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Research Article

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Commutator of fractional integral with Lipschitz functions related to Schrödinger operator on local generalized mixed Morrey spaces

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Abstract: Let $L = -\triangle + V$ be the Schrödinger operator on $\mathbb{R}^n$, where $V \neq 0$ is a non-negative function satisfying the reverse Hölder class RH_{q_1} for some $q_1 > n/2$. $\triangle$ is the Laplacian on $\mathbb{R}^n$. Assume that b is a member of the Campanato space $\Lambda_\nu^\theta(\rho)$ and that the fractional integral operator associated with L is $\mathcal{I}_\beta^L$. We study the boundedness of the commutators $[b, \mathcal{I}_\beta^L]$ with $b \in \Lambda_\nu^\theta(\rho)$ on local generalized mixed Morrey spaces. Generalized mixed Morrey spaces $M_{\vec{p},\varphi}^{\alpha,V}$, vanishing generalized mixed Morrey spaces $VM_{\vec{p},\varphi}^{\alpha,V}$, and $LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}$ are related to the Schrödinger operator, in that order. We demonstrate that the commutator operator $[b, \mathcal{I}_\beta^L]$ is satisfied when b belongs to $\Lambda_\nu^\theta(\rho)$ with $\theta > 0$, $0 < \nu < 1$, and (φ_1, φ_2) satisfying certain requirements are bounded from $LM_{\vec{p},\varphi_1}^{\alpha,V,\{x_0\}}$ to $LM_{\vec{q},\varphi_2}^{\alpha,V,\{x_0\}}$; from $M_{\vec{p},\varphi_1}^{\alpha,V}$ to $M_{\vec{q},\varphi_2}^{\alpha,V}$, and from $VM_{\vec{p},\varphi_1}^{\alpha,V}$ to $VM_{\vec{q},\varphi_2}^{\alpha,V}$, $\sum_{i=1}^n 1/p_i - \sum_{i=1}^n 1/q_i = \beta + \nu$.

Keywords: Schrödinger operator, fractional integral, commutator, Lipschitz function, local generalized mixed Morrey space

MSC 2020: 42B35, 35J10, 47H50

1 Introduction and main results

This work studies the Schrödinger differential operator of second order in $\mathbb{R}^n$, where $n \geq 3$, as described by

$$L = -\Delta + V.$$

For some exponent $q \geq n/2$, the function V is non-negative, $V \neq 0$, and it belongs to a reverse Hölder class RH_q. Assume that V is a nonnegative locally $L_q(\mathbb{R}^n)$ integrable function on $\mathbb{R}^n$, then we say that V belongs to RH_q $(1 < p \leq \infty)$ if there exists a positive constant C such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)\mathrm{d}x\right)^{1/q} \leq \frac{C}{|B|}\int_B V(x)\mathrm{d}x$$

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holds for every $x \in \mathbb{R}^n$, where $B(x,r)$ denotes the ball centered at x with radius r. The nonnegative polynomial $V \in RH_\infty$, and specifically $|x| \in RH_\infty$, should be noted (see, e.g., [1,2]).

Let the potential $V \in RH_q$ with $q \geq n/2$, and the critical radius function $\rho(x)$ be defined as

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\mathrm{d}y \leq 1\right\}, \quad x \in \mathbb{R}^n.$$

Clearly, $0 < m_V(x) <$ ß when $V \neq 0$, and $m_V(x) = 1$ when $V = 1$ and $m_V(x) \approx 1 + |x|$ with $V(x) = |x|^2$.

