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Semisimple modules that are small cyclic in their injective envelopes

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This paper presents the fundamental characteristics of s-cosingular modules, which constitute semisimple and small submodules within an injective module. We establish that over a commutative Kasch ring S, each (semi) simple S-module is s-cosingular if and only if each maximal ideal of S is essential in S. Furthermore, we delve into the examination of modules that fulfill the condition of $(\mathcal{S}_s^*)$. We provide several characterizations of rings using these modules. Specifically, we show that a ring S is left ss-Harada if and only if each left S-module verifies $(\mathcal{S}_s^*)$.

Keywords: Module with $(\mathcal{S}_s^*)$; s-cosingular module; left ss-Harada rings; left V-rings; QF-rings.

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1. Introduction

Using the concepts of (semi) simple modules, which are one of the most important concepts of module theory and thus concepts of ring theory, some characterizations of the class of rings are given. In homological algebra, the subject of (semi) simple modules is the basic structure that any researcher should know. A ring S is a division ring if and only if ${}_SS$ is *simple*. A ring S is *semisimple* if and only if each left (or right) S-module is semisimple if and only if each left S-module is projective (or injective) if and only if each (semi) simple S-module is projective. As an injective module is the dual concept of a projective module, one can consider that whole (semi) simple S-modules are injective, the ring S may be semisimple but this does not hold, in general. Therefore, the class of rings whose (semi) simple modules are injective is defined as follows. A ring S is termed as *left V-ring* in case each left simple S-module is injective, and it is termed as *left SSI-ring* provided each left semisimple S-module is injective (see [3] and [23]). It is shown in [3] that a ring S is a left SSI-ring if and only if it is a left noetherian ring and a left V-ring. The fact that a module W is injective if and only if it is a direct summand of each extension of W is widely known. So a simple module W is either injective or a small submodule of its injective envelope $E(W)$. The injectivity of (semi) simple modules has been explored extensively by many authors in recent years. Therefore, it is natural to study the notion of (semi) simple modules which are small.

The main purpose of this paper is to develop (semi) simple and small submodules of injective modules. We designate the concept of *s-cosingular* modules and concentrate on their fundamental properties. Among other results, we show that a module W is s-cosingular if and only if $W \subseteq \mathrm{Soc}_s(E(W))$, where $\mathrm{Soc}_s(E(W))$ is the sum of whole simple and small submodules of $E(W)$. We establish that the property of being s-cosingular module whose class is $\mathcal{SC}(S)$ is carried by the terms submodules, quotient modules, direct sums and sums. Additionally, we demonstrate that over a commutative Kasch ring S each (semi) simple S-module is s-cosingular if and only if each maximal ideal of S is essential in S. In Sec. 3 of this paper, we deal the modules verifying $(\mathcal{S}_s^*)$. We provide certain characterizations of rings through these modules. Specifically, we demonstrate that each left S-module verifies $(\mathcal{S}_s^*)$ if and only if the ring S is left ss-Harada ring.

In this paper, we assess the rings, represented as S, which are associative and have a unit, along with the unital left modules over S. Let W be an S-module as mentioned. We use the notation $Y \leq W$ to mean Y is a submodule of W. We denote $\mathrm{Rad}(W)$, $\mathrm{Soc}(W)$, $\mathrm{Soc}_s(W)$, $E(W)$ and $Z(W)$ as the radical, the socle, the sum of whole simple submodules of W that are small in W, the injective envelope and the singular submodule of W, respectively (see [23] and [24]). A submodule Y of W is termed as *essential* in W, notated as $Y \trianglelefteq W$, if $Y \cap H \neq 0$ for each nonzero $H \leq W$. A module W is said to be *singular* provided $W \cong T/Y$ for some module T and $Y \trianglelefteq T$. A submodule Y of W is *small* in W, denoted by the notation $Y \ll W$, if $W \neq Y + H$ for each proper submodule H of W. In [14], a module W is termed as *small* if W is a small submodule of some S-module and the author showed that W is a small module if and only if W is small submodule of $E(W)$. It is evident that each small submodule of W is a small module. We will refer to [5], [12] and [23] for whole undefined notions used in this paper and also for further properties of small and semisimple submodules.

The subsequent lemma is derived from [13, Lemma 2], and we will utilize it along the paper.

Lemma 1.1. *Suppose that W is a module. Then* $\mathrm{Soc}_s(W) = \mathrm{Soc}(W) \cap \mathrm{Rad}(W)$.

2. s-Cosingular Modules

In this section, we define the concept of s-cosingular modules and acquire their basic properties. We prove that a module W is s-cosingular if and only if it is a semisimple and small submodule of its injective envelope. In particular, we show that over a commutative Kasch ring S each (semi) simple S-module is s-cosingular if and only if each maximal ideal of S is essential in S.

In [19], we encounter the submodule below of a module W:

$$\mathcal{Z}^*(W) = \{w \in W \mid Sw \text{ is a small module}\}.$$

Since $\mathrm{Rad}(W)$ is the sum of whole small submodules of W, we obtain that $\mathrm{Rad}(W) \leq \mathcal{Z}^*(W)$. According to [19], a module W is termed as *cosingular* when $\mathcal{Z}^*(W) = W$, which means that $W \leq \mathrm{Rad}(E(W))$.

Influenced by Özcan's concept, for a module W, we contemplate the following subset:

$$\mathcal{Z}_s^*(W) = \{w \in W \mid Sw \text{ is a semisimple and small module}\}.$$

Lemma 2.1. *Suppose that* W *is a module. Then* $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathcal{Z}^*(W)$.

Proof. Let $w \in \mathcal{Z}_s^*(W)$. Then Sw is a semisimple and small module. Thus $w = 1_S w \in Sw \leq \mathrm{Soc}(W) \cap \mathcal{Z}^*(W)$. This implies that $\mathcal{Z}_s^*(W) \leq \mathrm{Soc}(W) \cap \mathcal{Z}^*(W)$. Conversely, let $w \in \mathrm{Soc}(W) \cap \mathcal{Z}^*(W)$. Then Sw is a semisimple module, and also it is small module according to the definition of $\mathcal{Z}^*(W)$. Hence, $w \in \mathcal{Z}_s^*(W)$. □

The fact below arises from [19] that for a module W,

$$\mathcal{Z}^*(W) = W \cap \mathrm{Rad}(E(W)).$$

Using this fact, we acquire the result below.

Corollary 2.1. *Suppose that* W *is a module. Then* $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathrm{Rad}(E(W))$.

Proof. By Lemma 2.1, we have that $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathcal{Z}^*(W)$. The definition of $\mathcal{Z}^*(W)$ leads to the conclusion that $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathrm{Rad}(E(W))$. □

Definition 2.1. Let S be an arbitrary ring and W be an S-module. W is termed as an *s-cosingular* module when $\mathcal{Z}_s^*(W) = W$.

The subsequent result demonstrates that an s-cosingular module is a semisimple module that is small contained within its injective envelope.

Proposition 2.1. *Suppose that* W *is a module. Then* W *is s-cosingular if and only if* $W \subseteq \mathrm{Soc}_s(E(W))$, *that is,* W *is semisimple and small.*

Proof. ($\Longrightarrow$) By Corollary 2.1, it is deduced that $W = \mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathrm{Rad}(E(W)) \subseteq \mathrm{Soc}(E(W)) \cap \mathrm{Rad}(E(W)) = \mathrm{Soc}_s(E(W))$, and so W is semisimple and small.

($\Longleftarrow$) This direction of the proof is evident, because the class of both semisimple and small modules is closed under submodules. □

As we mentioned in introduction, a simple module is either injective or small. Employing this fact along with Proposition 2.1, we derive the result below.

