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Maximal operator and Calderon–Zygmund operators in local Morrey–Lorentz spaces

V. S. Guliyev[a,b], C. Aykol[c], A. Kucukaslan[d] and A. Serbetci[c]

[a]Institute of Mathematics and Mechanics, Baku, Azerbaijan; [b]Department of Mathematics, Ahi Evran University, Kirsehir, Turkey; [c]Department of Mathematics, Ankara University, Ankara, Turkey; [d]Ministry of Health, Central Organization, Ankara, Turkey

ABSTRACT
In this paper we proved the boundedness of the Hardy–Littlewood maximal operator M, the Calderon–Zygmund operators T and the maximal Calderon–Zygmund operators $\mathcal{T}$ on the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$. Finally, we give some applications of these results.

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1. Introduction and main results

Recently, in [1], the boundedness of the Hilbert transform H on the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}$ was extensively studied. In the present paper we study the boundedness of the Hardy–Littlewood maximal operator M, the Calderon–Zygmund operators T and the maximal Calderon–Zygmund operators $\mathcal{T}$ on the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ by using related rearrangement inequalities. As applications, we obtain the boundedness of the Bochner–Riesz operator B^{δ}_{r} on $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$. Further, we get the boundedness of the operators B^{δ}_{r}, M, T and $\mathcal{T}$ on the Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ including weak versions and on the weak L_p spaces $WL_p(\mathbb{R}^n)$. The local Morrey–Lorentz spaces denoted by $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ are a very natural generalization of the Lorentz spaces such that $M^{\mathrm{loc}}_{p,q;0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$ (see [2]).

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centred at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Therefore $|B(x,r)| = \omega_n r^n$, ω_n denotes the volume of unit sphere S^{n-1} in $\mathbb{R}^n$. For $f \in L^{\mathrm{loc}}_{1}(\mathbb{R}^n)$, the Hardy–Littlewood maximal function Mf of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

CONTACT A. Serbetci serbetci@ankara.edu.tr

In the following theorem we prove the boundedness of maximal operator M on the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda} \equiv M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$.

Theorem 1.1: *Let* $1 \le q \le \infty$, $0 \le \lambda < 1$ *and* $q/(q+\lambda) \le p \le \infty$.

(i) *If* $q/(q+\lambda) < p < \infty$, *then the maximal operator* M *is bounded on the local Morrey–Lorentz space* $M^{\mathrm{loc}}_{p,q;\lambda}$.
(ii) *If* $p = q/(q+\lambda)$, *then the operator* M *is bounded from* $M^{\mathrm{loc}}_{p,q;\lambda}$ *to the weak local Morrey–Lorentz space* $WM^{\mathrm{loc}}_{p,q;\lambda}$.
(iii) *If* $p = q = \infty$, *then the operator* M *is bounded on* $L_\infty(\mathbb{R}^n)$.

Suppose that $K \in L^{\mathrm{loc}}_1(\mathbb{R}^n \setminus \{0\})$ and satisfies the following conditions:

$$\text{(i) } |K(x)| \le \frac{C}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

$$\text{(ii) } \int_{r_1<|x|<r_2} K(x)\,\mathrm{d}x = 0, \quad 0 < r_1 < r_2,$$

$$\text{(iii) } |K(x-y) - K(x)| \le C|y|/|x|^{n+1} \quad \text{for } 2|y| \le |x|.$$

Then K is called the Calderon–Zygmund kernel, where C is a constant independent of x and y. Set

$$T_\varepsilon f(x) = \int_{\complement B(x,\varepsilon)} K(x-y) f(y)\,\mathrm{d}y.$$

We define the Calderon–Zygmund singular integral associated to K as

$$Tf(x) = (K * f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x)$$

and the maximal singular integral by

$$\mathcal{T} f(x) = \sup_{\epsilon > 0} |T_\varepsilon f(x)|.$$

It is well known that Tf exists almost everywhere whenever f is a step function. The almost everywhere existence of the limit (of certain integral averages) was known for dense subset of L_1 and the result was extended to all of L_1 by establishing control over the corresponding maximal operator. For the Calderon–Zygmund operator T, the dense subset of L_1 consists of the step functions, and in order to extend to all of L_1 the almost everywhere existence of the limit of $T_\varepsilon f(x)$, $x \in \mathbb{R}^n$ as $\varepsilon \to 0$, we need to consider the maximal Calderon–Zygmund operator $\mathcal{T} f$ of f.