For sufficient good function f, the heat diffusion semigroup e^{-tL} allows the negative powers $L^{-\beta/2}$ $(\beta > 0)$ associated with the Schrödinger operators L to be written as

$$\mathcal{I}_\beta^L f(x) = L^{-\beta/2} f(x) = \int_0^\infty e^{-tL}(f)(x)\, t^{\beta/2-1}\mathrm{d}t, \quad 0 < \beta < n.$$

Applying Lemma 3.3 in [3] for enough good function f holds that

$$\mathcal{I}_\beta^L f(x) = \int_{\mathbb{R}^n} K_\beta(x,y) f(y)\mathrm{d}y, \quad 0 < \beta < n.$$

The commutator of $\mathcal{I}_\beta^L$ is defined by

$$[b, \mathcal{I}_\beta^L] f(x) = b(x)\mathcal{I}_\beta^L f(x) - \mathcal{I}_\beta^L(bf)(x).$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $\mathcal{I}_\beta^L$ and $[b, \mathcal{I}_\beta^L]$ are the Riesz potential I_β and the commutator of the Riesz potential $[b, I_\beta]$, respectively, i.e.,

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n}\mathrm{d}y, \quad [b, I_\beta] f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^n} f(y)\mathrm{d}y.$$

Let $\theta > 0$ and $0 < \nu < 1$; in view of [4], the Campanato class, associated with Schrödinger operator, $\Lambda_\nu^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\nu/n}} \int_{B(x,r)} |b(y) - b_B|\mathrm{d}y \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta, \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\nu^\theta(\rho)$, denoted by $[b]_\nu^\theta$, is given by the infimum of the constants in the aforementioned inequality.

Note that if $\theta = 0$, $\Lambda_\nu^\theta(\rho)$ is the classical Campanato space; if $\nu = 0$, $\Lambda_\nu^\theta(\rho)$ is exactly the space $\mathrm{BMO}_\theta(\rho)$ introduced in [1].

Throughout this article, the letter $\vec{p}$ denotes n-tuples of the numbers in $(0, \infty]$, $n \geq 1$, $\vec{p} = (p_1, p_2, \ldots, p_n)$, $1 \leq \vec{p} < \infty$ means $1 \leq p_i < \infty$ for each i. For $1 \leq \vec{p} \leq \infty$, we denote $\vec{p}\,' = (p_1', \ldots, p_n')$, where p_i' satisfies $1/p_i + 1/p_i' = 1$.

In 2019, Nogayama [5] considered a new Morrey space, with the L_p norm replaced by the mixed Lebesgue norm $L_{\vec{p}}(\mathbb{R}^n)$, which is called mixed Morrey spaces.

We first recall the definition of mixed Lebesgue space defined in [6].

Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then, the mixed Lebesgue norm $\|\cdot\|_{L_{\vec{p}}}$ or $\|\cdot\|_{L_{(p_1,\ldots,p_n)}}$ is defined by

$$\|f\|_{L_{\vec{p}}} \equiv \|f\|_{L_{(p_1,\ldots,p_n)}} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1}\mathrm{d}x_1\right)^{\frac{p_2}{p_1}} \mathrm{d}x_2\right)^{\frac{p_3}{p_2}} \ldots \mathrm{d}x_n\right)^{\frac{1}{p_n}},$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a measurable function. If $p_j=\infty$ for some $j=1,\dots,n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L_{\vec{p}}(\mathbb{R}^n)=L_{(p_1,\dots,p_n)}(\mathbb{R}^n)$ to be the set of all $f\in L_0(\mathbb{R}^n)$ with $\|f\|_{L_{\vec{p}}}<\infty$, where $L_0(\mathbb{R}^n)$ denotes the set of measureable functions on $\mathbb{R}^n$.

The following analog of Hölder's inequality for $L_{\vec{p}}$ is well known (see, e.g., [7]).

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^n$ be a measurable set, $1\le\vec{p}\le\infty$ and $\frac{1}{\vec{p}}+\frac{1}{\vec{p}\,'}=1$. Then, for any $f\in L_{\vec{p}}(\Omega)$, and $g\in L_{\vec{p}\,'}(\Omega)$, the following inequality is valid*:

$$\int_{\Omega}|f(x)g(x)|\mathrm{d}x\le\|f\|_{L_{\vec{p}}(\Omega)}\|g\|_{L_{\vec{p}\,'}(\Omega)}.$$

By elementary calculations, we have the following property.