Corollary 2.2. *Each simple module which is not injective is s-cosingular.*

Let S be a ring. By $\mathcal{I}(S)$ and $\mathcal{SC}(S)$ we signify the class of whole injective S-modules and the class of whole s-cosingular S-modules, respectively. It follows that a ring S is a left SSI-ring if and only if $\mathcal{SC}(S) = \{0\}$. The subsequent fact is a direct outcome of Proposition 2.1.

Corollary 2.3. *For an arbitrary ring S,* $\mathcal{I}(S) \cap \mathcal{SC}(S) = \{0\}$.

Prior to delving into the study of s-cosingular modules, we shall present some facts that play a pivotal role in this investigation.

Lemma 2.2. *For a module W, the statements below hold*:

(1) *When* $f : W \to T$ *is a homomorphism,* $f(\mathcal{Z}_s^*(W)) \leq \mathcal{Z}_s^*(T)$.
(2) *When* Y *is a submodule of W,* $\mathcal{Z}_s^*(Y) = Y \cap \mathcal{Z}_s^*(W)$.
(3) *When* $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$, $\mathcal{Z}_s^*(W) = \bigoplus_{\lambda\in\Lambda} \mathcal{Z}_s^*(W_\lambda)$.

Proof. (1) Let $f : W \to T$ be a homomorphism and $a \in f(\mathcal{Z}_s^*(W))$ be arbitrary. Then $a = f(w)$ for some element w of $\mathcal{Z}_s^*(W)$. Since $w \in \mathcal{Z}_s^*(W)$, then $Sw \leq \mathrm{Soc}(W) \cap \mathcal{Z}^*(W)$. It follows from [19, Lemma 2.1; 12, 8.1.5] that $Sa = Sf(w) \leq \mathrm{Soc}(T) \cap \mathcal{Z}^*(T)$. Hence, $a \in \mathrm{Soc}(T) \cap \mathcal{Z}^*(T) = \mathcal{Z}_s^*(T)$.

(2) It is evident.

(3) It follows from (2) that for whole $\lambda \in \Lambda$, $\mathcal{Z}_s^*(W_\lambda) \leq \mathcal{Z}_s^*(W)$. Thus $\bigoplus_{\lambda\in\Lambda} \mathcal{Z}_s^*(W_\lambda) \leq \mathcal{Z}_s^*(W)$. Let $w \in \mathcal{Z}_s^*(W)$. Then $w = w_{\lambda_1} + w_{\lambda_2} + \cdots + w_{\lambda_n}$ for some elements $w_{\lambda_j} \in W_{\lambda_j}$ where $j = 1, 2, \ldots, n$. For each $j = 1, 2, \ldots, n$, $w_{\lambda_j} = \pi_{\lambda_j}(w) \in \pi_{\lambda_j}(\mathcal{Z}_s^*(W)) \leq \mathcal{Z}_s^*(W_{\lambda_j})$ where $\pi_{\lambda_j} : W \to W_{\lambda_j}$ is the canonical projection by (1). Thus $w \in \bigoplus_{\lambda\in\Lambda} \mathcal{Z}_s^*(W_\lambda)$. Hence, $\mathcal{Z}_s^*(W) = \bigoplus_{\lambda\in\Lambda} \mathcal{Z}_s^*(W_\lambda)$. □

Corollary 2.4. *The class of s-cosingular S-modules, denoted as* $\mathcal{SC}(S)$ *for any ring S, is closed under direct sums, submodules and quotient modules.*

Proof. The proof is evident by Lemma 2.2. □

Observe from Corollary 2.4 that the class $\mathcal{SC}(S)$ of s-cosingular S-modules is pretorsion and hereditary in S-Mod. On the other hand, this class does not define torsion theory in S-Mod since $\mathcal{SC}(S)$ may not be closed under extensions.

Example 2.1. Given the local ring $S = \mathbb{Z}_4$ and consider the module $W = {}_SS$. Then $\mathrm{Rad}(W)$ and $W/\mathrm{Rad}(W)$ are s-cosingular modules. However, W is not s-cosingular since it is not semisimple.

Proposition 2.2. *For any ring S and any S-module W, $\mathcal{Z}_s^*(S)W \leq \mathcal{Z}_s^*(W)$.*

Proof. Suppose that $w \in W$. Let $f : S \to W$ be any map defined by $f(a) = aw$ for whole $a \in S$. Then f is a homomorphism and so, by Lemma 2.2-(1), $f(\mathcal{Z}_s^*({}_SS)) \leq \mathcal{Z}_s^*(W)$. Now let $s \in \mathcal{Z}_s^*(S)$. Then $sw = f(s) \in f(\mathcal{Z}_s^*(S)) \leq \mathcal{Z}_s^*(W)$, and so $sw \in \mathcal{Z}_s^*(W)$. It follows that $\mathcal{Z}_s^*(S)w \leq \mathcal{Z}_s^*(W)$. Hence, $\mathcal{Z}_s^*(S)W \leq \mathcal{Z}_s^*(W)$. □

Corollary 2.5. *Suppose that S is a ring. Then $\mathcal{Z}_s^*(S)$ is an ideal of S.*

Proof. By Proposition 2.2. □

Remember that a module W is termed as *locally projective*, under the condition that, when $g : X \to T$ is an epimorphism and $f : W \to T$ is a homomorphism of the modules X and T, for any finitely generated submodule K of W, there exists a homomorphism $h : W \to X$ with the property that $(gh)|_K = f|_K$. It is evident that each projective module is inherently a locally projective module.

Theorem 2.1. *Suppose that S is a ring and W is a locally projective S-module. Then $\mathcal{Z}_s^*(W) = \mathcal{Z}_s^*(S)W$.*

Proof. It is evident that $\mathcal{Z}_s^*(S)W \leq \mathcal{Z}_s^*(W)$ as indicated by Proposition 2.2. Let $w \in \mathcal{Z}_s^*(W)$. As W is locally projective, this implies the existence of a finite set of homomorphisms $f_\lambda : W \to S$ and elements $w_\lambda \in W$ where $\lambda = 1, 2, \ldots, n$ such that the expression $f_1(w)w_1 + f_2(w)w_2 + \cdots + f_n(w)w_n = w$ holds. By using Lemma 2.2-(1), we conclude that $f_\lambda(w) \in \mathcal{Z}_s^*(S)$, and so $\mathcal{Z}_s^*(W) \leq \mathcal{Z}_s^*(S)W$. Hence, $\mathcal{Z}_s^*(W) = \mathcal{Z}_s^*(S)W$. □

Explicitly, each s-cosingular module is cosingular. Also, a semisimple cosingular module is s-cosingular. But for instance, suppose that W is a radical module with zero socle. Then W is a cosingular module but not s-cosingular. Because, $\mathcal{Z}^*(W) = W \cap \mathrm{Rad}(E(W)) = \mathrm{Rad}(W) \cap \mathrm{Rad}(E(W)) = \mathrm{Rad}(W) = W$. However, W is not s-cosingular since $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathcal{Z}^*(W) = 0$.

We will now provide a characterization of rings that possess the property that each small module is s-cosingular, subject to a specific condition.

Suppose that W is a module. W is termed as a *good module* provided $f(\mathrm{Rad}(W)) = \mathrm{Rad}(f(W))$ for any homomorphism $f : W \to T$. A ring S is termed as a *left good ring* in case ${}_SS$ is a good module (see [23, Section 23]).

