It has a wide range of uses such as comparison and benchmarking.)

Algorithm 1:

Step 1 : Determine the vaccines as V_i $(i = 1, 2, \ldots)$, and the different influenza viruses in the human body as f_q $(q = 1, 2, \ldots)$ that antibiotics caused by these vaccines can protect against.

Step 2 : Choose a model strait fuzzy set $\Lambda^{\mathfrak{s}}$ with which similarity to be computed.

Step 3 : Create the strait fuzzy set $\Psi_i^{\mathfrak{s}}$ for each vaccine V_i.

Step 4 : Calculate similarity measure $\Im(\Psi_i^{\mathfrak{s}}, \Lambda^{\mathfrak{s}})$ of each of the strait fuzzy set $\Psi_i^{\mathfrak{s}}$ with the model strait fuzzy set $\Lambda^{\mathfrak{s}}$.

Step 5 : Compare all the results and rank all vaccines.

Figure 4 illustrates the step by step procedure of Algorithm 1.

Example 5.1: Influenza viruses cause mild to severe respiratory infections in humans and are a major public health problem. Seasonal influenza viruses cause

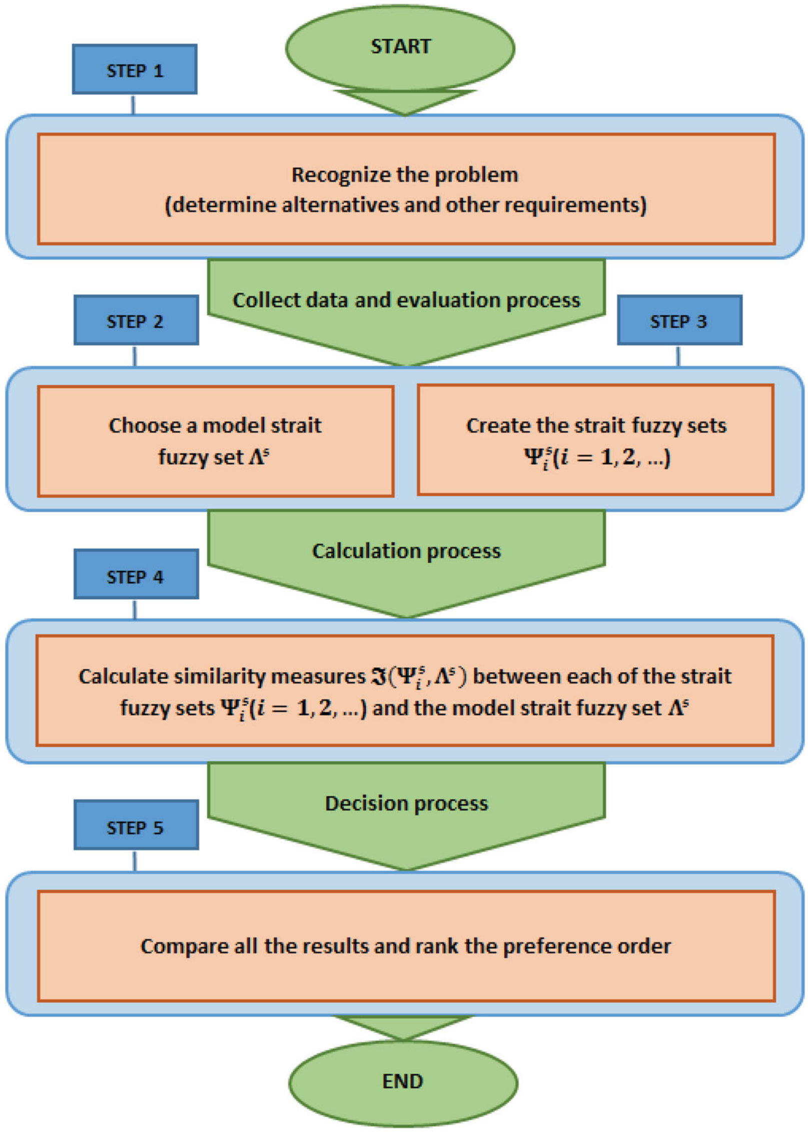

Figure 4. Flow chart of Algorithm 1.

millions of serious cases and many deaths each year worldwide. Influenza vaccines cause antibodies to develop in the body about two-three weeks after vaccination. Antibody responses to the influenza virus have long been known to be protective against influenza virus infection.