Lemma 1.1. *Let $0<\vec{p}<\infty$ and B be a ball in $\mathbb{R}^n$. Then,*

$$\|\chi_B\|_{L_{\vec{p}}}=\|\chi_B\|_{WL_{\vec{p}}}=|B|^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}}.$$

By Theorem 1.1 and Lemma 1.1, we obtain the following estimate.

Lemma 1.2. *For $1\le\vec{p}<\infty$ and for the balls $B=B(x,r)$, the following inequality is valid*:

$$\int_{B}|f(y)|\mathrm{d}y\le|B|^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p'}}\ \|f\|_{L_{\vec{p}}(B)}.$$

The following lemma shows the Lebesgue differential theorem in mixed-norm Lebesgue spaces as follows.

Lemma 1.3. [7, Lemma 2.4] *Let $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $0<\vec{p}<\infty$, then*

$$\lim_{r\to0}\|\chi_{B(x,r)}\|_{L_{\vec{p}}}^{-1}\|f\|_{L_{\vec{p}}(B(x,r))}=|f(x)|,\quad a.e.\ x\in\mathbb{R}^n.$$

Morrey [8] proposed the traditional Morrey spaces $L_{p,\lambda}$ to examine the local behavior of solutions to elliptic partial differential equations (PDEs). Actually, a higher degree of regularity in the solutions to certain elliptic and parabolic boundary problems may be obtained thanks to the improved inclusion between the Morrey and the Hölder spaces. Generalized Morrey spaces $M_{p,\varphi}$ were separately introduced by Guliyev et al. [9–11] (see also [12–14]). Generally speaking, local Morrey spaces were also introduced separately by Guliyev [9] and Garcia-Cuerva and Herrero [15] (see also [16]). It should be noted that Guliyev introduced and analyzed integral operators in local Morrey-type spaces, including generalized local Morrey spaces, in [9].

We now present the definition of local generalized mixed Morrey space $LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}$ and generalized mixed Morrey spaces $M_{\vec{p},\varphi}^{\alpha,V}(\mathbb{R}^n)$ associated with Schrödinger operator, which in the case of $\vec{p}=(p,\dots,p)$ introduced by Guliyev [17].

For brevity, in the sequel, we use the notations

$$\mathfrak{A}_{\vec{p},\varphi}^{\alpha,V}(f;\,x,r)\coloneqq\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}\|\chi_{B(x,r)}\|_{L_{\vec{p}}(\mathbb{R}^n)}^{-1}\|f\chi_{B(x,r)}\|_{L_{\vec{p}}(\mathbb{R}^n)}.$$

Definition 1.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$, $1\le p<\infty$, $\alpha\ge0$, and $V\in RH_q$, $q\ge1$. For any fixed $x_0\in\mathbb{R}^n$, we denote by $LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}=LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$, $M_{\vec{p},\varphi}^{\alpha,V}=M_{\vec{p},\varphi}^{\alpha,V}(\mathbb{R}^n)$ the local generalized mixed Morrey space, and the generalized mixed Morrey space associated with Schrödinger operator, and the space of all functions $f\in L_{\vec{p}}^{\mathrm{loc}}(\mathbb{R}^n)$ with finite norms:

$$\|f\|_{LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}}=\sup_{r>0}\mathfrak{A}_{\vec{p},\varphi}^{\alpha,V}(f;\,x_0,r),$$

$$\|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{\alpha,V}_{\vec{p},\varphi}(f;\, x, r),$$

respectively.

Remark 1.1.