For each measurable function φ on $(0,\infty)$ and each $t > 0$, let

$$(S\varphi)(t) = \int_0^\infty \min\left(1, \frac{s}{t}\right) \varphi(s) \frac{\mathrm{d}s}{s} = \frac{1}{t}\int_0^t \varphi(s)\,\mathrm{d}s + \int_t^\infty \varphi(s)\frac{\mathrm{d}s}{s}.$$

It is clear that S is linear. For the aim, its importance based on the fact that it dominates the maximal Calderon–Zygmund operator.

Theorem A ([3,4]): *Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ and suppose*

$$(Sf^*)(1) = \int_0^1 f^*(s)\,\mathrm{d}s + \int_1^\infty f^*(s)\frac{\mathrm{d}s}{s} < \infty. \tag{1.1}$$

Then

$$(\mathcal{T}f)^*(t) \le CS(f^*)(t), \quad 0 < t < \infty, \tag{1.2}$$

where C is a constant independent of f and t.

Theorem B: *Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ and f satisfies (1.1). Then the Calderon–Zygmund operator T exists almost everywhere $x \in \mathbb{R}^n$. Furthermore,*

$$(Tf)^*(t) \le CS(f^*)(t), \quad 0 < t < \infty, \tag{1.3}$$

where C is a constant independent of f and t.

Remark 1.1: Note that, the inequality (1.2) is due to Bennett and Rudnick [5], the integrated form (1.3) was known previously to O'Neil and Weiss [6] and Calderon [7].

The Calderon–Zygmund operator T extends to the whole space L_p, $1 \le p < \infty$, by continuity. In the case $p = \infty$ we need a renormalization of T (see [8]). For this reason let us choose a point $x_0 \in \mathbb{R}^n$ and let $f \in L_\infty$. Set

$$T^0 f(x) = T(f\chi_{2B})(x) - T(f\chi_{2B})(x_0) + \int_{\complement_{B(x,r)}} [K(x-y) - K(x_0-y)]f(y)\,\mathrm{d}y,$$

where $x_0 \in B(x,r)$. If $f \in L_p(\mathbb{R}^n)$, $p < \infty$, then obviously

$$T^0 f(x) = T(f)(x) - T(f)(x_0).$$

In the following theorem we give the boundedness of the Calderon–Zygmund operator T on the spaces $M_{p,q;\lambda}^{\text{loc}}$.

Theorem 1.2: *Suppose that $f \in M_{p,q;\lambda}^{\text{loc}}$, $1 \le q \le \infty$, $0 \le \lambda < 1$, $q/(q+\lambda) \le p \le q/\lambda$ and the inequality (1.1) holds, then the Calderon–Zygmund integral $Tf(x)$ exists almost every $x \in \mathbb{R}^n$. Furthermore,*

(i) *If $1 \le q < \infty$, $q/(q+\lambda) < p < q/\lambda$, then the Calderon–Zygmund operator T is bounded on the local Morrey–Lorentz space $M_{p,q;\lambda}^{\text{loc}}$.*
(ii) *If $1 < q < \infty$, $p = q/(q+\lambda)$, then the operator T is bounded from $M_{p,q;\lambda}^{\text{loc}}$ to the weak local Morrey–Lorentz space $WM_{p,q;\lambda}^{\text{loc}}$.*
(iii) *If $1 \le q \le \infty$, $p = q/\lambda$, then the operator T^0 is bounded from $M_{p,q;\lambda}^{\text{loc}}$ to BMO.*

In the following theorem we give the boundedness of the maximal Calderon–Zygmund operator $\mathcal{T}$ on the spaces $M_{p,q;\lambda}^{\text{loc}}$.