Assume that X, Y, Z companies give the antibody production ratios of the vaccines V_1, V_2, and V_3 produced against 10 different types $f_1, f_2, \ldots, f_{10}$ of influenza viruses in the human body as follows:

For V_1 Strait fuzzy set Ψ_1^s	$[0, 0.2)$	$[0.2, 0.4)$	$[0.4, 0.6)$	$[0.6, 0.8)$	$[0.8, 1]$
	f_3, f_4	f_1, f_5, f_6	f_2	f_8, f_{10}	f_7, f_9

For V_2 Strait fuzzy set Ψ_2^s	$[0, 0.25)$	$[0.25, 0.5)$	$[0.5, 0.75)$	$[0.75, 1]$
	f_1, f_3, f_6	f_2, f_4	f_5, f_7, f_9	f_8, f_{10}

For V_3 Strait fuzzy set Ψ_3^s	$[0, 0.4)$	$[0.4, 0.6)$	$[0.6, 0.9)$	$[0.9, 1]$
	f_5, f_8	f_2, f_3, f_6	f_1, f_4	f_7, f_9, f_{10}

A new company W produces the vaccine V_4 and wants to measure the similarity of the antibody production rate of the vaccine against these 10 different viruses with other vaccine types. The antibody production ratios of the vaccine V_4 against these influenza viruses in the human body as below:

For V_4 Strait fuzzy set Λ^s	$[0, 0.35)$	$[0.35, 0.55)$	$[0.55, 0.7)$	$[0.7, 0.85)$	$[0.85, 1]$
	f_1, f_4	f_5, f_6	f_2	f_3, f_{10}	f_7, f_8, f_9

Then the similarity measures of these strait fuzzy sets are $\Im(\Psi_1^s, \Lambda^s) = 0.68$, $\Im(\Psi_2^s, \Lambda^s) = 0.62$ and $\Im(\Psi_3^s, \Lambda^s) = 0.63$. As a result, the order of these vaccines according to their similarity rate to the vaccine V_4 is $V_1 > V_3 > V_2$.

6. Strait fuzzy rough sets

Since strait fuzzy sets contain a partition of the interval $[0,1]$ and each partition corresponds to an equivalence relation on $[0,1]$, a rough set structure can naturally be created on strait fuzzy sets.

Definition 6.1: Let $\Psi \in \widetilde{F}(\mathcal{A})$ and $\Psi^{\mathfrak{s}} \in SF(\Psi)$ such that $\Psi^{\mathfrak{s}} = \{Y_a(x_q)|\ a \in \{1,2,\ldots,r\}, x_q \in \mathcal{A}, \psi(x_q) \in Y_a\}$, where $\mathfrak{P}[0,1] = \{\alpha_1,\alpha_2,\ldots,\alpha_k,\ldots\}$ and $\alpha_k = \{Y_1,Y_2,\ldots,Y_r\}$. Suppose that $\emptyset \neq X \subseteq [0,1]$. Then,

(1) The **lower approximation** of X depending on $\Psi^{\mathfrak{s}}$ is the set

$$\Psi_*(X) = \{\bigcup Y_a|\ \psi(x) \in Y_a \quad \text{and} \quad Y_a \subseteq X\} \tag{2}$$

(2) The **upper approximation** of X depending on $\Psi^{\mathfrak{s}}$ is the set

$$\Psi^*(X) = \{\bigcup Y_a|\ \psi(x) \in Y_a \quad \text{and} \quad Y_a \cap X \neq \emptyset\} \tag{3}$$

(3) The **boundary region** of X depending on $\Psi^{\mathfrak{s}}$ is the set

$$\mathcal{B}_\Psi(X) = \Psi^*(X) - \Psi_*(X) \tag{4}$$

(4) If $\Psi_*(X) = \Psi^*(X)$, X is said to be strait definable, otherwise, i.e. if $\mathcal{B}_\Psi(X) \neq \emptyset$, then X is called a **strait fuzzy rough set** depending on $\Psi^{\mathfrak{s}}$.

