

ABSTRACT KOROVKIN TYPE THEOREMS ON
MODULAR SPACES BY \mathcal{A} -SUMMABILITY

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Abstract. Our aim is to change classical test functions of Korovkin theorem on modular spaces by using \mathcal{A} -summability.

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1. INTRODUCTION

Approximation theory is one of the most thriving areas within functional analysis. Korovkin has proved a well known approximation theorem which states the uniform convergence in $C[a, b]$, the space of continuous real functions defined on $[a, b]$, of a sequence of positive linear operators by stating the convergence only on three test functions $\{1, x, x^2\}$. Korovkin theory provides a useful technique for approaching behavior of positive linear operators within the area of approximation theory. This theory has been studied by many authors in various directions. There is a deep insight into the relation between summability theory and approximation theory. Based on this relation, we give some abstract Korovkin type theorems via modular convergence in the sense of \mathcal{A} -summability and strong convergence in the sense of \mathcal{A} -summability. These notions enable us to give generalizations of the Korovkin theorem. Our aim is to change classical test functions of Korovkin theorem on modular spaces by using \mathcal{A} -summability. Similar problems have been studied in [1], [2], [3], [4].

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We recall the foundations of the theory of modular function spaces and some notions which are needed. We refer the reader to [11], [17].

Let us start by considering the notion of \mathcal{A} -summability of a sequence introduced by Bell (see [12]). Assume that $\mathcal{A} = \{A^{(n)}\} = (a_{kj}^{(n)})$, $j, k, n \in \mathbb{N}$ is a sequence of infinite matrices. $(Ax)_k^{(n)} := \sum_j a_{kj}^{(n)} x_j$ is said to be the \mathcal{A} -transform of x whenever the series converges for all k and n . Then a sequence x is said to be \mathcal{A} -summable (or \mathcal{A} -convergent) to some number L provided that

$$\lim_{k \rightarrow \infty} (Ax)_k^{(n)} = L \quad \text{uniformly in } n \in \mathbb{N}.$$

Also, \mathcal{A} is said to be a regular method of matrices if $\lim_{j \rightarrow \infty} x_j = L$ implies $\lim_{k \rightarrow \infty} (Ax)_k^{(n)} = L$ uniformly in $n \in \mathbb{N}$. This method has the advantage of summing some divergent sequences and has been used in approximation theory (see [21]).

Let I be a locally compact Hausdorff topological space, endowed with a uniform structure $\mathcal{U} \subset 2^{I \times I}$ which generates the topology of I . Let μ be a regular measure defined on \mathcal{B} which is the σ -algebra of all Borel sets of I . Then, by $X(I)$ we denote the space of all real-valued μ -measurable functions on I equipped with the equality μ -a.e. As usual, let $C(I)$ denote the space of all continuous real valued functions on I . The space of all real-valued continuous and bounded functions on I is denoted by $C_b(I)$ and also the subspace of $C_b(I)$ of all functions with compact support on I is denoted by $C_c(I)$. We say that a functional $\varrho: X(I) \rightarrow [0, \infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

- (i) $\varrho[f] = 0$ if and only if $f = 0$ μ -almost everywhere on I ,
- (ii) $\varrho[-f] = \varrho[f]$ for every $f \in X(I)$,
- (iii) $\varrho[\alpha f + \beta g] \leq \varrho[f] + \varrho[g]$ for every $f, g \in X(I)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ϱ is said to be Q -quasi convex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[\alpha f + \beta g] \leq Q\alpha\varrho[Qf] + Q\beta\varrho[Qg]$$

holds for every $f, g \in X(I)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $Q = 1$, then ϱ is called convex.

A modular ϱ is said to be Q -quasi semiconvex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[af] \leq Qa\varrho[Qf]$$

holds for every nonnegative function $f \in X(I)$ and $a \in (0, 1]$.