(i) When $\alpha = 0$, and $\varphi(x_0, r) = r^{(\frac{\lambda}{n}-1)\sum_{i=1}^{n}\frac{1}{p_i}}$, $LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi}(\mathbb{R}^n)$ is the local (central) mixed Morrey space $LM^{\{x_0\}}_{\vec{p},\lambda}(\mathbb{R}^n)$ studied in [9,16,18] in the case of $\vec{p} = (p, \ldots, p)$;

(ii) When $V \equiv 0$ ($\alpha = 0$), $LM^{0,0,\{x_0\}}_{\vec{p},\varphi}(\mathbb{R}^n)$ is the local generalized mixed Morrey space, and $LM^{\{x_0\}}_{\vec{p},\varphi}(\mathbb{R}^n)$ was introduced by Guliyev [9] in the case of $\vec{p} = (p, \ldots, p)$ (see also [19–22]).

Definition 1.2. The vanishing generalized Morrey space associated with Schrödinger operator $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{\vec{p},\varphi}(f;\, x, r) = 0. \tag{1.2}$$

The vanishing space $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is Banach space with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{\vec{p},\varphi}} \equiv \|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{\alpha,V}_{\vec{p},\varphi}(f;\, x, r).$$

In the case of $\alpha = 0$, $\vec{p} = (p, \ldots, p)$, and $\varphi(x, r) = r^{(\lambda-n)/p}$, $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [23], where applications to PDE were considered.

We refer to [24–27] for some properties of vanishing generalized Morrey spaces.

When $b \in \mathrm{BMO}$, Chanillo proved in [28] that $[b, I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$. When b belongs to the Campanato space Λ_ν, $0 < \nu < 1$, Paluszyński [29] showed that $[b, I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - (\beta + \nu)/n$, $1 < p < n/(\beta + \nu)$. When $b \in \mathrm{BMO}_\theta(\rho)$, Bui [30] obtained the boundedness of $[b, \mathcal{I}^L_\beta]$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$.

Inspired by the aforementioned results, we are interested in the boundedness of $[b, \mathcal{I}^L_\beta]$ generalized mixed Morrey spaces $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ and the vanishing generalized mixed Morrey spaces $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$, when b belongs to the new Campanato class $\Lambda^\theta_\nu(\rho)$.

In this article, we consider the boundedness of the commutator of $\mathcal{I}^L_\beta$ on the local generalized mixed Morrey spaces $LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi}$, the generalized mixed Morrey spaces $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$, and the vanishing generalized mixed Morrey spaces $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$. When b belongs to the new Campanato space $\Lambda^\theta_\nu(\rho)$, $0 < \nu < 1$, we show that $[b, \mathcal{I}^L_\beta]$ are bounded from $LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}$ to $LM^{\alpha,V,\{x_0\}}_{\vec{q},\varphi_2}$, from $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ to $M^{\alpha,V}_{\vec{q},\varphi}(\mathbb{R}^n)$, and from $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ to $VM^{\alpha,V}_{\vec{q},\varphi}(\mathbb{R}^n)$ with $\sum_{i=1}^{n} 1/p_i - \sum_{i=1}^{n} 1/q_i = \beta + \nu$, $1 < \vec{p} < n/(\beta + \nu)$.

Our main results are the following.

Theorem 1.2. *Let $x_0 \in \mathbb{R}^n$, $b \in \Lambda^\theta_\nu(\rho)$, $V \in RH_{q_1}$, $q_1 > n/2$, $0 < \nu < 1$, $\alpha \geq 0$, $1 < \vec{p} < n/(\beta + \nu)$, $\sum_{i=1}^{n} 1/p_i - \sum_{i=1}^{n} 1/q_i = \beta + \nu$, and $\varphi_1, \varphi_2 \in \Omega^{\alpha,V}_{\vec{p},\mathrm{loc}}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t} \leq c_0\, \varphi_2(x_0, r), \tag{1.3}$$

where c_0 does not depend on x_0 and r, and for the definition of $\Omega^{\alpha,V}_{\vec{p},\mathrm{loc}}$, see Remark 2.1. Then, the operator $[b, \mathcal{I}^L_\beta]$ is bounded from $M^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}$ to $M^{\alpha,V,\{x_0\}}_{\vec{q},\varphi_2}$. Moreover,

$$\|[b, \mathcal{I}^L_\beta] f\|_{M^{\alpha,V,\{x_0\}}_{\vec{q},\varphi_2}} \leq C[b]^\theta_\nu \|f\|_{M^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}},$$

where C does not depend on f.