We give the properties of approximations depending on a strait fuzzy set by the following Theorem. Since the proof is straitforward by Definition 6.1, it is omitted.

Theorem 6.2: *If X is a strait fuzzy rough set depending on $\Psi^{\mathfrak{s}} \in SF(\Psi)$, then we have*

(a) $\Psi_*(X) \subseteq X \subseteq \Psi^*(X)$.
(b) $\Psi_*(\Psi^*(X)) = \Psi^*(\Psi^*(X)) = \Psi^*(X)$.
(c) $\Psi_*(\Psi_*(X)) = \Psi^*(\Psi_*(X)) = \Psi_*(X)$.

As Pawlak's rough set, if X is the interval of alternatives that satisfy the property $\mathcal{P}$, then we have

$\psi(x) \in \psi_*(X)$ means that x certainly has property $\mathcal{P}$,
$\psi(x) \in \psi^*(X)$ means that x possibly has property $\mathcal{P}$ and
$\psi(x) \in [0,1] - \Psi^*(X)$ means that x definitely does not have property $\mathcal{P}$.

We can characterise the parameters for a strait fuzzy rough set X with the following definition:

Definition 6.3: Let X be a strait fuzzy rough set depending on $\Psi^{\mathfrak{s}} \in SF(\Psi)$.

(1) The **definitely set of parameters** for X depending on $\Psi^{\mathfrak{s}}$ is the set

$$\Delta_* = \{x \in \mathcal{A}\,|\quad \psi(x) \in Y_a \subseteq \psi_*(X)\} \tag{5}$$

(2) The **possibly set of parameters** for X depending on $\Psi^{\mathfrak{s}}$ is the set

$$\Delta^* = \{x \in \mathcal{A}\,|\quad \psi(x) \in Y_a \subseteq \Psi^*(X)\} \tag{6}$$

(3) The **boundary set of parameters** for X depending on $\Psi^{\mathfrak{s}}$ is the set

$$\mathfrak{B}(X) = \Delta^* - \Delta_* \tag{7}$$

Clearly, X is a strait definable set depending on $\Psi^{\mathfrak{s}}$ iff $\mathfrak{B}(X) = \emptyset$ and X is a strait fuzzy rough set depending on $\Psi^{\mathfrak{s}}$ iff $\mathfrak{B}(X) \neq \emptyset$.

Example 6.4: Let us consider the fuzzy set $\Psi^{\mathfrak{s}}$ of fuzzy set Ψ on the set $\mathcal{A}$ given in Example 3.7. Also, let $X = [0.3, 0.9) \subset [0,1]$. Then, we have

$\Psi_*(X) = [0.4, 0.8)$
is the lower approximation of X,
$\Psi^*(X) = Y_2 \cup Y_3 \cup Y_4 = [0.2, 1]$
is the upper approximation of X,
$\mathcal{B}_\Psi(X) = \Psi^*(X) - \Psi_*(X) = [0.2, 0.4) \cup [0.8, 1]$
is the boundary set of X

depending on $\Psi^{\mathfrak{s}}$. Since $\mathcal{B}_\Psi(X) \neq \emptyset$, then X is a strait fuzzy rough set depending on $\Psi_{\mathfrak{s}}$.

$\Delta_* = \{x_6\}$
is the definitely set of parameters for X,
$\Delta^* = \{x_1, x_2, x_4, x_5, x_6, x_7\}$
is the possibly set of parameters for X and
$\mathfrak{B}(X) = \{x_1, x_2, x_4, x_5, x_7\}$
is the boundary set of parameters for X

depending on $\Psi^{\mathfrak{s}}$. Since $\mathfrak{B}(X) \neq \emptyset$, X is a strait fuzzy rough set depending on $\Psi^{\mathfrak{s}}$.