It is clear that every Q -quasi convex modular is Q -quasi semiconvex. We now consider some subspaces of $X(I)$ by means of a modular ϱ as follows:

$$L^{\varrho}(I) := \left\{ f \in X(I) : \lim_{\lambda \rightarrow 0^+} \varrho[\lambda f] = 0 \right\}$$

and

$$E^\varrho(I) := \{f \in L^\varrho(I) : \varrho[\lambda f] < \infty \text{ for all } \lambda > 0\}$$

is called the modular space generated by ϱ and the space of the finite elements of $L^\varrho(I)$, respectively. Observe that if ϱ is Q -quasi semiconvex, then the space

$$\{f \in X(I) : \varrho[\lambda f] < \infty \text{ for some } \lambda > 0\}$$

coincides with $L^\varrho(I)$. The notions about modulars have been introduced in [19] and have been widely discussed in [4], [5], [7], [9]–[11], [13], [14], [16]–[18], and [20].

We need some of the following assumptions on modulars:

- ▷ ϱ is monotone, i.e. for $f, g \in X(I)$ if $|f| \leq |g|$, then $\varrho[f] \leq \varrho[g]$.
- ▷ ϱ is strongly finite, i.e. $\chi_A \in E^\varrho(I)$ for all $A \in \mathcal{B}$ with $\mu(A) < \infty$.
- ▷ ϱ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every $f \in X(I)$ with $\varrho[f] < \infty$:
 - ▷ for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ with $\mu(A) < \infty$ and $\varrho[\alpha f \chi_{I \setminus A}] \leq \varepsilon$,
 - ▷ for each $\varepsilon > 0$ there is $\delta > 0$ with $\varrho[\alpha f \chi_B] \leq \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

According to [8], recall that $\{f_j\}$ is modularly convergent to a function $f \in L^\varrho(I)$ if and only if

$$\lim_{j \rightarrow \infty} \varrho[\lambda_0(f_j - f)] = 0 \quad \text{for some } \lambda_0 > 0,$$

also $\{f_j\}$ is strongly convergent to a function $f \in L^\varrho(I)$ if and only if

$$\lim_{j \rightarrow \infty} \varrho[\lambda(f_j - f)] = 0 \quad \text{for every } \lambda > 0.$$

Moreover, we recall the following convergences in modular spaces which have also been studied in [15]. Let $\{f_j\}$ be a function sequence whose terms belong to $L^\varrho(I)$. Then $\{f_j\}$ is modularly convergent to a function $f \in L^\varrho(I)$ in the sense of \mathcal{A} -summability if and only if

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda_0(f_j - f)] = 0 \quad \text{for some } \lambda_0 > 0 \text{ uniformly in } n.$$

Also, $\{f_j\}$ is strongly convergent to a function $f \in L^\varrho(I)$ in the sense of \mathcal{A} -summability if and only if

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(f_j - f)] = 0 \quad \text{for every } \lambda > 0 \text{ uniformly in } n.$$

If there exists a constant $M > 0$ such that for all $u \geq 0$

$$\varrho[2u] \leq M\varrho[u]$$

holds, then it is said that ϱ satisfies the Δ_2 -condition. The key property of the Δ_2 -condition is the following theorem.

Theorem 1. *Let $L^\varrho(I)$ be a modular space. Δ_2 -condition is sufficient in order that strong convergence in the sense of \mathcal{A} -summability and modular convergence in the sense of \mathcal{A} -summability be equivalent in $L^\varrho(I)$.*

Proof. Obviously, strong convergence of $\{f_j\}$ to f in the sense of \mathcal{A} -summability is equivalent to the condition $\lim_k \sum_{j=1}^\infty a_{kj}^{(n)} \varrho[2^N \lambda(f_j - f)] = 0$ uniformly in n for some $\lambda > 0$ and all $N = 1, 2, \dots$. Let $\{f_j\}$ be modularly convergent to f in the sense of \mathcal{A} -summability. Then there exists $\lambda > 0$ such that $\lim_k \sum_{j=1}^\infty a_{kj}^{(n)} \varrho[\lambda(f_j - f)] = 0$ uniformly in n . Δ_2 -condition implies by induction that $\varrho[2^N \lambda(f_j - f)] \leq M^N \varrho[\lambda(f_j - f)]$. Therefore we get

$$\lim_k \sum_{j=1}^\infty a_{kj}^{(n)} \varrho[2^N \lambda(f_j - f)] = 0.$$

This completes the proof. □

2. MAIN RESULTS

In this section we give some Korovkin-type theorems by using different test functions from the ordinary ones $\{1, x, x^2\}$ in the sense of \mathcal{A} -summability.