Theorem 1.3: *Suppose that $f \in M^{\text{loc}}_{p,q;\lambda}$, $1 \le q \le \infty$, $0 \le \lambda < 1$, $q/(q+\lambda) \le p \le q/\lambda$ and the inequality (1.1) holds, then the maximal Calderon–Zygmund integral $\mathcal{T}f(x)$ is finite almost every $x \in \mathbb{R}^n$. Furthermore,*

(i) *If $1 \le q < \infty$, $q/(q+\lambda) < p < q/\lambda$, then the operator $\mathcal{T}$ is bounded in the local Morrey–Lorentz space $M^{\text{loc}}_{p,q;\lambda}$.*
(ii) *If $1 < q < \infty$, $p = q/(q+\lambda)$, then the operator $\mathcal{T}$ is bounded from $M^{\text{loc}}_{p,q;\lambda}$ to the weak local Morrey–Lorentz space $WM^{\text{loc}}_{p,q;\lambda}$.*
(iii) *If $1 \le q \le \infty$, $p = q/\lambda$, then the operator $\mathcal{T}$ is bounded from $M^{\text{loc}}_{p,q;\lambda}$ to BMO.*

Remark 1.2: For the limiting case $\lambda = 1$, in the classical Lorentz space $M^{\text{loc}}_{p,q;1} = \Lambda_{\infty,t^{1/p-1/q}}$ the boundedness of Calderon–Zygmund operator T is given in [9].

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. If $p \in [1,\infty]$, the conjugate number p' is defined by $pp' = p + p'$.

2. Preliminaries

We shall use the following notation. For a Lebesgue measurable set $E \subset \mathbb{R}^n$ and $0 < p \le \infty$, $L_p(E)$ is the standard Lebesgue space of all functions f Lebesgue measurable on E for which

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p \,\mathrm{d}y \right)^{1/p} < \infty,$$

if $0 < p < \infty$ and

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \ge \alpha\}| > 0\},$$

if $p = \infty$. Also, for an open set $E \subset \mathbb{R}^n$, $L^{\text{loc}}_p(E)$ is the set of all functions f such that $f \in L_p(K)$ for any compact $K \subset E$. If $E = \mathbb{R}^n$, then, for brevity, we write L_p for $L_p(\mathbb{R}^n)$ and L^{loc}_p for $L^{\text{loc}}_p(\mathbb{R}^n)$. The same convention refers to the case of weak Lebesgue spaces $WL_p(E)$, the space of all functions f Lebesgue measurable on E for which

$$\|f\|_{WL_p(E)} := \sup_{0<t\le|E|} t^{1/p} f^*(t), \quad 1 \le p < \infty$$

and

$$\|f\|_{WL_\infty} \equiv \|f\|_{L_\infty}, \quad p = \infty.$$

Here $|E|$ is the Lebesgue measure of E, and f^* denotes the non-increasing rearrangement of f:

$$f^*(t) := \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \quad \forall t \in (0,\infty),$$

with

$$\mu_f(\lambda) := |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}|.$$

It is well known that for the classical Hardy–Littlewood maximal operator the rearrangement inequality

$$cf^{**}(t) \le (Mf)^*(t) \le Cf^{**}(t), \quad t \in (0,\infty) \tag{2.1}$$

holds [4], where the positive constants c,C are independent of f and t, and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s.$$

Lorentz spaces are introduced by Lorentz in the 1950. Lorentz spaces, which are Banach spaces and generalizations of the more familiar L_p spaces, appear to be useful in the general interpolation theory.

Definition 2.1: The Lorentz space $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$, $0 < p,\ q \le \infty$ is the space of all measurable functions f on $\mathbb{R}^n$ such the quantity

$$\|f\|_{L_{p,q}} := \|t^{1/p-1/q}f^*(t)\|_{L_q(0,\infty)} \tag{2.2}$$

is finite. Note that $L_{p,\infty} = WL_p$.

If $p = q = \infty$, then the space $L_{\infty,\infty}$ is denoted by L_∞.

Useful references for Lorentz spaces are for instance [4,10]. In the following we give the local Morrey spaces $LM_{p,\lambda}(0,\infty)$ which we use while proving of our main results.