Definition 6.5: Let $\Psi \in \widetilde{F}(\mathcal{A})$ and $\Psi^{\mathfrak{s}} \in SF(\Psi)$ such that $\Psi^{\mathfrak{s}} = \{Y_a(x_q)|\ a \in \{1, 2, \ldots, r\}, x_q \in \mathcal{A}, \psi(x_q) \in Y_a\}$, where $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$ and $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$ and let $\emptyset \neq X \subseteq [0, 1]$. Suppose that $\Psi_*(X) = \{\bigcup Y_i|\ \psi(x) \in Y_i \text{ and } Y_i \subseteq X\}$ and $\Psi^*(X) = \{\bigcup Y_j|\ \psi(x) \in Y_j \text{ and } Y_j \cap X \neq \emptyset\}$.

(1) The **internal measure** of X depending on $\Psi^{\mathfrak{s}}$ is the number

$$(M_\Psi)_*(X) = \begin{cases} \sum_i (supY_i - infY_i) & \text{if } \sup \Psi_*(X) \neq inf\,\Psi_*(X) \\ \Psi_*(X) & \text{if } \sup \Psi_*(X) = inf\,\Psi_*(X) \end{cases} \tag{8}$$

(2) The **external measure** of X depending on $\Psi^{\mathfrak{s}}$ is the number

$$(M_\Psi)^*(X) = \begin{cases} \sum_j (supY_j - infY_j) & \text{if } \sup \Psi^*(X) \neq inf\,\Psi^*(X) \\ \Psi^*(X) & \text{if } \sup \Psi^*(X) = inf\,\Psi^*(X) \end{cases} \tag{9}$$

(3) The **distance measure** of X depending on $\Psi^{\mathfrak{s}}$ is the number

$$d_\Psi(X) = (M_\Psi)^*(X) - (M_\Psi)_*(X) \tag{10}$$

(4) The **accuracy** of X depending on $\Psi^{\mathfrak{s}}$ is the number

$$\mathfrak{A}_\Psi(X) = \frac{(M_\Psi)_*(X)}{(M_\Psi)^*(X)} \tag{11}$$

(5) X is **measurable** depending on $\Psi^{\mathfrak{s}}$, if $(M_\Psi)_*(X) = (M_\Psi)^*(X)$ or $\mathfrak{A}_\Psi(X) = 1$.

Let $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$, $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$ and let $\Psi \in \widetilde{F}(\mathcal{A})$. If $\Psi^{\mathfrak{s}} \in SF(\Psi)$, then by Definition 3.2, there exists at least $x \in \mathcal{A}$ such that $\psi(x) \in Y_i$ for all $i \in \{1, 2, \ldots, r\}$. Then, $(M_\Psi)^*(X) > 0$, if $\emptyset \neq X \subseteq [0, 1]$. Therefore, $\mathfrak{A}_\Psi(X)$ is always determined. But, in Pawlak's rough set, if the external measure of X is different from 0, its accuracy was defined.

The measurement definitions for the parameters are as follows:

Definition 6.6: Let $\Psi \in \widetilde{F}(\mathcal{A})$, $\Psi^{\mathfrak{s}} \in SF(\Psi)$ and let $\emptyset \neq X \subseteq [0, 1]$. If Δ_* and Δ^* are definitely and possibly sets of parameters for X depending on $\Psi^{\mathfrak{s}}$, respectively, then

(1) The **lower parameter effect factor** for X depending on $\Psi^{\mathfrak{s}}$ is the number

$$\mu_*(\Psi(X)) = |\Delta_*| \tag{12}$$

(2) The **upper parameter effect factor** for X depending on $\Psi^{\mathfrak{s}}$ is the number

$$\mu^*(\Psi(X)) = |\Delta^*| \tag{13}$$

(3) The **parameter effect ratio** for X depending on $\Psi^{\mathfrak{s}}$ is the number

$$\wp_\Psi(X) = \frac{\mu_*(\Psi(X))}{\mu^*(\Psi(X))} \tag{14}$$

By Definitions 6.5 and 6.6, we immediately have;

Proposition 6.7: *Let $\Psi \in \widetilde{F}(\mathcal{A})$ and $\Psi^{\mathfrak{s}} \in SF(\Psi)$ such that $\Psi^{\mathfrak{s}} = \{Y_a(x_q)|\ a \in \{1, 2, \ldots, r\}, x_q \in \mathcal{A}, \psi(x_q) \in Y_a\}$, where $\mathfrak{P}[0, 1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$ and $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$ and let $\emptyset \neq X \subseteq [0, 1]$. Then,*