Observe now that if a modular ϱ is monotone and finite, then we have $C(I) \subset L^\varrho(I)$ (see [11]). In a similar manner, if ϱ is monotone and strongly finite, then $C(I) \subset E^\varrho(I)$. Let ϱ be monotone and finite modular on $X(I)$. Assume that D is a set satisfying $C_b(I) \subset D \subset X(I)$. Assume further that $T := \{T_j\}$ is a sequence of positive linear operators from D into $X(I)$. Also we say that the sequence T satisfies condition:

- (*) If there exists a subset $X_T \subset D \cap L^\varrho(I)$ with $C_b(I) \subset X_T$ and a positive real constant R with $T_j f \in L^\varrho(I)$ for all $f \in X_T$ and $j \in \mathbb{N}$ such that

$$\limsup_k \sum_j a_{kj}^{(n)} \varrho[\tau(T_j f)] \leq R \varrho[\tau f]$$

for every $f \in X_T$ and $\tau > 0$.

Assume that $e_0(t) = 1$ for all $t \in I$ and let e_i, a_i be functions in $C_b(I)$ for $i = 0, 1, \dots, m$. Put

$$(1) \quad P_s(t) = \sum_{i=0}^m a_i(s) e_i(t), \quad s, t \in I$$

and suppose that $P_s(t), s, t \in I$, satisfies the following conditions:

(K1) $P_s(s) = 0$ for all $s \in I$,

(K2) for every neighbourhood $U \in \mathcal{U}$ there is a positive real number η with $P_s(t) \geq \eta$ whenever $s, t \in I, (s, t) \notin U$.

Some examples of P_s for which (K1) and (K2) are satisfied have been given in [4].

Theorem 2. Let $\mathcal{A} = \{A^{(n)}\}$ be a sequence of infinite nonnegative real matrices and let ϱ be a strongly finite, monotone and Q -quasi semiconvex modular. Assume that e_i and $a_i, i = 0, 1, \dots, m$ satisfy properties (K1) and (K2). Let $\{T_j\}, j \in \mathbb{N}$ be a sequence of positive linear operators satisfying condition (*). If

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \text{uniformly in } n$$

for some $\lambda > 0$ and $i = 0, 1, \dots, m$, then for every $f \in C_c(I)$

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \text{uniformly in } n$$

for some $\gamma > 0$. Moreover, if

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \text{uniformly in } n$$

for every $\lambda > 0$ and $i = 0, 1, \dots, m$, then for every $f \in C_c(I)$

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] = 0 \quad \text{uniformly in } n$$

for every $\lambda > 0$.

Proof. Let $f \in C_c(I)$. Since I is endowed with \mathcal{U} uniformity, f is uniformly continuous and bounded on I . Let $\varepsilon > 0$. Without loss of generality we can choose $0 < \varepsilon \leq 1$. From the uniform continuity of f there exists $U \in \mathcal{U}$ such that

$$|f(s) - f(t)| \leq \varepsilon, \quad s, t \in I, (s, t) \in U.$$

For every $s, t \in I$ and in correspondence with U let $P_s(t)$ be as in (1) and $\eta > 0$ satisfy condition (K2). If $M = \sup_{t \in I} |f(t)|$, for $s, t \in I, (s, t) \notin U$, we have

$$|f(s) - f(t)| \leq 2M \leq \frac{2M}{\eta} P_s(t).$$

For every $s, t \in I$ we obtain

$$|f(s) - f(t)| \leq 2M \leq \varepsilon + \frac{2M}{\eta} P_s(t).$$

Therefore for every $s, t \in I$ we get

$$(2) \quad -\varepsilon - \frac{2M}{\eta} P_s(t) \leq f(s) - f(t) \leq \varepsilon + \frac{2M}{\eta} P_s(t).$$