Theorem 2.2. *Suppose that S is a left good ring. Then the statements below are equivalent*:

(1) $\mathrm{Rad}(S) \leq \mathrm{Soc}({}_SS)$.
(2) *Each cosingular S-module is s-cosingular.*
(3) *Each small S-module is semisimple.*

Proof. (1) $\Longrightarrow$ (2) Suppose that W is an injective S-module. Now consider the epimorphism $\psi : S^{(\Lambda)} \to W$ for some index set Λ. (1) implies that $\mathrm{Rad}(S^{(\Lambda)}) \leq \mathrm{Soc}(S^{(\Lambda)})$. So $\mathrm{Rad}(S^{(\Lambda)})$ is semisimple. Since S is a left good ring, $\psi(\mathrm{Rad}(S^{(\Lambda)})) = \mathrm{Rad}(\psi(S^{(\Lambda)})) = \mathrm{Rad}(W)$ is semisimple. Let A be a cosingular S-module, then $A = \mathcal{Z}^*(A) \leq \mathrm{Rad}(E(A))$, and so A is semisimple. Thus $\mathcal{Z}_s^*(A) = \mathrm{Soc}(A) \cap \mathrm{Rad}(E(A)) = \mathrm{Soc}(A) = A$.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are evident. □

It is widely known that $\mathrm{Rad}(W) = 0$ for any S-module W if and only if the ring S is a left V-ring. In [11], left WV-rings are introduced extending the concept of left V-rings. A ring S is termed as *left WV-ring* if each simple S-module is S/J-injective for any left ideal J of S such that S/J is proper.

Corollary 2.6. *Suppose that S is a left WV-ring which is not left V-ring. Then each cosingular S-module is s-cosingular.*

Proof. As left WV-rings are left good and $\mathrm{Rad}(S)$ is simple by [21, Lemma 3] then the result can be seen from Theorem 2.2. □

Lemma 2.3. *Suppose that S is a left WV-ring which is not left V-ring. Then* $\mathrm{Rad}(S)$ *is s-cosingular S-module.*

Theorem 2.3. *Suppose that S is a left WV-ring that is not left V-ring and W is an S-module. Then* $\mathrm{Rad}(W)$ *is s-cosingular.*

Proof. Since $\mathrm{Rad}(W)$ is cosingular, then we can conclude from Corollary 2.6 that $\mathrm{Rad}(W)$ is s-cosingular module as S is left WV-ring which is not left V-ring. □

Proposition 2.3. *The statements below are equivalent for a ring S*:

(1) *S is a left V-ring.*
(2) *For each left S-module W, $\mathcal{Z}_s^*(W) = 0$.*
(3) *For each simple left S-module W, $\mathcal{Z}_s^*(W) = 0$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that W is an S-module. By [18, Theorem 12] $\mathcal{Z}^*(W) = 0$. Hence, $\mathcal{Z}_s^*(W) = 0$.

(2) $\Longrightarrow$ (3) It is evident.

(3) $\Longrightarrow$ (1) Suppose that W is a simple S-module. Then by hypothesis, $\mathcal{Z}_s^*(W) = 0$. Since W is simple, then W is either injective or small. When W is small, then by [19] we conclude that $\mathcal{Z}^*(W) = W$. This is a contradiction. Thus W is injective, and hence S is a left V-ring. □

Remember that a submodule H is termed as a *Rad-supplement* of Y in a module W if $W = H + Y$ and $H \cap Y \leq \mathrm{Rad}(H)$ (see [5, 10.14]).

Proposition 2.4. *Let W be a module. Assume that W is a Rad-supplement in $E(W)$. Then $\mathcal{Z}_s^*(W) = \mathrm{Soc}_s(W)$.*

Proof. Since W is a *Rad*-supplement in $E(W)$, then we infer from [2, Corollary 4.2] that $\mathrm{Rad}(W) = W \cap \mathrm{Rad}(E(W))$. Therefore, we get $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathrm{Rad}(E(W)) = (\mathrm{Soc}(W) \cap W) \cap \mathrm{Rad}(E(W)) = \mathrm{Soc}(W) \cap (W \cap \mathrm{Rad}(E(W))) = \mathrm{Soc}(W) \cap \mathrm{Rad}(W) = \mathrm{Soc}_s(W)$. □

Proposition 2.5. *Suppose that W is an injective module. Then $\mathcal{Z}_s^*(W) = \mathrm{Soc}_s(W)$.*

Proof. As W is injective, then it is deduced from [19] that $\mathcal{Z}^*(W) = \mathrm{Rad}(W)$. Thus $\mathcal{Z}_s^*(W) = \mathrm{Soc}(W) \cap \mathcal{Z}^*(W) = \mathrm{Soc}(W) \cap \mathrm{Rad}(W) = \mathrm{Soc}_s(W)$. □

We will now provide a characterization of the rings for which their semisimple modules are s-cosingular. Remember that a module W is *flat* provided each exact sequence $0 \to W \xrightarrow{\psi} T \xrightarrow{\phi} K \to 0$ is pure exact, that is, $\psi(W)$ is a pure submodule of T. Each projective module falls under the class of flat modules. A ring S is termed as *left Kasch ring* in case each simple left S-module embeds into S.

Lemma 2.4. *Suppose that S is a ring. Then the statements below are equivalent*:

(1) *Each semisimple S-module is s-cosingular.*
(2) *Each simple S-module is s-cosingular.*

Proof. (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (1) Let $W = \sum_{\lambda\in\Lambda} W_\lambda$ where each W_λ is a simple submodule of W and Λ is any index set. Then $W_\lambda \leq W \leq E(W)$, and so, by (2), $W_\lambda \leq \mathrm{Soc}_s(E(W))$. It implies that $W = \sum_{\lambda\in\Lambda} W_\lambda \leq \mathrm{Soc}_s(E(W))$. Hence, W is s-cosingular by Proposition 2.1. □

The result below is due to Ware.

Lemma 2.5 ([22, Lemma 2.6]). *Suppose that S is a commutative ring and W is a simple S-module. Then W is injective if and only if it is flat.*

Theorem 2.4. *Suppose that S is a commutative Kasch ring. Then the statements below are equivalent*:

(1) *Each semisimple S-module is s-cosingular.*
(2) *Each simple S-module is s-cosingular.*
(3) *Each simple S-module is singular.*
(4) *If an S-module W has a nonzero socle, the socle is singular.*
(5) *Each maximal ideal of S is essential in W.*

Proof. (1) $\Longleftrightarrow$ (2) By Lemma 2.4.

(2) $\Longrightarrow$ (3) Suppose that W is a simple S-module. We infer from [9, Proposition 1.24] that W is projective or singular. Assume that W is projective. Since projective modules are flat, it is flat. By Lemma 2.5, W is an injective module and so $W \in \mathcal{I}(S) \cap \mathcal{SC}(S)$. This is a contradiction according to Corollary 2.3. Thus W is singular.

(3) $\Longrightarrow$ (4) By (3), it is evident.

(4) $\Longrightarrow$ (5) Let J be a maximal ideal of S and $J \cap A = 0$ for some ideal A of S. Therefore, the sum $S = J + A$ is direct and so A is isomorphic to the simple S-module S/J. Thus it brings about the result that A is simple and $A \subseteq \mathrm{Soc}({}_S S) \neq 0$. By [23, 18.1], we get that A is projective as a direct summand of the projective module ${}_S S$. This contradicts the fact that $\mathrm{Soc}({}_S S)$ is singular. Hence, J is an essential ideal of S.

(5) $\Longrightarrow$ (2) Let W be a simple S-module. Then there exists a maximal ideal J of S such that $W \cong S/J$. Since S is a Kasch ring, W can be embedded in the ring S. With the similar method in the proof of (2) $\Longrightarrow$ (3), we deduce that W is not injective by (5). This implies that W is an s-cosingular module according to Corollary 2.2. □

3. Harada Rings

Consider a module W. In [7], W is termed as *ss-lifting* when each submodule Y of W contains a direct summand W_1 of W such that $W = W_1 \oplus W_2$ and $Y \cap W_2 \leq \mathrm{Soc}_s(W)$. The same paper provides various properties of these modules. In particular, it has been shown in [7, Theorem 6] that a ring S is artinian serial with semisimple radical if and only if each left S-module is ss-lifting.