(1) $0 \leq (M_\Psi)_*(X) \leq supX - infX \leq (M_\Psi)^*(X) \leq 1$.
(2) $|\alpha_k| = r \leq |\mathcal{A}|$.
(3) $0 \leq (M_\Psi)_*(X) \leq 1$.
(4) $0 < (M_\Psi)^*(X) \leq 1$.
(5) $0 \leq \mathfrak{A}_\Psi(X) \leq 1$.
(6) $\emptyset \subseteq \Delta_* \subseteq \Delta^* \subseteq \mathcal{A}$.
(7) $\Delta^* \neq \emptyset$.
(8) $0 \leq \mu_*(\Psi(X)) \leq |\mathcal{A}|$.
(9) $0 < \mu^*(\Psi(X)) \leq |\mathcal{A}|$.
(10) $0 \leq \mu_*(\Psi(X)) \leq \mu^*(\Psi(X)) \leq |\mathcal{A}|$.
(11) $0 \leq \wp_\Psi(X) \leq 1$.

Proof: They can easily be seen from the above equations, so they are omitted. ∎

Example 6.8: Let $\Psi^{\mathfrak{s}} = \{Y_1(x_3), Y_2(x_1, x_2, x_4), Y_3(x_6), Y_4(x_5, x_7)\}$ and $X = [0.3, 0.9) \subset [0, 1]$ given in Example 6.4. Then, we obtain $(M_\Psi)_*(X) = 0.4$, $(M_\Psi)^*(X) = 0.8$, $d_\Psi(X) = 0.8 - 0.4 = 0.4$, $\mathfrak{A}_\Psi(X) = \frac{0.4}{0.8} = \frac{1}{2}$. $\mu^*(\Psi(X)) = 1$, $\mu^*(\Psi(X)) = 6$, $\wp_\Psi(X) = \frac{1}{6}$.

Now, we show that if $X_k \in V_4$ for the strait fuzzy sets $\Psi_i^{\mathfrak{s}}$ $(i = 1, 2, 3)$ given in Example 5.1 are also strait fuzzy rough sets then the similarity between the strait fuzzy set $\Psi_i^{\mathfrak{s}}$ and $\Lambda^{\mathfrak{s}}$ can be determined by using the following algorithm.

Algorithm 2:

Step 1 : Repeat Step 1-Step 3 in Algorithm 1.

Step 2 : Determine X_k $(k = 1, 2, \ldots, r)$, such that $V_4 = \{X_1, X_2, \ldots, X_r\}$.

Step 3 : Compute the lower approximation $(\Psi_i)_*(X_k)$, the upper approximation $(\Psi_i)^*(X_k)$ and the boundary region $\mathcal{B}_{\Psi_i}(X_k)$ for each X_k $(k = 1, 2, \ldots, r)$ depending on $\Psi_i^{\mathfrak{s}}$ $(i = 1, 2, \ldots)$.

Step 4 : Calculate the internal measure $(M_{\Psi_i})_*(X_k)$, the external measure $(M_{\Psi_i})^*(X_k)$ and the distance measure $d_{\Psi_i}(X_k)$ for each X_k $(k = 1, 2, \ldots, r)$ depending on $\Psi_i^{\mathfrak{s}}$ $(i = 1, 2, \ldots)$.

Step 5 : Calculate the similarity coefficient

$$S_{\Psi_i^{\mathfrak{s}}} = 1 - \frac{1}{n}\sum_{q=1}^{n} d_{\Psi_i}(X_k)_q, \quad (0 \leq S_{\Psi_i^{\mathfrak{s}}} \leq 1) \tag{15}$$

where $(X_k)_q$ represents X_k providing $\Psi_i(f_q) \in X_k$.

(1) Compare all the results and rank the preference order.

Note that the collect data and evaluation processes of Algorithm 1 and Algorithm 2 are the same, but the calculation processes are different (see: Step 4 in Algorithm 1 and Steps 2,3,4,5 in Algorithm 2).

Let us give the following example to detail the steps of Algorithm 2.