Since T_j is a linear positive operator, using (2) for each $j \in \mathbb{N}$ and every $s \in I$ we have

$$-\varepsilon(T_j e_0)(s) - \frac{2M}{\eta}(T_j P_s)(s) \leq f(s)(T_j e_0)(s) - (T_j f)(s) \leq \varepsilon(T_j e_0)(s) + \frac{2M}{\eta}(T_j P_s)(s)$$

and hence

$$\begin{aligned} |(T_j f)(s) - f(s)| &\leq |(T_j f)(s) - f(s)(T_j e_0)(s)| + |f(s)(T_j e_0)(s) - f(s)| \\ &\leq \varepsilon(T_j e_0)(s) + \frac{2M}{\eta}(T_j P_s)(s) + M|(T_j e_0)(s) - e_0(s)|. \end{aligned}$$

Let $\gamma > 0$. Using the modular ϱ in the last inequality, for each $j \in \mathbb{N}$ we have

$$(3) \quad \begin{aligned} \varrho[\gamma(T_j f - f)] &\leq \varrho[3\gamma\varepsilon(T_j e_0)] + \varrho[3\gamma M(T_j e_0 - e_0)] + \varrho\left[6\gamma\frac{M}{\eta}(T_j P_{(\cdot)})(\cdot)\right] \\ &= J_1 + J_2 + J_3. \end{aligned}$$

So to prove the theorem it is sufficient to show that there exists a positive real number γ such that $\lim_{k \rightarrow \infty} \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0$ uniformly in n . From hypothesis there exists $\lambda > 0$ such that for each $i = 0, 1, \dots, m$

$$\lim_{k \rightarrow \infty} \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \text{uniformly in } n.$$

For each $i = 0, 1, \dots, m$ and $s \in I$ choose $N > 0$ and $\gamma > 0$ such that $|a_i(s)| \leq N$ and $\max\{3\gamma M, 6\gamma M\eta^{-1}(m+1)N\} \leq \lambda$. Consider condition (K1), for each $j \in \mathbb{N}$ and $i = 0, 1, \dots, m$ we get

$$\begin{aligned} J_3 &= \varrho\left[6\gamma\frac{M}{\eta}(T_j P_{(\cdot)})(\cdot)\right] = \varrho\left[6\gamma\frac{M}{\eta}(T_j P_{(\cdot)})(\cdot) - P_{(\cdot)}(\cdot)\right] \\ &\leq \sum_{i=0}^m \varrho\left[6\gamma\frac{M}{\eta}(m+1)N(T_j e_i - e_i)\right] \leq \sum_{i=0}^m \varrho[\lambda(T_j e_i - e_i)]. \end{aligned}$$

Hence we obtain

$$\lim_{k \rightarrow \infty} \sum_j a_{kj}^{(n)} J_3 = 0 \quad \text{uniformly in } n.$$

Moreover, from choosing λ and γ it is clear that $\lim_{k \rightarrow \infty} \sum_j a_{kj}^{(n)} J_2 = 0$. Since ϱ is Q -quasi semiconvex and $0 < \varepsilon \leq 1$, we have

$$(4) \quad \varrho[3\gamma\varepsilon e_0] \leq Q\varepsilon\varrho[3\gamma Qe_0].$$

If condition $(*)$ is considered in (3) and (4), we get uniformly in n

$$(5) \quad 0 \leq \limsup_k \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] \leq \limsup_k \sum_j a_{kj}^{(n)} \varrho[3\gamma\varepsilon(T_j e_0)] \\ \leq N\varrho[3\gamma\varepsilon e_0] \leq NQ\varepsilon\varrho[3\gamma Qe_0].$$

Since ε is arbitrary positive real number and ϱ is strongly finite using (5), we have

$$\limsup_k \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \text{uniformly in } n$$

and hence

$$\lim_k \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \text{uniformly in } n.$$

This means that $\{T_j f\}$ is modularly convergent to f in the sense of \mathcal{A} -summability on $L^\varrho(I)$. The second part can be proved similarly to the first one. \square

The next theorem is similar to Theorem 2.1 of [15] (see also [4]) under weaker condition by using different test functions.

Theorem 3. *Let $\mathcal{A} = \{A^{(n)}\}$ be a sequence of infinite nonnegative real matrices and let ϱ be a strongly finite, monotone, absolutely continuous and Q -quasi semi-convex modular on $X(I)$. Let $T_j, j \in \mathbb{N}$ be a sequence of positive linear operators satisfying condition $(*)$. If*

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \text{uniformly in } n$$

for every $\lambda > 0$ and $i = 0, 1, \dots, m$, then for every $f \in L^\varrho(I) \cap D$ with $f - C_b(I) \subset X_T$,

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \text{uniformly in } n$$

for some $\gamma > 0$, where X_T and D are as before.