In this section, by the motivation of Özcan's modules with $(\mathcal{S}^*)$, we generalize ss-lifting modules to the modules that verify $(\mathcal{S}^*_s)$ since $\mathrm{Soc}_s(W) \leq \mathcal{Z}^*_s(W)$ for any module W. Our main purpose is to state that the necessary and the sufficient condition for a ring S to be a left Harada with semisimple radical is that each left S-module verifies the property $(\mathcal{S}^*_s)$.

Recall from [19] that a module W holds $(\mathcal{S}^*)$, provided for each submodule Y of W, there exists a direct summand D of W such that $D \leq Y$ and $\mathcal{Z}^*(Y/D) = Y/D$.

Definition 3.1. Let S be a ring and W be an S-module. We refer to a module W exhibits the property $(\mathcal{S}_s^*)$ when for each submodule Y of W, there exists a direct summand D of W such that $D \leq Y$ and Y/D is s-cosingular.

Theorem 3.1. *Suppose that W is an S-module. The statements below are equivalent*:

(1) *W exhibits the property $(\mathcal{S}_s^*)$.*
(2) *For each submodule Y of W, W has a decomposition $W = W_1 \oplus W_2$ such that $W_1 \leq Y$ and $Y \cap W_2$ is s-cosingular.*
(3) *Each submodule Y of W has a decomposition $Y = D \oplus D'$ such that D is a direct summand of W and D' is an s-cosingular module.*

Proof. The proof can be readily constructed in a similar manner to the one in [5, 22.1]. □

Proposition 3.1. *When a module W exhibits the property $(\mathcal{S}_s^*)$, each submodule of W does, too.*

Proof. Let $Y \leq W$. Suppose that V is any submodule of Y. Therefore, as W possesses $(\mathcal{S}_s^*)$, then it implies the existence of the decomposition $W = W_1 \oplus W_2$ of W such that $W_1 \leq V$ and $V \cap W_2$ is s-cosingular. Thus we get $Y = W_1 \oplus (Y \cap W_2)$. Hence, Y exhibits the property $(\mathcal{S}_s^*)$ by Theorem 3.1. □

Proposition 3.2. *For a module W, the statements below are equivalent*:

(1) *$W/\mathcal{Z}_s^*(W)$ is semisimple.*
(2) *For each $Y \leq W$, there exists a submodule H of W such that $W = Y + H$ and $Y \cap H$ is s-cosingular.*
(3) *$W = H \oplus Y$ where H is semisimple, $Y/\mathrm{Soc}(Y)$ is semisimple and $\mathrm{Soc}(Y) \trianglelefteq Y$.*

Proof. (1) $\Longrightarrow$ (3) Suppose that K is a maximal submodule of W with respect to the property $K \cap \mathcal{Z}_s^*(W) = 0$. Then $K \oplus \mathcal{Z}_s^*(W) \trianglelefteq W$. Moreover, $K \cong (K \oplus \mathcal{Z}_s^*(W))/\mathcal{Z}_s^*(W)$ is a direct summand of $W/\mathcal{Z}_s^*(W)$. Thus K is semisimple and there is a semisimple submodule $Y/\mathcal{Z}_s^*(W)$ such that $(K + Y)/\mathcal{Z}_s^*(W) = W/\mathcal{Z}_s^*(W)$. Hence, $W = K + Y$ and $K \cap Y \leq K \cap \mathcal{Z}_s^*(W) = 0$ as $K \oplus \mathcal{Z}_s^*(W) \trianglelefteq W$ and $\mathcal{Z}_s^*(W) \trianglelefteq Y$. Note that $\mathcal{Z}_s^*(W) = \mathcal{Z}_s^*(K) \oplus \mathcal{Z}_s^*(Y) = \mathcal{Z}_s^*(Y)$. Hence, $\mathcal{Z}_s^*(Y) = \mathrm{Soc}(Y)$.

(3) $\Longrightarrow$ (1) It can be observed that $W/\mathrm{Soc}(Y) \cong H \oplus (Y/\mathrm{Soc}(Y))$. Note that

$$W/\mathcal{Z}_s^*(W) \cong (W/\mathrm{Soc}(Y))/(\mathcal{Z}_s^*(W)/\mathrm{Soc}(Y)).$$

Since each quotient module of a semisimple module is semisimple by [12, 8.1.5], then $W/\mathcal{Z}_s^*(W)$ is semisimple.

(1) $\Longrightarrow$ (2) By the hypothesis, $(Y + \mathcal{Z}_s^*(W))/\mathcal{Z}_s^*(W)$ is a direct summand of $W/\mathcal{Z}_s^*(W)$ for any submodule Y of W. Thus there exists a submodule $H/\mathcal{Z}_s^*(W)$

of $W/\mathcal{Z}_s^*(W)$ such that $(Y+\mathcal{Z}_s^*(W))/\mathcal{Z}_s^*(W)\oplus H/\mathcal{Z}_s^*(W) = W/\mathcal{Z}_s^*(W)$. Hence, we get $W = Y + H$ and $H \cap Y$ is s-cosingular.

(2) $\Longrightarrow$ (1) Let $Y/\mathcal{Z}_s^*(W)$ be a submodule of $W/\mathcal{Z}_s^*(W)$. Then by the hypothesis, there exists a submodule H of W such that $W = H + Y$ and $H \cap Y$ is s-cosingular. Thus $Y/\mathcal{Z}_s^*(W)\oplus(H+\mathcal{Z}_s^*(W))/\mathcal{Z}_s^*(W) = W/\mathcal{Z}_s^*(W)$. Hence, $W/\mathcal{Z}_s^*(W)$ is semisimple. □

The subsequent result can be derived from Proposition 3.2.

Corollary 3.1. *Suppose that a module W exhibits the property $(\mathcal{S}_s^*)$. Then $W/\mathcal{Z}_s^*(W)$ is semisimple and* $\mathrm{Rad}(W)$ *is s-cosingular.*

Proof. By Theorem 3.1 and Proposition 3.2, we conclude that $W/\mathcal{Z}_s^*(W)$ is semisimple. Therefore

$$\mathrm{Rad}(W/\mathcal{Z}_s^*(W)) = 0$$

and so $\mathrm{Rad}(W) \subseteq \mathcal{Z}_s^*(W)$. It means that $\mathrm{Rad}(W)$ is s-cosingular. □

It is evident that each module with $(\mathcal{S}_s^*)$ verifies $(\mathcal{S}^*)$. The example below shows that the converse of this is not generally true. First, we need the following lemma.

Lemma 3.1. *A module W verifies $(\mathcal{S}_s^*)$ if and only if it verifies $(\mathcal{S}^*)$ and* $\mathrm{Rad}(W)$ *is s-cosingular.*

Proof. By Corollary 3.1. □

Example 3.1. Consider the local $\mathbb{Z}$-module $W = \mathbb{Z}_8$. Then the only submodules of W are $\{\overline{0}\}$, $\{\overline{0}, \overline{2}\}$, $\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$ and $W = \mathbb{Z}_8$, and so $\mathrm{Soc}(W) = \{\overline{0}, \overline{2}\} \subseteq \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} = \mathrm{Rad}(W)$. Since local modules are lifting, W exhibits the property $(\mathcal{S}^*)$. Moreover, W does not verify $(\mathcal{S}_s^*)$ by Lemma 3.1.