Corollary 1.1. *Let* $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > n/2$, $0 < \nu < 1$, $\alpha \geq 0$, $1 < \vec{p} < n/(\beta + \nu)$, $\sum_{i=1}^n 1/p_i - \sum_{i=1}^n 1/q_i = \beta + \nu$, *and* $\varphi_1 \in \Omega_{\vec{p}}^{\alpha,V}$, $\varphi_2 \in \Omega_{\vec{q}}^{\alpha,V}$ *satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t} \leq c_0\, \varphi_2(x,r), \tag{1.4}$$

where c_0 *does not depend on* x *and* r, *and for the definition of* $\Omega_{\vec{p}}^{\alpha,V}$, *see Remark* 2.2. *Then, the operator* $[b, \mathcal{I}_\beta^L]$ *is bounded from* $M_{\vec{p},\varphi_1}^{\alpha,V}$ *to* $M_{\vec{q},\varphi_2}^{\alpha,V}$. *Moreover,*

$$\|[b, \mathcal{I}_\beta^L]f\|_{M_{\vec{q},\varphi_2}^{\alpha,V}} \leq C[b]_\theta \|f\|_{M_{\vec{p},\varphi_1}^{\alpha,V}},$$

where C *does not depend on* f.

Theorem 1.3. *Let* $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > n/2$, $0 < \nu < 1$, $\alpha \geq 0$, $b \in \Lambda_\nu^\theta(\rho)$, $1 < \vec{p} < n/(\beta + \nu)$, $\sum_{i=1}^n 1/p_i - \sum_{i=1}^n 1/q_i = \beta + \nu$, *and* $\varphi_1 \in \Omega_{\vec{p},1}^{\alpha,V}$, $\varphi_2 \in \Omega_{\vec{q},1}^{\alpha,V}$ *satisfy the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t} < \infty,$$

for every $\delta > 0$, *and*

$$\int_r^\infty \frac{\varphi_1(x,t)}{t^{1-\beta-\nu}} dt \leq C_0 \varphi_2(x,r), \tag{1.5}$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$, $r > 0$ *and for the definition of* $\Omega_{\vec{p},1}^{\alpha,V}$, *see Remark* 2.3. *Then, the operator* $[b, \mathcal{I}_\beta^L]$ *is bounded from* $VM_{\vec{p},\varphi_1}^{\alpha,V}$ *to* $VM_{\vec{q},\varphi_2}^{\alpha,V}$ *for* $\vec{p} > 1$.

Remark 1.2. Note that in the case of $\vec{p} = (p, \ldots, p)$, $V \equiv 0$, $\nu = 0$ Corollary 1.1 and Theorem 1.3 were proved in [31, Corollary 5.5 and 7.5] and in the case of $\vec{p} = (p, \ldots, p)$, $\varphi(x,r) = r^{(\lambda-n)/p}$, $\nu = 0$ in [32, Theorems 1.3 and 1.4].

In this article, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some technical lemmas and propositions

We would like to recall the important properties concerning the critical function.

Lemma 2.1. [2] *Suppose* $V \in RH_{q_1}$ *with* $q_1 > n/2$. *Then, there exist* C *and* $k_0 \geq 1$ *such that*

$$C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}, \tag{2.1}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.2. [33] *Suppose* $x \in B(x_0, r)$. *Then, for* $k \in N$, *we have*

$$\frac{1}{\left(1 + \frac{2^k r}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}.$$

According to [1], the new BMO space $\mathrm{BMO}_\theta(\rho)$ with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |b(y) - b_B|dy \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)dy$ and $\mathrm{BMO} \equiv \mathrm{BMO}_\theta(\rho)$. A norm for $b \in \mathrm{BMO}_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the aforementioned inequalities. Clearly, $BMO \subset \mathrm{BMO}_\theta(\rho)$.