Example 6.9: We consider the strait fuzzy sets $\Psi_i^{\mathfrak{s}}$ for V_i $(i = 1, 2, 3)$ and the strait fuzzy set $\Lambda^{\mathfrak{s}}$ for V_4 in Example 5.1. Also, we take $V_4 = \{X_1 = [0, 0.35), X_2 = [0.35, 0.55), X_3 = [0.55, 0.7), X_4 = [0.7, 0.85), X_5 = [0.85, 1]\}$.

For each X_k $(k = 1, 2, 3, 4, 5)$ depending on $\Psi_1^{\mathfrak{s}}$, we obtain the lower/upper approximations and the inter nal/external/distance measures as in Table 4.

From Table 4, we can say that X_k $(k = 1, 2, 3, 4, 5)$ are strait fuzzy rough sets depending on $\Psi_1^{\mathfrak{s}}$ since $\mathcal{B}_{\Psi_1}(X_k) \neq \emptyset$ for each $k = 1, 2, 3, 4, 5$.

For each X_k $(k = 1, 2, 3, 4, 5)$ depending on $\Psi_2^{\mathfrak{s}}$, the lower/upper approximations and the inter nal/external/distance measures are in Table 5.

Table 4. The internal/external/distance measures for X_k depending on $\Psi_1^{\mathfrak{s}}$.

	X_1	X_2	X_3	X_4	X_5
$(\Psi_1)_*(X_k)$	[0,0.2)	∅	∅	∅	∅
$(\Psi_1)^*(X_k)$	[0,0.4)	[0.2,0.6)	[0.4,0.8)	[0.6,1]	[0.8,1]
$\mathcal{B}_{\Psi_1}(X_k)$	[0.2,0.4)	[0.2,0.6)	[0.4,0.8)	[0.6,1]	[0.8,1]
$(M_{\Psi_1})_*(X_k)$	0.2	0	0	0	0
$(M_{\Psi_1})^*(X_k)$	0.4	0.4	0.4	0.4	0.2
$d_{\Psi_1}(X_k)$	0.2	0.4	0.4	0.4	0.2

Table 5. The internal/external/distance measures for X_k depending on $\Psi_2^{\mathfrak{s}}$.

	X_1	X_2	X_3	X_4	X_5
$(\Psi_2)_*(X_k)$	[0,0.25)	∅	∅	∅	∅
$(\Psi_2)^*(X_k)$	[0,0.5)	[0.25,0.75)	[0.5,0.75)	[0.5,1]	[0.75,1]
$\mathcal{B}_{\Psi_2}(X_k)$	[0.25,0.5)	[0.25,0.75)	[0.5,0.75)	[0.5,1]	[0.75,1]
$(M_{\Psi_2})_*(X_k)$	0.25	0	0	0	0
$(M_{\Psi_2})^*(X_k)$	0.5	0.5	0.25	0.5	0.25
$d_{\Psi_2}(X_k)$	0.25	0.5	0.25	0.5	0.25

Table 6. The internal/external/distance measures for X_k depending on $\Psi_3^{\mathfrak{s}}$.

	X_1	X_2	X_3	X_4	X_5
$(\Psi_3)_*(X_k)$	∅	∅	∅	∅	[0.9,1]
$(\Psi_3)^*(X_k)$	[0,0.4)	[0.4,0.6)	[0.4,0.9)	[0.6,0.9)	[0.6,1]
$\mathcal{B}_{\Psi_3}(X_k)$	[0,0.4)	[0.4,0.6)	[0.4,0.9)	[0.6,0.9)	[0.6,0.9)
$(M_{\Psi_3})_*(X_k)$	0	0	0	0	0.1
$(M_{\Psi_3})^*(X_k)$	0.4	0.2	0.5	0.3	0.4
$d_{\Psi_3}(X_k)$	0.4	0.2	0.5	0.3	0.3

From Table 5, we can say that X_k $(k = 1, 2, 3, 4, 5)$ are strait fuzzy rough sets depending on $\Psi_2^{\mathfrak{s}}$ since $\mathcal{B}_{\Psi_2}(X_k) \neq \emptyset$ for each $k = 1, 2, 3, 4, 5$.

For each X_k $(k = 1, 2, 3, 4, 5)$ depending on $\Psi_3^{\mathfrak{s}}$, the lower/upper approximations and the internal/exter nal/distance measures are given in Table 6.