Proposition 3.3. *Suppose that a ring S exhibits the property $(\mathcal{S}_s^*)$ and W is an S-module. Then $W/\mathcal{Z}_s^*(W)$ is semisimple.*

Proof. By Proposition 3.2 and Theorem 3.1, we have $S/\mathcal{Z}_s^*(S)$ is a semisimple ring. Since $\mathcal{Z}_s^*(S)W \leq \mathcal{Z}_s^*(W)$ for any S-module W, by Proposition 2.2, then $W/\mathcal{Z}_s^*(W)$ is an $S/\mathcal{Z}_s^*(S)$-module. Consequently, $W/\mathcal{Z}_s^*(W)$ is a semisimple module. □

Proposition 3.4. *Suppose that W is a module such that $\mathcal{Z}_s^*(W) \leq \mathrm{Rad}(W)$. If W exhibits the property $(\mathcal{S}_s^*)$, then W is an ss-lifting module.*

Proof. Let $Y \leq W$. Then there are $W_1, W_2 \leq W$ such that $W = W_1 \oplus W_2$, $W_1 \leq Y$ and $Y \cap W_2$ is s-cosingular. Since $\mathcal{Z}_s^*(Y \cap W_2) = Y \cap W_2$, then by assumption, $Y \cap W_2 \leq \mathrm{Rad}(W)$. Because of the semisimplicity of $Y \cap W_2$, W is an ss-lifting module. □

Corollary 3.2. *For an injective module W, W is ss-lifting if and only if W exhibits the property* $(\mathcal{S}_s^*)$.

Proof. The proof stems from Propositions 2.5 and 3.4. □

Theorem 3.2. *An injective module W verifies $(\mathcal{S}_s^*)$ if and only if each submodule of W can be represented as a direct sum of an s-cosingular module and an injective injective module.*

Proof. ($\Longrightarrow$) Let $Y \leq W$. Then by the hypothesis, there exist submodules H and H' of W such that $W = H \oplus H'$, $H \leq Y$ and Y/H is s-cosingular. Then $Y = H \oplus (Y \cap H')$ where H is injective and $Y \cap H'$ is s-cosingular as $Y/H \cong Y \cap H'$.

($\Longleftarrow$) Suppose that Y is a submodule of the injective module W. Then by the hypothesis, there exists an injective module Y_1 and an s-cosingular module Y_2 such that $Y = Y_1 \oplus Y_2$. Since Y_1 is injective, then Y_1 is a direct summand of W by [20, Theorem 2.15]. Moreover, since $Y_2 \cong Y/Y_1$, then $Y/Y_1 = \mathcal{Z}_s^*(Y/Y_1)$. Hence, W exhibits the property $(\mathcal{S}_s^*)$, by Theorem 3.1. □

In general, over a ring S, when an S-module W verifies $(\mathcal{S}_s^*)$, then each injective S-module does not need to verify $(\mathcal{S}_s^*)$.

Recall from [5, 28.1] that a ring S is termed as *left Harada ring*, in short *left H-ring*, if each injective left S-module is a lifting module. A ring S is termed as *quasi-Frobenius ring*, abbreviated *QF-ring*, if S is noetherian and injective as a left (or right) S-module. QF-rings are left and right artinian (see [5, 28.11]).

Example 3.2. Consider the ring $S = F[x, y]/(x^2, y^2)$ where F is a field. Let $J = \text{Rad}(S)$, $A = \text{Soc}(_S S)$ and $\overline{S} = S/A$. Note that S is a local QF-ring according to [17, p. 336]. Set

$$T = \begin{bmatrix} S & \overline{S} \\ J & \overline{S} \end{bmatrix} = \left\{ \begin{bmatrix} s & \overline{b} \\ k & \overline{z} \end{bmatrix} \middle| \, s, b, z \in S, k \in J \right\}.$$

T forms a ring through the standard matrix addition and multiplication operations. $\begin{bmatrix} 1 & \overline{0} \\ 0 & \overline{1} \end{bmatrix}$ is the identity element of T, and also $\begin{bmatrix} 1 & \overline{0} \\ 0 & \overline{0} \end{bmatrix}$, $\begin{bmatrix} 0 & \overline{0} \\ 0 & \overline{1} \end{bmatrix}$ are orthogonal primitive idempotents whose sums is equal to the identity element of T. In [17, pp. 336–337], it is showed that T is a left artinian ring although it is not left H-ring. Thus there is at least one injective left T-module W that is not lifting. It is evident that $\text{Soc}(_T T)$ verifies $(\mathcal{S}_s^*)$. However, if W exhibits the property $(\mathcal{S}_s^*)$, then W must be lifting module by the virtue of Corollary 3.2. This is a contradiction. Hence, W does not exhibit the property $(\mathcal{S}_s^*)$.

Theorem 3.3. *Suppose that S is a ring. Then the statements below are equivalent*:

(1) *Each S-module exhibits the property* $(\mathcal{S}_s^*)$.
(2) *Each injective S-module is ss-lifting.*

(3) *Each S-module can be represented as a direct sum of an s-cosingular module and an injective module.*

Proof. (1) $\Longrightarrow$ (2) By the hypothesis, each injective left S-module exhibits the property $(\mathcal{S}_s^*)$. Then by Corollary 3.2 that each injective left S-module is ss-lifting.

(2) $\Longrightarrow$ (1) Let W be an S-module. By the hypothesis, $E(W)$ is an ss-lifting module.
Thus $E(W)$ exhibits the property $(\mathcal{S}_s^*)$. Then by Proposition 3.1, W exhibits the property $(\mathcal{S}_s^*)$.

(2) $\Longrightarrow$ (3) Let W be an S-module. Then by the hypothesis, $E(W)$ exhibits the property $(\mathcal{S}_s^*)$. By Theorem 3.2, W can be represented as a direct sum of an s-cosingular module and an injective module.

(3) $\Longrightarrow$ (2) Suppose that W is an injective S-module. Then by the hypothesis, each submodule of W can be represented as a direct sum of an s-cosingular module and an injective module. Thus W exhibits the property $(\mathcal{S}_s^*)$ according to Theorem 3.2. Hence, W is ss-lifting by Corollary 3.2. □

A ring S is labeled as *left ss-Harada* in the realm of left Harada rings when every injective left S-module is an ss-lifting module (see [16]).

Corollary 3.3. *The statements below are equivalent for a ring S:*

(1) *S is a left H-ring with semisimple radical.*
(2) *S is a left ss-Harada ring.*
(3) *Each left S-module exhibits the property $(\mathcal{S}_s^*)$.*

Proof. (1) $\Longrightarrow$ (2) Assume that W is an injective S-module. Since S is left H-ring, W is lifting. The fact that W is ss-lifting can be deduced from [7, Lemma 4], as $\mathrm{Rad}(W) = \mathrm{Rad}(S)W \leq \mathrm{Soc}({}_SS)W \leq \mathrm{Soc}(W)$. Hence, S is a left ss-Harada ring.

(2) $\Longrightarrow$ (1) By (2), S is a left H-ring. On the other hand, S has semisimple radical by [16, Corollary 2.6].

(2) $\Longrightarrow$ (3) Since S is left ss-Harada ring, each injective S-module is ss-lifting, and hence each left S-module exhibits the property $(\mathcal{S}_s^*)$ by Theorem 3.3.

(3) $\Longrightarrow$ (2) Let W be an injective S-module. Then by (3), W exhibits the property $(\mathcal{S}_s^*)$. It follows from Corollary 3.2 that W is an ss-lifting module. Hence, S is a left ss-Harada ring. □

It is important to note that each supplement of a semisimple submodule in the module W also serves as an ss-supplement for the submodule within W.