Let $\theta > 0$ and $0 < \nu < 1$; a seminorm on Campanato class $\Lambda_\nu^\theta(\rho)$ is denoted by $[b]_\nu^\theta$

$$[b]_\nu^\theta \coloneqq \sup_{x\in\mathbb{R}^n, r>0} \frac{\frac{1}{|B(x,r)|^{1+\nu/n}}\int_{B(x,r)}|b(y) - b_B|dy}{\left(1 + \frac{r}{\rho(x)}\right)^\theta} < \infty. \tag{2.2}$$

The Lipschitz space, associated with Schrödinger operator (see [4]), consists of the functions f satisfying

$$\|f\|_{\mathrm{Lip}_\nu^\theta(\rho)} \coloneqq \sup_{x\in\mathbb{R}^n, r>0} \frac{|f(x) - f(y)|}{|x - y|^\nu\left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^\theta} < \infty.$$

It is easy to see that this space is exactly the Lipschitz space when $\theta = 0$.

Note that if $\theta = 0$ in (2.2), $\Lambda_\nu^\theta(\rho)$ is exactly the classical Campanato space; if $\nu = 0$, $\Lambda_\nu^\theta(\rho)$ is exactly the space $\mathrm{BMO}_\theta(\rho)$; if $\theta = 0$ and $\nu = 0$, it is exactly the John-Nirenberg space BMO.

The following relation between $\mathrm{Lip}_\nu^\theta(\rho)$ and $\Lambda_\nu^\theta(\rho)$ was proved in [4, Theorem 5].

Lemma 2.3. [4] *Let $\theta > 0$ and $0 < \nu < 1$. Then, the following embedding is valid*:

$$\Lambda_\nu^\theta(\rho) \subseteq \mathrm{Lip}_\nu^\theta(\rho) \subseteq \Lambda_\nu^{(k_0+1)\theta}(\rho),$$

where k_0 is the constant appearing in Lemma 2.1.

We list some properties involving Campanato space, associated with Schrödinger operator $\Lambda_\nu^\theta(\rho)$.

Lemma 2.4. [4] *Let $\theta > 0$ and $1 \leq s < \infty$. If $b \in \Lambda_\nu^\theta(\rho)$, then there exists a positive constant C such that*

$$\left(\frac{1}{|B|}\int\limits_B |b(y) - b_B|^s dy\right)^{1/s} \leq C[b]_\nu^\theta r^\nu\left(1 + \frac{r}{\rho(x)}\right)^{\theta'},$$

for all $B = B(x,r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0 + 1)\theta$ and k_0 is the constant appearing in (2.1).

Let K_β be the kernel of $\mathcal{I}_\beta^L$. The following result gives the estimate on the kernel $K_\beta(x,y)$.

Lemma 2.5. [30] *If $V \in RH_{q_1}$ with $q_1 > n/2$, then for every N, there exists a constant C such that*

$$|K_\beta(x,y)| \leq \frac{C}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N}\frac{1}{|x - y|^{n-\beta}}, \quad 0 < \beta < n. \tag{2.3}$$

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 2.6. [34] *Let f be a real-valued nonnegative function and measurable on E. Then,*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E}\frac{1}{f(x)}.$$

It is natural, first of all, to find conditions ensuring that the spaces $LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}$ and $M_{\vec{p},\varphi}^{\alpha,V}$ are nontrivial, which consist not only of functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.7. *Let $x_0 \in \mathbb{R}^n$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le \vec{p} < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. If*

$$\sup_{t<r<\infty}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\frac{r^{-\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x_0,r)}=\infty,\quad \text{for some } t>0, \tag{2.4}$$

then $LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi}(\mathbb{R}^n)=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Proof. Let (2.4) be satisfied and f be not equivalent to zero. Then, $\|f\|_{L_{\vec{p}}(B(x_0,t))}>0$; hence,

$$\begin{aligned}\|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi}} &\ge \sup_{t<r<\infty}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi(x_0,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|f\|_{L_{\vec{p}}(B(x_0,r))}\\ &\ge \|f\|_{L_{\vec{p}}(B(x_0,t))}\sup_{t<r<\infty}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi(x_0,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}.\end{aligned}$$