Definition 2.2: Let $0 \le p < \infty$ and $0 \le \lambda \le 1$. We denote by $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ the local Morrey space, the space of all functions $\varphi \in L_p^{\mathrm{loc}}(0,\infty)$ with finite quasinorm

$$\|\varphi\|_{LM_{p,\lambda}} = \sup_{r>0} r^{-\lambda/p}\|\varphi\|_{L_p(0,r)}.$$

Also by $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ we denote the weak local Morrey space of all functions $\varphi \in WL_p^{\mathrm{loc}}(0,\infty)$ for which

$$\|\varphi\|_{WLM_{p,\lambda}} = \sup_{r>0} r^{-\lambda/p}\|\varphi\|_{WL_p(0,r)} < \infty.$$

The local Morrey-type spaces $LM_{p\theta,w}$, $0 < p,\theta \le \infty$, were introduced by Guliyev in the doctoral thesis [11] (see, also [12]) defined by

$$\|\varphi\|_{LM_{p\theta,w}} = \|w(r)\|\varphi\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. If $\theta = \infty$, it denotes $LM_{p,w} \equiv LM_{p\infty,w}$. The boundedness of the classical operators in $LM_{p\theta,w}$ was intensively studied in [11–15], etc.

In [16, Section 4.1], Mingione studied the boundedness of the restricted fractional maximal operator in the restricted Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(B)$, where B is any ball. Ragusa [17] defined the Morrey–Lorentz spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ and studied some embeddings between these spaces.

The boundedness of the classical integral operators on Morrey–Lorentz spaces was studied by Mingione [16], Ragusa [17], etc.

In the following definition we give the local Morrey–Lorentz spaces denoted by $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ which are a very natural generalization of the Lorentz spaces such that $M^{\mathrm{loc}}_{p,q;0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$.

Definition 2.3 ([2]): Let $0 < p, q \le \infty$ and $0 \le \lambda \le 1$. We denote by $M^{\mathrm{loc}}_{p,q;\lambda} \equiv M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$ the local Morrey–Lorentz space, the space of all measurable functions with finite quasi-norm

$$\|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}} := \sup_{r>0} r^{-\lambda/q} \|t^{1/p-1/q} f^*(t)\|_{L_q(0,r)}.$$

In the cases $\lambda < 0$ or $\lambda > 1$, we have $M^{\mathrm{loc}}_{p,q;\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Also $M^{\mathrm{loc}}_{p,q;0} = L_{p,q}$ and $M^{\mathrm{loc}}_{p,p;\lambda} \equiv M^{\mathrm{loc}}_{p;\lambda}$. In the limiting case $\lambda = 1$ the space $M^{\mathrm{loc}}_{p,q;1}$ is the classical Lorentz space $\Lambda_{\infty,t^{1/p-1/q}}$. For $0 < q \le p < \infty$ and $0 < \lambda \le q/p$, the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}$ are equal to weak Lebesgue spaces $WL_{1/p-\lambda/q}$. Note that, in the case $q = \infty$ we have $M^{\mathrm{loc}}_{p,\infty;\lambda} = \Lambda_{\infty,t^{1/p}} = WL_p$.

We denote by $WM^{\mathrm{loc}}_{p,q;\lambda}$ the weak local Morrey–Lorentz space of all measurable functions with finite quasinorm

$$\|f\|_{WM^{\mathrm{loc}}_{p,q;\lambda}} := \sup_{r>0} r^{-\lambda/q} \|t^{1/p-1/q} f^*(t)\|_{WL_q(0,r)}.$$

Remark 2.1: We have that $M^{\mathrm{loc}}_{\infty,q;\lambda} = \Theta$ for any $0 < q < \infty$. Indeed, assume that $M^{\mathrm{loc}}_{\infty,q;\lambda} \neq \Theta$. Then there exists a non-zero function $f \in M^{\mathrm{loc}}_{\infty,q;\lambda}$ which means that there exists $c > 0$ and a positive measurable set A such that $|f(x)| \ge c$ for all $x \in A$. Then

$$\begin{aligned}
\|f\|_{M^{\mathrm{loc}}_{\infty,q;\lambda}} &= \sup_{r>0} r^{-\lambda/q} \|t^{-1/q} f^*(t)\|_{L_q(0,r)} \\
&\ge \sup_{r>0} r^{-\lambda/q} \|t^{-1/q} (f\chi_A)^*(t)\|_{L_q(0,r)} \\
&\ge c \sup_{r>0} r^{-\lambda/q} \|t^{-1/q}\|_{L_q(0,\min\{|A|,r\})} = \infty.
\end{aligned}$$

Lemma 2.1 ([2]): *Let* $0 < q \le p < \infty$, $1/s = 1/p - \lambda/q$ *and* $0 < \lambda \le q/p$. *Then*

$$\left(\frac{q}{p}\right)^{-1/q} \|f\|_{WL_s} \le \|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}} \le \lambda^{-1/q} \|f\|_{WL_s}.$$

In particular, $\|f\|_{WL_\infty} = \|f\|_{M^{\mathrm{loc}}_{q/\lambda,q;\lambda}}$.