Proof. Let $f \in L^\varrho(I) \cap D$ such that $f - C_b(I) \subset X_T$. From Proposition 3.2 of [4] there exist $\lambda > 0$ and a sequence (f_m) in $C_c(I)$ such that $\varrho[3\lambda f] < \infty$ and $\lim_m \varrho[3\lambda(f_m - f)] = 0$. Take arbitrary fixed $\varepsilon > 0$ and choose a positive integer \overline{m} such that

$$(6) \quad \varrho[3\lambda(f_{\overline{m}} - f)] \leq \varepsilon.$$

For each $j \in \mathbb{N}$ we have

$$(7) \quad \varrho[\lambda(T_j f - f)] \leq \varrho[3\lambda(T_j f - T_j f_{\overline{m}})] + \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] + \varrho[3\lambda(f_{\overline{m}} - f)].$$

Using a similar technique as in the previous theorem, we obtain

$$(8) \quad \begin{aligned} 0 &= \lim_k \sum_j a_{kj}^{(n)} \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] \\ &= \limsup_k \sum_j a_{kj}^{(n)} \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] \quad \text{uniformly in } n. \end{aligned}$$

From condition (*) there exists $R > 0$ such that

$$(9) \quad \lim_k \sum_j a_{kj}^{(n)} \varrho[3\lambda(T_j f - T_j f_{\overline{m}})] \leq R\varrho[3\lambda(f - f_{\overline{m}})] \leq R\varepsilon \quad \text{uniformly in } n.$$

From (6)–(9) and the subadditivity of the operator \limsup we have

$$(10) \quad 0 \leq \limsup_k \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] \leq \varepsilon(R + 1) \quad \text{uniformly in } n.$$

From (10) and the arbitrariness of ε we get for $\gamma = \lambda$

$$\limsup_k \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] = 0 \quad \text{uniformly in } n.$$

This implies $\lim_k \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] = 0$ uniformly in n . □

3. CONCLUDING REMARKS AND EXAMPLES

In this section we give some remarks and an example to show that our theorems are generalizations of known theorems. We remark that if $A^{(n)}$ equals to identity matrix for every $n \in \mathbb{N}$, then \mathcal{A} -summability reduces to the ordinary convergence. In this case our Theorem 3 is similar to Theorem 3.2 of [8].

Take $I = [0, 1]$ and let $\varphi: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be a convex continuous function with $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, $\varphi(u) \rightarrow \infty$ as $u \rightarrow \infty$. Then it is easily shown that

$$\varrho[f] = U_\varphi[f] = \int_I \varphi(|f(t)|) d\mu(t)$$

is a convex modular on the space $X(I)$. U_φ is known as an Orlicz modular in $X(I)$. The respective modular space $L_\varphi^g(I)$ is called the Orlicz space. Now let us consider the following linear positive operator on the space $L_\varphi^g(I)$ which is defined as

$$(11) \quad B_j(f; x) := s_j \sum_{r=0}^j \binom{j}{r} x^r (1-x)^{j-r} (j+1) \int_{r/(j+1)}^{(r+1)/(j+1)} f(t) dt \quad \text{for } x \in I,$$

where $\{s_j\}$ is a sequence of zeros and ones which is \mathcal{A} -summable to 1, but not ordinary convergent. Also we assume that \mathcal{A} is a regular method of matrices. Observe that the operators B_j map the Orlicz space L_φ^g into itself. By Lemma 5.1 of [8], for every $h \in X_B := L_\varphi^g$, all $\lambda > 0$ and for a positive constant N we get

$$U_\varphi[\lambda B_j h] \leq s_j N U_\varphi[\lambda h].$$

Then we have

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda B_j h] \leq N U_\varphi[\lambda h].$$