From Table 6, we can say that X_k $(k = 1, 2, 3, 4, 5)$ are strait fuzzy rough sets depending on $\Psi_3^{\mathfrak{s}}$ since $\mathcal{B}_{\Psi_3}(X_k) \neq \emptyset$ for each $k = 1, 2, 3, 4, 5$.

Consequently, we obtain the similarity coefficients as

$$S_{\Psi_1^{\mathfrak{s}}} = 1 - \frac{1}{10}(0.2 + 0.4 + 0.4 + 0.2 + 0.4 + 0.4 + 0.2 + 0.2 + 0.2 + 0.4) = 0.7,$$

$$S_{\Psi_2^{\mathfrak{s}}} = 1 - \frac{1}{10}(0.25 + 0.25 + 0.5 + 0.25 + 0.5 + 0.5 + 0.25 + 0.25 + 0.25 + 0.5) = 0.65,$$

$$S_{\Psi_3^{\mathfrak{s}}} = 1 - \frac{1}{10}(0.4 + 0.5 + 0.3 + 0.4 + 0.2 + 0.2 + 0.3 + 0.3 + 0.3 + 0.3) = 0.68.$$

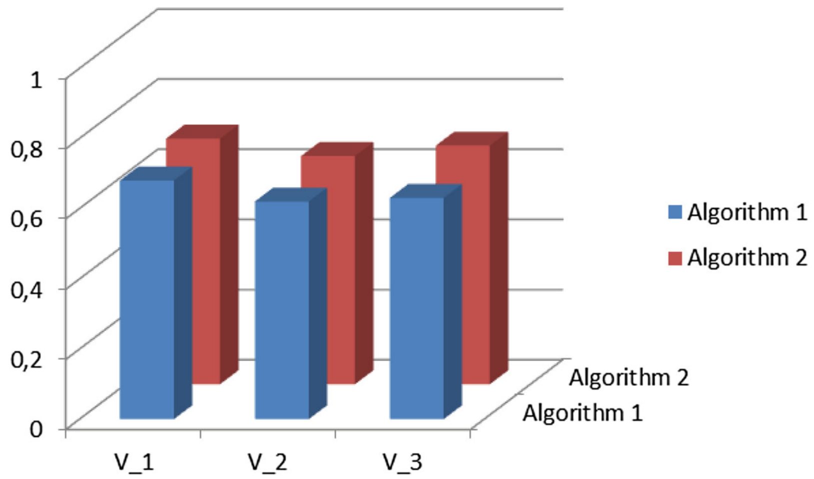

Figure 5. The ranking order of vaccines according to Algorithm 1 and Algorithm 2.

By using these coefficients, we deduce that the ranking order of these vaccines is $V_1 > V_3 > V_2$.

According to the Algorithms 1 and 2, the ranking order of vaccines V_1, V_2, and V_3 is illustrated in Figure 5. Thus, it can be seen that although the similarity coefficients are different (but these values are very close to each other as matched) according to the two algorithms, the ranking order of the vaccines is the same.

7. Conclusions

In this study, we put forward the idea that the representation of memberships of some objects using fuzzy membership grades (exact values) is insufficient. Relatedly, we have studied the fundamentals of strait fuzzy set that permits us to represent the situation in which different membership functions are considered, and thus have constructed the basic theoretical framework of the proposed set. Besides, we have created a rough set structure based on the strait fuzzy sets, named strait fuzzy rough set, and presented its theoretical results. Also, we have developed the evaluation approaches based on the similarity measures of strait fuzzy (rough) sets that can effectively deal with complex problems under fuzzy environment. Then, we illustrated the efficiency of proposed strait fuzzy (rough) similarity measures in handling with real-life problems.

In the future, new similarity and distance measures of strait fuzzy (rough) sets and their practical directions will be an in-depth study in order to solve more and more complex problems. Moreover, research will be conducted on different extensions and properties of strait fuzzy (rough) sets. Eventually, the new strait fuzzy systems and their potential application will be further studied. Our future work will aim to overcome these open problems.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Data availability statement

The data used the support the findings of this study are included within the article.

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