Definition 2.4: The space of functions with bounded mean oscillation, $\mathrm{BMO} \equiv \mathrm{BMO}(\mathbb{R}^n)$, consists of those functions f for which

$$\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B(x)|\, \mathrm{d}x$$

is finite, where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and

$$f_B(x) = \frac{1}{|B|} \int_B f(x)\, \mathrm{d}x.$$

We will use the boundedness of the following Hardy operators to obtain the boundedness of the maximal operator M, Calderon–Zygmund operator T and maximal Calderon–Zygmund operator $\mathcal{T}$ in the local Morrey–Lorentz spaces $M^{\rm loc}_{p,q;\lambda}$.

Definition 2.5 ([18]): Let φ be a measurable function on $(0,\infty)$ and β be a real number. The weighted Hardy operators A_β and $\mathcal{A}_\beta$ with power weights acting on φ are defined by

$$A_\beta\varphi(t) = t^{\beta-1}\int_0^t \frac{\varphi(s)}{s^\beta}\,{\rm d}s, \quad \mathcal{A}_\beta\varphi(t) = t^\beta\int_t^\infty \frac{\varphi(s)}{s^{\beta+1}}\,{\rm d}s. \tag{2.3}$$

The following theorem was proved in [18] by N. Samko.

Theorem C: *Let $\beta\in\mathbb{R}$, $0\le\lambda<1$ and $1\le q<\infty$. If $\beta<\lambda/q+1/q'$ and $\beta>\lambda/q-1/q$, then the operators A_β and $\mathcal{A}_\beta$ are bounded on the local Morrey space $LM_{q,\lambda}(0,\infty)$, respectively.*

The following theorem was proved in [1].

Theorem D: *Let $\beta\in\mathbb{R}$, $0\le\lambda<1$ and $1<q<\infty$. If $\beta=\lambda/q+1/q'$ and $\beta=\lambda/q-1/q$, then the operators A_β and $\mathcal{A}_\beta$ are bounded from the local Morrey space $LM_{q,\lambda}(0,\infty)$ to the weak local Morrey space $WLM_{q,\lambda}(0,\infty)$, respectively.*

3. Proof of Theorems

Proof of Theorem 1.1: Let $1\le q\le\infty$, $0\le\lambda<1$ and $q/(q+\lambda)\le p\le\infty$.

(i) Suppose $q/(q+\lambda)<p<\infty$ and $f\in M^{\rm loc}_{p,q;\lambda}$. From the definition in local Morrey–Lorentz spaces and inequality (2.1) we get

$$\|Mf\|_{M^{\rm loc}_{p,q;\lambda}} \le C\sup_{r>0} r^{-\lambda/q}\left\| t^{1/p-1/q-1}\int_0^t f^*(s)\,{\rm d}s\right\|_{L_q(0,r)} = C\|A_{(1/p-1/q)}g\|_{LM_{q,\lambda}(0,\infty)},$$

where $g(t)=t^{1/p-1/q}f^*(t)$. Since $1/p-\lambda/q<1$, for $\beta=1/p-1/q$ the inequality $\beta<\lambda/q+1/q'$ holds. By Theorem C we get

$$\|A_{(1/p-1/q)}g\|_{LM_{q,\lambda}(0,\infty)} \le C\|g\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{\rm loc}_{p,q;\lambda}}. \tag{3.1}$$

Therefore we obtain the boundedness of M in $M^{\rm loc}_{p,q;\lambda}$ for $q/(q+\lambda)<p<\infty$.