It is easily seen that

$$\begin{aligned} B_j(e_0; x) &= s_j, \\ B_j(e_1; x) &= s_j \left(\frac{jx}{j+1} + \frac{1}{2(j+1)} \right), \\ B_j(e_2; x) &= s_j \left(\frac{j(j-1)x^2}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2} \right), \end{aligned}$$

where $e_i(t) = t^i$, $i = 0, 1, 2$. Therefore we can observe for any $\lambda > 0$ that

$$\lambda |B_j(e_0; x) - e_0(x)| = \lambda(1 - s_j)$$

which implies

$$\begin{aligned} U_\varphi[\lambda(B_j e_0 - e_0)] &= U_\varphi[\lambda(1 - s_j)] = \int_0^1 \varphi[\lambda(1 - s_j)] dx \\ &= \varphi[\lambda(1 - s_j)] = (1 - s_j)\varphi(\lambda) \end{aligned}$$

because of the definition of $\{s_j\}$. Now we get for any $\lambda > 0$

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda(B_j e_0 - e_0)] = 0 \quad \text{uniformly in } n.$$

Also since

$$\lambda|B_j(e_1; x) - e_1(x)| \leq \lambda \left\{ (1 - s_j) + \frac{3s_j}{2(j+1)} \right\}$$

by the definition of $\{s_j\}$ and U_φ , we may write that

$$\begin{aligned} U_\varphi[\lambda(B_j e_1 - e_1)] &\leq U_\varphi \left[\lambda \left\{ (1 - s_j) + \frac{3s_j}{2(j+1)} \right\} \right] \leq U_\varphi[2\lambda(1 - s_j)] + U_\varphi \left[\frac{3\lambda s_j}{j+1} \right] \\ &= \varphi[2\lambda(1 - s_j)] + \varphi \left[\frac{3\lambda s_j}{j+1} \right], \end{aligned}$$

which implies for any $\lambda > 0$ that

$$U_\varphi[\lambda(B_j e_1 - e_1)] \leq (1 - s_j)\varphi[2\lambda] + s_j \varphi \left[\frac{3\lambda}{j+1} \right].$$

Since φ is continuous, we have $\lim_j \varphi[3\lambda/(j+1)] = \varphi[\lim_j 3\lambda/(j+1)] = \varphi(0) = 0$.

Therefore we have for every $\lambda > 0$

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda(B_j e_1 - e_1)] = 0 \quad \text{uniformly in } n.$$

Finally, since

$$\lambda|B_j(e_2; x) - e_2(x)| \leq \lambda \left\{ (1 - s_j) + s_j \frac{15j+4}{3(j+1)^2} \right\},$$

we get

$$\begin{aligned} U_\varphi[\lambda(B_j e_2 - e_2)] &\leq U_\varphi[2\lambda(1 - s_j)] + U_\varphi \left[\lambda s_j \frac{30j+8}{3(j+1)^2} \right] \\ &= \varphi[2\lambda(1 - s_j)] + \varphi \left[\lambda s_j \frac{30j+8}{3(j+1)^2} \right], \end{aligned}$$

which yields

$$(12) \quad U_\varphi[\lambda(B_j e_2 - e_2)] \leq (1 - s_j)\varphi[2\lambda] + s_j \varphi \left[\lambda \frac{30j+8}{3(j+1)^2} \right].$$

Considering the continuity of φ , it follows from (12) for any $\lambda > 0$ that

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda(B_j e_2 - e_2)] = 0 \quad \text{uniformly in } n.$$

The sequence of operators $\{B_j\}$ defined by (11) satisfies all conditions of Theorem 3. So we conclude that

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda_0(B_j(f) - f)] = 0 \quad \text{uniformly in } n$$

holds for $\lambda_0 > 0$ and every $f \in L_\varphi^g(I)$. However, since $\{s_j\}$ is not convergent to zero, it is clear that $\{B_j\}$ is not modularly convergent to f .

Also remark that if we assume $I = [0, 1]$, $e_0(t) = 1$, $e_1(t) = t$, $e_2(t) = t^2$, $a_0(s) = s^2$, $a_1(s) = -2s$, $a_2(s) = 1$, $s, t \in I$ in equation (1), $A^{(n)} = I$ for every $n \in \mathbb{N}$ and that ϱ is a sup-norm on $C(I)$ which is the set of all continuous functions on I in Theorem 3, the classical Korovkin theorem is obtained.

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