Proposition 3.5. *Suppose that a module W exhibits the property $(\mathcal{S}_s^*)$. Assume that $\mathcal{Z}_s^*(W)$ possesses a supplement in W. Then $W = D \oplus H$ where D is an ss-lifting module and H is an s-cosingular module.*

Proof. By the hypothesis, W possesses a submodule Y such that $W = Y + \mathcal{Z}_s^*(W)$ and $Y \cap \mathcal{Z}_s^*(W) \leq \mathrm{Soc}_s(Y)$. Then $\mathcal{Z}_s^*(Y) = \mathrm{Soc}_s(Y)$. Since W verifies $(\mathcal{S}_s^*)$, then

$W = W_1 \oplus W_2$ where $W_1 \leq Y$ and $Y \cap W_2 = \mathcal{Z}_s^*(Y \cap W_2)$. Then we deduce that $Y = W_1 \oplus (Y \cap W_2)$. Since $\mathcal{Z}_s^*(Y \cap W_2) \leq \mathcal{Z}_s^*(Y) = \mathrm{Soc}_s(Y)$ and $Y \cap W_2$ is a direct summand of Y, then $Y \cap W_2$ has to be zero. Thus $W = Y \oplus W_2$. According to Propositions 3.1 and 3.4, we conclude that Y is an ss-lifting module. Moreover, we have $W = Y + \mathcal{Z}_s^*(W) = Y + \mathcal{Z}_s^*(Y) + \mathcal{Z}_s^*(W_2) = Y \oplus \mathcal{Z}_s^*(W_2)$, and so $\mathcal{Z}_s^*(W_2) = W_2$. □

Corollary 3.4. *Suppose that* W *exhibits the property* $(\mathcal{S}_s^*)$. *Then there exists a decomposition* $W = D \oplus H$ *such that* $\mathcal{Z}_s^*(D) = 0$ *and* $\mathrm{Soc}(H) \trianglelefteq H$.

Proof. Let K be submodule of W that is maximal with respect to the property $K \cap \mathcal{Z}_s^*(W) = 0$. Since W exhibits the property $(\mathcal{S}_s^*)$, then $W = W_1 \oplus W_2$ where $W_1 \leq K$ and $K \cap W_2$ is s-cosingular. Thus $K = W_1 \oplus (K \cap W_2)$. Since $K \cap \mathcal{Z}_s^*(W) = 0$, then $\mathcal{Z}_s^*(K \cap W_2) = K \cap W_2 = 0$. So, we deduce that $W = K \oplus W_2$. Therefore, $\mathcal{Z}_s^*(W) = \mathcal{Z}_s^*(W_2)$ and $K \oplus \mathcal{Z}_s^*(W) \trianglelefteq W$. Thus $\mathcal{Z}_s^*(W_2) \trianglelefteq W_2$. This leads to the conclusion that $\mathrm{Soc}(W_2) \trianglelefteq W_2$. □

As indicated in [6] that a module W is Σ-*injective* in case, for any index set Λ, $W^{(\Lambda)}$ is injective.

Proposition 3.6. *Suppose that* S *is a ring with* $\mathcal{Z}_s^*({}_SS) = \mathrm{Rad}(S)$ *and that the left* S*-module* $E(S^{(\mathbb{N})})$ *exhibits the property* $(\mathcal{S}_s^*)$. *Then the left* S*-module* S *is* Σ-*injective, and so* S *is a* QF*-ring.*

Proof. Suppose that $W = S \oplus S \oplus \cdots$ is a free left S-module represented as the direct sum of a countably infinite number of copies of S, that is $W = S^{(\mathbb{N})}$. By the hypothesis, $E(W)$ exhibits the property $(\mathcal{S}_s^*)$. By Theorem 3.2, $W = K \oplus H$ for an s-cosingular submodule H and some injective submodule K. According to Lemma 2.2-(3) we can denote that

$$H = \mathcal{Z}_s^*(H) \leq \mathcal{Z}_s^*(W) = \mathrm{Rad}(S) \oplus \mathrm{Rad}(S) \cdots = \mathrm{Rad}(S)W.$$

Note that $W/K = \mathrm{Rad}(S)(W/K)$. Since $H \cong W/K$ and H is projective, then $W/K = 0$ by [23, 22.3(1)]. Thus W is injective. By [8, 20.3A] ${}_SS$ is Σ-injective. Hence, we conclude that S is a QF-ring by [6, 18.1]. □

As stated in [23, 42.6] that a ring S is labeled as *semiperfect* if each finitely generated S-module has a projective cover, that is, for any S-module W, there exist a projective module T and an epimorphism $f : T \to W$ such that $\mathrm{Ker}(f) \ll T$.

Proposition 3.7. *Suppose that* S *is a semiperfect ring with* $\mathrm{Rad}(S) \leq \mathrm{Soc}({}_SS)$. *If* $\mathcal{Z}_s^*({}_SS) = Z({}_SS)$, *then* $\mathcal{Z}_s^*({}_SS) = \mathrm{Soc}_s({}_SS)$. *The converse holds in case* S *is a right or left perfect left quasi-continuous ring with* $\mathrm{Rad}(S) \leq \mathrm{Soc}({}_SS)$.

Proof. Let S be a semiperfect ring with $\mathrm{Rad}(S) \leq \mathrm{Soc}({}_SS)$ and $\mathcal{Z}_s^*({}_SS) = Z({}_SS)$. There is an idempotent element e such that $eS \leq Z({}_SS)$ and $(1-e)S \cap Z({}_SS)$

$\leq \text{Soc}_s({}_SS)$ by [7, Lemma 3]. Since $Z({}_SS)$ includes no nonzero idempotent, it leads to the conclusion that $Z({}_SS) \leq \text{Soc}_s({}_SS)$. Hence, $\mathcal{Z}_s^*({}_SS) = \text{Soc}_s({}_SS)$.

Conversely, let S be a right or left perfect left quasi-continuous ring with $\text{Rad}(S) \leq \text{Soc}({}_SS)$ and $\mathcal{Z}_s^*({}_SS) = \text{Soc}_s({}_SS)$. By [4, Lemma 6] , $Z({}_SS) = \text{Rad}(S)$. Thus we conclude that $\mathcal{Z}_s^*({}_SS) = \text{Soc}_s({}_SS) = \text{Rad}(S) = Z({}_SS)$, by assumption. □

A ring S is termed as *co-H-ring* if for each projective left S-module W, each submodule of W is essential in a direct summand of W (see [5, 28.11]). Also note that S is a QF-ring if and only if S is left H-ring with $\text{Rad}(S) = Z(S)$ (see [5, 28.16]).

Theorem 3.4. *The statements below are equivalent for a ring S with* $\text{Rad}(S) \leq \text{Soc}({}_SS)$:

(1) *S is a QF-ring.*
(2) *Each left S-module exhibits the property $(\mathcal{S}_s^*)$ and S is a left self injective ring.*
(3) *$E(S^{(\mathbb{N})})$ exhibits the property $(\mathcal{S}_s^*)$ and S is a left self injective ring.*
(4) *$\mathcal{Z}_s^*({}_SS) = \text{Rad}(S)$ and either of the conditions below hold*:
   (a) *Each left S-module exhibits the property $(\mathcal{S}_s^*)$ or,*
   (b) *$E(S^{(\mathbb{N})})$ verifies $(\mathcal{S}_s^*)$ or,*
   (c) *S is a left co-H-ring or,*
   (d) *S is a left H-ring.*

Proof. (1) $\Longrightarrow$ (2) Since S is QF-ring, then S is self injective by [5, 28.11]. Note that S is left H-ring by [5, 28.16]. Thus each left S-module verifies $(\mathcal{S}_s^*)$ by Corollary 3.3 as S is left H-ring with $\text{Rad}(S) \leq \text{Soc}({}_SS)$.