(ii) For the limiting case $p=q/(q+\lambda)$ suppose $f\in M^{\rm loc}_{p,q;\lambda}$. From the definition of norm in weak local Morrey–Lorentz space and by using the inequality (2.1) we get

$$\|Mf\|_{WM^{\rm loc}_{q/(q+\lambda),q;\lambda}} \le C\sup_{r>0} r^{-\lambda/q}\left\| t^{(\lambda-1)/q}\int_0^t f^*(s)\,{\rm d}s\right\|_{WL_q(0,t)} = C\|A_\beta h\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta=1+(\lambda-1)/q$ and $h(t)=t^{1+(\lambda-1)/q}f^*(t)$. Therefore we get from Theorem D

$$\|A_\beta h\|_{WLM_{q,\lambda}(0,\infty)} \le C\|h\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{\rm loc}_{q/(q+\lambda),q;\lambda}}. \tag{3.2}$$

Then we obtain the boundedness of the operator M from the space $M^{\rm loc}_{q/(q+\lambda),q;\lambda}$ to the weak space $WM^{\rm loc}_{q/(q+\lambda),q;\lambda}$.

(iii) In the limiting case $p=\infty$, suppose $f \in M^{\rm loc}_{\infty,q;\lambda}$. Remark 2.1 implies that for any $0<q<\infty$ the space $M^{\rm loc}_{\infty,q;\lambda}$ is trivial. Therefore, we must consider the case $q=\infty$. Since the operator M is bounded on L_∞ we get the statement. ■

Proof of Theorem 1.2: Let $1 \le q \le \infty$, $0 \le \lambda < 1$ and $q/(q+\lambda) \le p \le q/\lambda$. Since f satisfies (1.1), by Theorem B the Calderon–Zygmund operator Tf exists almost every $x \in \mathbb{R}^n$.

(i) Suppose that $1 \le q < \infty$, $0 \le \lambda < 1$, $q/(q+\lambda) < p < q/\lambda$ and $f \in M^{\rm loc}_{p,q;\lambda}$. From the definition of norm in local Morrey–Lorentz spaces, by using the inequality (1.3) and Minkowski's inequality we get

$$\begin{aligned}\|Tf\|_{M^{\rm loc}_{p,q,\lambda}} &\le C \sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q-1} \int_0^t f^*(s)\,{\rm d}s \right\|_{L_q(0,r)} \\ &\quad + \sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q} \int_t^\infty \frac{f^*(s)}{s}\,{\rm d}s \right\|_{L_q(0,r)} = I_1 + I_2.\end{aligned}$$

I_1 can be estimated using the same method as in the proof of the boundedness of the maximal operator on $M^{\rm loc}_{p,q;\lambda}$ in Theorem 1.1.

Let us estimate I_2 :

$$I_2 = C \sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q} \int_t^\infty \frac{f^*(s)}{s}\,{\rm d}s \right\|_{L_q(0,r)} = C\|\mathcal{A}_{(1/p-1/q)} g\|_{LM_{q,\lambda}(0,\infty)}, \tag{3.3}$$

where $g(t) = t^{1/p-1/q} f^*(t)$. Since $1/p - \lambda/q > 0$, for $\beta = 1/p - 1/q$ the inequality $\beta > \lambda/q - 1/q$ holds. By Theorem C we get

$$\|\mathcal{A}_{(1/p-1/q)} g\|_{LM_{q,\lambda}(0,\infty)} \le C\|g\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{\rm loc}_{p,q;\lambda}}.$$

Therefore we get $I_2 \le C\|f\|_{M^{\rm loc}_{p,q;\lambda}}$. Consequently we obtain the boundedness of T in $M^{\rm loc}_{p,q;\lambda}$ from the inequalities (3.1) and (3.3).