The implications (2) $\Longrightarrow$ (3) and ($4a$) $\Longrightarrow$ ($4b$) are straightforward.

(1) $\Longrightarrow$ ($4a$) By (1) and [19, Theorem 5.5], we obtain that $\mathcal{Z}_s^*({}_SS) = \text{Soc}({}_SS) \cap \mathcal{Z}^*({}_SS) = \text{Rad}(S)$. Since S is left H-ring with semisimple radical, then by Corollary 3.3 each left S-module verifies $(\mathcal{S}_s^*)$.

(1) $\Longrightarrow$ ($4c$) By [5, 28.16], S is left co-H-ring. On the other hand, similar to the proof of (1) $\Rightarrow$ ($4a$) we get that $\mathcal{Z}_s^*({}_SS) = \text{Rad}(S)$.

The proofs of (3) $\Longrightarrow$ (1) and ($4b$) $\Longrightarrow$ (1) follow from Proposition 3.6.

($4c$) $\Longrightarrow$ ($4b$) Since S is left perfect ring with $\text{Rad}(S) \leq \text{Soc}({}_SS)$, then each projective S-module is ss-supplemented by [13, Theorem 41]. Thus each projective S-module is ss-lifting by [7 Theorem 2], and so each projective S-module verifies $(\mathcal{S}_s^*)$. By [17 Theorem 3.18], since the class of projective S-modules is closed under essential extensions, then $E(S^{(\mathbb{N})})$ verifies $(\mathcal{S}_s^*)$.

(1) $\Longleftrightarrow$ ($4d$) The proof follows from [10, Corollary p. 673] and [19, Theorem 5.5].

□

Corollary 3.5. *Let S be a commutative ring. If each S-module exhibits the property $(\mathcal{S}_s^*)$, then $S = S_1 \oplus S_2$ where S_1 is a QF-ring and S_2 is a cosingular ring.*

Proof. Since modules with $(\mathcal{S}_s^*)$ exhibit the property $(\mathcal{S}^*)$, then the result can be deduced from [19, Corollary 5.6]. □

Theorem 3.5. *Suppose that $W = W_1 \oplus W_2$ where W_1 is semisimple and W_2 exhibits the property $(\mathcal{S}_s^*)$. Then W exhibits the property $(\mathcal{S}_s^*)$.*

Proof. Let $Y \leq W$. Then $W_1 = (Y \cap W_1) \oplus W'$ for some submodule W' of W_1 as W_1 is semisimple. Thus $W = (Y \cap W_1) \oplus W' \oplus W_2$ and $Y = (Y \cap W_1) \oplus X$ where $X = Y \cap (W' \oplus W_2)$. Since $(W_2 \oplus W')/W'$ verifies $(\mathcal{S}_s^*)$, then $(X + W')/W' = (K/W') \oplus (H/W')$ for some submodules H and K including W' such that H/W' is s-cosingular and K/W' is a direct summand of $(W_2 \oplus W')/W'$. Therefore, K is a direct summand of W. Since $K = W' \oplus (K \cap X)$, then $K \cap X$ is a direct summand of W. Hence, $(Y \cap W_1) \oplus (K \cap X)$ is a direct summand of W. Moreover,

$$Y/[(Y \cap W_1) \oplus (K \cap X)] \cong X/(K \cap X) \cong (X + K)/K = (X + W')/K \cong H/W'$$

is s-cosingular. Consequently, W exhibits the property $(\mathcal{S}_s^*)$. □

Corollary 3.6. *Suppose that $W = W_1 \oplus W_2$ where W_1 is semisimple and W_2 is s-cosingular. Then W exhibits the property $(\mathcal{S}_s^*)$.*

Consider the $\mathbb{Z}$-module $W = \mathbb{Z} \oplus \mathbb{Z}_{t^\infty}$ for any prime integer t. Then it is well known that $\mathbb{Z}$ and $\mathbb{Z}_{t^\infty}$ are relatively projective, $\mathbb{Z}$ is not semisimple, W and $\mathbb{Z}_{t^\infty}$ are not ss-lifting module.

Corollary 3.7. *Suppose that $W = W_1 \oplus W_2$ where W_1 is semisimple and W_2 is ss-lifting. Then W exhibits the property $(\mathcal{S}_s^*)$.*

Let W and T be modules. T is termed as *W-projective* provided for a module L with an epimorphism $\pi : W \to L$ and a homomorphism $f : T \to L$, there is a homomorphism $g : T \to W$ such that $\pi g = f$. If T is T-projective, T is termed as *quasi-(or self-)projective.* A class of modules $\{W_\lambda\}_{\lambda \in \Lambda}$ is termed as *relatively projective* if W_λ is W_γ-projective for whole distinct $\lambda, \gamma \in \Lambda$.

Theorem 3.6. *Suppose that $W = W_1 \oplus W_2$ where W_1 and W_2 are quasi-projective, relatively projective modules such that W_1 and W_2 exhibit the property $(\mathcal{S}_s^*)$. Then W exhibits the property $(\mathcal{S}_s^*)$.*

Proof. Let $Y \leq W$.

Case 1. If $W_1 \cap (Y + W_2) = 0$, then Y is a submodule of W_2. Since W_2 verifies $(\mathcal{S}_s^*)$, there exists $K_1 \leq Y$ such that $W_2 = K_1 \oplus K_2$ and $Y \cap K_2$ is s-cosingular for some submodule K_2 of W_2. Hence, $W = W_1 \oplus K_1 \oplus K_2$ and $Y \cap (W_1 \oplus K_2) = Y \cap K_2$ is s-cosingular. Thus W verifies $(\mathcal{S}_s^*)$.

Case 2. If $W_1 \cap (Y + W_2) \neq 0$, then there exists $K_1 \leq W_1 \cap (Y + W_2)$ such that $W_1 = K_1 \oplus K_2$ and $W_1 \cap (Y + W_2) \cap K_2 = K_2 \cap (Y + W_2)$ is s-cosingular as W_1

verifies $(\mathcal{S}_s^*)$. Then $W = K_1 \oplus K_2 \oplus W_2 = Y + (W_2 \oplus K_2)$. Now if $W_2 \cap (Y + K_2) = 0$, then $Y \cap K_2 \leq K_2$ and since K_2 verifies $(\mathcal{S}_s^*)$, there exists $X_1 \leq Y \cap K_2$ such that $K_2 = X_1 \oplus X_2$, $Y \cap K_2 \cap X_2 = Y \cap X_2$ is s-cosingular. Then $W = (K_1 \oplus X_1) \oplus (X_2 \oplus W_2) = Y + (X_2 \oplus W_2)$. Since W_1 is $(W_1 \oplus W_2)$-projective, K_1 is $(X_2 \oplus W_2)$-projective and X_1 is $(X_2 \oplus W_2)$-projective by [15]. Then $(K_1 \oplus X_1)$ is $(X_2 \oplus W_2)$-projective. Thus by [23, 41.14] there exists $Y' \leq Y$ such that $W = Y' \oplus X_2 \oplus W_2$, $Y \cap (X_2 \oplus W_2) \leq X_2 \cap (Y + W_2) = Y \cap X_2$ is s-cosingular. Hence, W exhibits the property $(\mathcal{S}_s^*)$.