(ii) For the limiting case $p = q/(q+\lambda)$, $1 < q < \infty$, suppose $f \in M^{\rm loc}_{p,q;\lambda}$. From the definition of norm in weak local Morrey–Lorentz spaces and by using the inequality (1.3) and Minkowski's inequality we get

$$\begin{aligned}\|Tf\|_{WM^{\rm loc}_{q/(q+\lambda),q;\lambda}} &\le C \sup_{r>0} r^{-\lambda/q} \left\| t^{(\lambda-1)/q} \int_0^t f^*(s)\,{\rm d}s \right\|_{WL_q(0,r)} \\ &\quad + C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+(\lambda-1)/q} \int_t^\infty \frac{f^*(s)}{s}\,{\rm d}s \right\|_{WL_q(0,r)} = N_1 + N_2.\end{aligned}$$

N_1 can be estimated using the same method as in the proof of the weak boundedness of the maximal operator on $M^{\rm loc}_{p,q;\lambda}$ in Theorem 1.1. Let us estimate N_2 :

$$N_2 = C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+(\lambda-1)/q} \int_t^\infty \frac{f^*(s)}{s}\,{\rm d}s \right\|_{WL_q(0,r)} = C\|\mathcal{A}_\beta h\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta = 1 + (\lambda - 1)/q$ and $h(t) = t^{1+(\lambda-1)/q} f^*(t)$. By Theorem D the operator $\mathcal{A}_\beta$ is bounded from the Morrey spaces $LM_{q,\lambda}(0,\infty)$ to $WLM_{q,\lambda}(0,\infty)$. Then,

$$N_2 \leq C\|h\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{\rm loc}_{q/(q+\lambda),q;\lambda}}. \tag{3.4}$$

From the inequalities (3.2) and (3.4) we obtain the boundedness of the operator T from $M^{\rm loc}_{q/(q+\lambda),q;\lambda}$ to $WM^{\rm loc}_{q/(q+\lambda),q;\lambda}$.

(iii) For the limiting case $p = q/\lambda$, $1 \leq q \leq \infty$ and $0 \leq \lambda < 1$, suppose $f \in M^{\rm loc}_{p,q;\lambda}$.

Since the operator T^0 is bounded from L_∞ to BMO and from Lemma 2.1 $M^{\rm loc}_{q/\lambda,q;\lambda} \equiv WL_\infty \equiv L_\infty$, then the inequality

$$\|T^0 f\|_{\rm BMO} \leq C\|f\|_{M^{\rm loc}_{q/\lambda,q;\lambda}} = C\|f\|_{L_\infty}$$

holds (see [8]) which proves that Calderon–Zygmund operator T^0 is bounded from $M^{\rm loc}_{q/\lambda,q;\lambda}$ to BMO.

Thus the proof of the theorem is completed. ■

Proof of Theorem 1.3: Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ and $q/(q+\lambda) \leq p \leq q/\lambda$. Since f satisfies (1.1), by Theorem B the maximal Calderon–Zygmund operator $\mathcal{T} f(x)$, $x \in \mathbb{R}^n$ is finite almost everywhere.

The proof of the statements (i) and (ii) of this corollary can be easily obtained from the inequality (1.3) and using the same method of Theorem 1.2.

(iii) For the limiting case $1 \leq q \leq \infty$, $p = q/\lambda$ and $0 \leq \lambda < 1$, suppose $f \in M^{\rm loc}_{p,q;\lambda}$. Since the operator $\mathcal{T}$ is bounded from L_∞ to BMO and $M^{\rm loc}_{q/\lambda,q;\lambda} \equiv L_\infty$, the inequality

$$\|\mathcal{T} f\|_{\rm BMO} \leq C\|f\|_{M^{\rm loc}_{q/\lambda,q;\lambda}}$$

holds (see [19]) which proves that the operator $\mathcal{T}$ is bounded from $M^{\rm loc}_{q/\lambda,q;\lambda}$ to BMO.

Thus the proof of the theorem is completed. ■

4. Some applications

In this section we give some applications of our results. Firstly, as an application of the boundedness of the maximal operator M we obtain the boundedness of Bochner–Riesz operator B^δ_r.

Let $\delta > (n-1)/2$, $B^\delta_r(f)\hat{}(\xi) = (1 - r^2|\xi|^2)^\delta_+ \hat{f}(\xi)$ and $B^\delta_r(x) = r^{-n} B^\delta(x/r)$ for $r > 0$. The maximal Bochner–Riesz operator is defined by (see [20,21])

$$B_{\delta,*}(f)(x) = \sup_{r>0} |B^\delta_r(f)(x)|.$$

It is clear that (see [22])

$$B_{\delta,*}(f)(x) \leq CMf(x).$$

Since the maximal operator M is bounded on the spaces $M^{\rm loc}_{p,q;\lambda}$, we get the following theorem.

Theorem 4.1: *Let* $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ *and* $q/(q+\lambda) \leq p \leq \infty$.

(i) *If* $q/(q+\lambda) < p < \infty$, *then the Bochner–Riesz operator* B_r^δ *is bounded on the local Morrey–Lorentz space* $M_{p,q;\lambda}^{\mathrm{loc}}$.
(ii) *If* $p = q/(q+\lambda)$, *then the Bochner–Riesz operator* B_r^δ *is bounded from* $M_{p,q;\lambda}^{\mathrm{loc}}$ *to the weak local Morrey–Lorentz space* $WM_{p,q;\lambda}^{\mathrm{loc}}$.
(iii) *If* $p = q = \infty$, *then the Bochner–Riesz operator* B_r^δ *is bounded on* $L_\infty(\mathbb{R}^n)$.

For the case $\lambda = 0$, from Theorem 4.1 we get the following.

Corollary 4.1: *Let* $1 \leq q \leq \infty$, *and* $1 \leq p \leq \infty$.

(i) *If* $1 < p < \infty$, *then the Bochner–Riesz operator* B_r^δ *is bounded on the Lorentz space* $L_{p,q}(\mathbb{R}^n)$.
(ii) *If* $p = 1$, *then the Bochner–Riesz operator* B_r^δ *is bounded from* $L_{p,q}(\mathbb{R}^n)$ *to the weak Lorentz space* $WL_{p,q}(\mathbb{R}^n)$.
(iii) *If* $p = q = \infty$, *then the Bochner–Riesz operator* B_r^δ *is bounded on* $L_\infty(\mathbb{R}^n)$.

For the case $\lambda = 0$, from Theorem 1.1 we get the following corollary.

Corollary 4.2: *Let* $1 \leq q \leq \infty$, *and* $1 \leq p \leq \infty$.

(i) *If* $1 < p < \infty$, *then the maximal operator* M *is bounded on the Lorentz space* $L_{p,q}(\mathbb{R}^n)$.
(ii) *If* $p = 1$, *then the operator* M *is bounded from* $L_{1,q}(\mathbb{R}^n)$ *to the weak Lorentz space* $WL_{1,q}(\mathbb{R}^n)$.
(iii) *If* $p = q = \infty$, *then the operator* M *is bounded on* $L_\infty(\mathbb{R}^n)$.

Remark 4.1: Note that, the statements (i) and (iii) of the Corollary 4.2 are known, see for example [23, pp. 76], but the statement (ii) of the Corollary 4.2 is new.

In the case $\lambda = 0$, from Theorems 1.2 and 1.3 we get the following corollary.

Corollary 4.3: *Let* $1 \leq q \leq \infty$, *and* $1 \leq p \leq \infty$.

(i) *If* $1 < p < \infty$, *then the operators* T *and* $\mathcal{T}$ *are bounded on the Lorentz space* $L_{p,q}(\mathbb{R}^n)$.
(ii) *If* $p = 1$, *then the operators* T *and* $\mathcal{T}$ *are bounded from* $L_{1,q}(\mathbb{R}^n)$ *to the weak Lorentz space* $WL_{1,q}(\mathbb{R}^n)$.
(iii) *If* $p = q = \infty$, *then the operators* T^0 *and* $\mathcal{T}$ *are bounded from* $L_\infty(\mathbb{R}^n)$ *to* $BMO(\mathbb{R}^n)$.

Remark 4.2: Note that, the statements (i) and (iii) of the Corollary 4.3 is known (see, e.g. [23]) but the statement (ii) is new.

From Lemma 2.1, since the norms $\|f\|_{WL_s}$ and $\|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}}$ are equivalent for the case $1 \leq q \leq p < \infty$, $1/s = 1/p - \lambda/q$ and $0 < \lambda \leq q/p$, we get the following corollaries.

Corollary 4.4: *Let* $0 < s < \infty$*. Then the operators* M *and* B_r^δ *are bounded on weak Lebesgue spaces* WL_s*.*

Corollary 4.5: *Let* $1 < s < \infty$*. Then the operators* T *and* $\mathcal{T}$ *are bounded on weak Lebesgue spaces* WL_s*.*

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