If $W_2 \cap (Y + K_2) \neq 0$, then there exists $X_1 \leq W_2 \cap (Y + K_2)$ such that $W_2 = X_1 \oplus X_2$ and $X_2 \cap (Y + K_2)$ is s-cosingular. Then $W = Y + (K_2 \oplus W_2) = (K_1 \oplus X_1) \oplus (K_2 \oplus X_2)$ and $Y \cap (K_2 \oplus X_2)$ is s-cosingular as $K_2 \cap (Y + W_2)$ and $X_2 \cap (Y + K_2)$ are s-cosingular. Since $(K_1 \oplus X_1)$ is $(K_2 \oplus X_2)$-projective, then by [23, 41.14] there exists $Y' \leq Y$ such that $W = Y' \oplus K_2 \oplus X_2$. Hence, W exhibits the property $(\mathcal{S}_s^*)$. □

Corollary 3.8. *Suppose that $W = W_1 \oplus W_2$ is a projective module such that W_1 and W_2 exhibit the property $(\mathcal{S}_s^*)$. Then W exhibits the property $(\mathcal{S}_s^*)$.*

A ring S is semiperfect with semisimple radical if and only if the left S-module ${}_SS$ is ss-lifting by [7, Lemma 3]. It is widely known that when S is semiperfect ring, then each projective finitely generated S-module is lifting. Thus over a semiperfect ring S with semisimple radical, each finitely generated projective S-module is ss-lifting by the virtue of [7, Lemma 4]. Hence, we give the next result of Theorem 3.6 for a ring S verifying $(\mathcal{S}_s^*)$.

Corollary 3.9. *Suppose that S is a ring verifying $(\mathcal{S}_s^*)$. Then each finitely generated projective S-module verifies $(\mathcal{S}_s^*)$.*

Proof. Suppose that W is a finitely generated projective S-module. Thus W is isomorphic to a direct summand of an S-module which is free. Hence, W verifies $(\mathcal{S}_s^*)$ by Theorem 3.6. □

Last, we present a decomposition for a module W verifying the property $(\mathcal{S}_s^*)$ under which condition its submodule $\mathcal{Z}_s^*(W)$ exhibits ascending chain condition (or descending chain condition) on its direct summands.

Lemma 3.2. *Let W be a module such that $\mathcal{Z}_s^*(W) \trianglelefteq W$. Let W_1 and W_2 be direct summands of W with $W_1 \leq W_2$. Then $\mathcal{Z}_s^*(W_1) = \mathcal{Z}_s^*(W_2)$ if and only if $W_1 = W_2$.*

Proof. Let $W = W_1 \oplus K$ for some submodule K of W. Then $W_2 = W_1 \oplus (W_2 \cap K)$ and $\mathcal{Z}_s^*(W_2) = \mathcal{Z}_s^*(W_1) \oplus \mathcal{Z}_s^*(W_2 \cap K)$. If $\mathcal{Z}_s^*(W_1) = \mathcal{Z}_s^*(W_2)$, then $\mathcal{Z}_s^*(W_2 \cap K) = (W_2 \cap K) \cap \mathcal{Z}_s^*(W) = 0$. It implies that $W_2 \cap K = 0$, by the hypothesis. Hence, $W_1 = W_2$. □

Proposition 3.8. *Let W be a module such that $\mathcal{Z}_s^*(W) \trianglelefteq W$. If $\mathcal{Z}_s^*(W)$ exhibits ascending chain condition (descending chain condition) on direct summands, then W exhibits ascending chain condition (descending chain condition) on direct summands.*

Proof. It is evident by Lemma 3.2. □

Theorem 3.7. *Suppose that W is a module that exhibits the property $(\mathcal{S}_s^*)$. Assume that $\mathcal{Z}_s^*(W)$ possesses ascending chain condition (descending chain condition) on direct summands. Then $W = W_1 \oplus W_2$ such that $\mathcal{Z}_s^*(W_1) = 0$ and W_2 is a finite direct sum of indecomposable modules K_λ where $\lambda \in \Lambda$, Λ is finite index set such that each proper submodule of K_λ is s-cosingular.*

Proof. Let W be a module verifying $(\mathcal{S}_s^*)$. Then W has a decomposition $W = W_1 \oplus W_2$ where $\mathcal{Z}_s^*(W_1) = 0$ and $\mathrm{Soc}(W_2) = \mathcal{Z}_s^*(W_2) \trianglelefteq W_2$ by Corollary 3.4. Then by Proposition 3.8, W_2 exhibits ascending chain condition on direct summands. Thus by [1, 10.14], W_2 is a finite direct sum of indecomposable modules. So each indecomposable direct summand verifies $(\mathcal{S}_s^*)$. Hence, each proper submodule of each summand is s-cosingular as each one is indecomposable.

The proof for the descending chain condition can be made similar method to the above. □

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References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules* (Springer, 1992).
2. E. Büyükaşık, E. Mermut and S. Özdemir, Rad-supplemented modules, *Rend. Sem. Mat. 'Univ.' Padova* **124** (2010) 157–177.
3. K. A. Byrd, Rings whose quasi-injective modules are injective, *Proc. Am. Math. Soc.* **33**(2) (1972) 235–240.
4. J. Clark and D. Huynh, When is a self injective semiperfect ring Quasi-Frobenius, *J. Algebra* **165** (1994) 531–542.
5. J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules* (Birkhäuser, Verlag, Basel, Boston, MA, 2006).
6. N. V. Dung, D. Huynh, P. F. Smith and R. Wisbauer, *Extending Modules*, Putman Research Notes in Mathematics Series (Longman, Harlow, 1994).

7. F. Eryilmaz, *ss*-Lifting modules and rings, *Miskolc Math. Notes* **22**(2) (2021) 655–662.
8. C. Faith, *Algebra II, Ring Theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1976).
9. K. R. Goodearl, *Ring Theory: Nonsingular Rings and Modules* (Marcel Dekker, New York, 1976).
10. M. Harada, Non-small modules and non-cosmall modules, in *Ring Theory*, ed. F. Van Oystaeyen (Marcel Dekker, New York, 1979), pp. 669–690.
11. C. J. Holston, S. K. Jain and A. Leroy, Rings over which cyclics are direct sums of projective and CS or noetherian, *Glasg. Math. J.* **52**(A) (2010) 103–110.
12. F. Kasch, *Modules and Rings* (Academic Press Inc. London, 1982).
13. E. Kaynar, H. Çalışıcı and E. Türkmen, *ss*-Supplemented modules, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **69**(1) (2020) 473–485.
14. W. W. Leonard, Small modules, *Proc. Amer. Math. Soc.* **17** (1966) 527–531.
15. S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules* (Cambridge, New York, Melbourne, Sydney, 1990).
16. B. Nişancı Türkmen and Y. M. Demirci, On a class of Harada rings, *Open Math.* **20** (2022) 1833–1837.
17. K. Oshiro, Lifting modules, extending modules and their applications to QF-rings, *Hokkaido Math. J.* **13** (1984) 310–338.
18. A. Ç. Özcan, Some characterizations of V-modules and rings, *Vietnam J. Math.* **26**(3) (1998) 253–258.
19. A. Ç. Özcan, Modules with small cyclic submodules in their injective hulls, *Commun. Algebra* **30**(4) (2002) 1575–1589.
20. D. W. Sharpe and P. Vamos, *Injective Modules* (Cambridge University Press, Cambridge, 1972).
21. İ. Soydan and E. Türkmen, Generalizations of *ss*-supplemented modules, *Carpathian Math. Publ.* **13**(1) (2021) 119–126.
22. R. Ware, Endomorphism rings of projective modules, *Trans. Amer. Math. Soc.* **155** (1971) 233–256.
23. R. Wisbauer, *Foundations of Module and Ring Theory* (Gordon and Breach, 1991).
24. D. X. Zhou and X. R. Zhang, Small-essential submodules and morita duality, *Southeast Asian Bull. Math.* **35** (2011) 1051–1062.