Therefore, $\|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi}}=\infty$. □

Remark 2.1. We denote by $\Omega^{\alpha,V}_{\vec{p},\mathrm{loc}}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n}\left\|\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{-\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)}\right\|_{L_\infty(t,\infty)}<\infty.$$

In what follows, keeping in mind Lemma 2.7, for the non-triviality of the space $LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi}(\mathbb{R}^n)$, we always assume that $\varphi \in \Omega^{\alpha,V}_{\vec{p},\mathrm{loc}}$.

Lemma 2.8. [33] *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le \vec{p} < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$.*

(*i*) *If*

$$\sup_{t<r<\infty}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{-\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)}=\infty,\quad \text{for some } t>0 \text{ and for all } x\in\mathbb{R}^n, \tag{2.5}$$

then $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)=\Theta$.

(*ii*) *If*

$$\sup_{0<r<\tau}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}=\infty,\quad \text{for some } \tau>0 \text{ and for all } x\in\mathbb{R}^n, \tag{2.6}$$

then $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)=\Theta$.

Remark 2.2. We denote by $\Omega^{\alpha,V}_{\vec{p}}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n}\left\|\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{-\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)}\right\|_{L_\infty(t,\infty)}<\infty,\quad\text{and}\quad \sup_{x\in\mathbb{R}^n}\left\|\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}\right\|_{L_\infty(0,t)}<\infty,$$

respectively. In what follows, keeping in mind Lemma 2.8, for the non-triviality of the space $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$, we always assume that $\varphi \in \Omega^{\alpha,V}_{\vec{p}}$.

Remark 2.3. We denote by $\Omega^{\alpha,V}_{\vec{p},1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n}\inf_{r>\delta}\left(1+\frac{r}{\rho(x)}\right)^{-\alpha}\varphi(x,r)>0,\quad \text{for some } \delta>0, \tag{2.7}$$

and

$$\lim_{r\to 0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)}=0.$$

For the non-triviality of the space $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$, we always assume that $\varphi\in\Omega^{\alpha,V}_{\vec{p},1}$.

3 Proof of Theorem 1.2

We state some properties, see, for example, [25].

Lemma 3.1. [25] *Let* $0<\nu<1$, $0<\beta+\nu<n$, *and* $b\in\Lambda^{\theta}_{\nu}(\rho)$; *then, the following pointwise estimate holds*:

$$|[b,\mathcal{I}^L_{\beta}]f(x)|\lesssim[b]^{\theta}_{\nu}I_{\beta+\nu}(|f|)(x).$$

From Lemma 3.1, we obtain the following.

Corollary 3.1. *Suppose* $V\in RH_{q_1}$ *with* $q_1>n/2$ *and* $b\in\Lambda^{\theta}_{\nu}(\rho)$ *with* $0<\nu<1$. *Let* $0<\beta+\nu<n$, *and let* $1<\vec{p}<\vec{q}<\infty$ *satisfy* $\sum_{i=1}^{n}1/p_i-\sum_{i=1}^{n}1/q_i=\beta+\nu$. *Then, for all* f *in* $L_{\vec{p}}(\mathbb{R}^n)$, *we have*

$$\|[b,\mathcal{I}^L_{\beta}]f\|_{L_{\vec{q}}(\mathbb{R}^n)}\lesssim\|f\|_{L_{\vec{p}}(\mathbb{R}^n)}.$$

In order to prove Theorem 1.2, we need the following.

Theorem 3.1. *Suppose* $V\in RH_{q_1}$ *with* $q_1>n/2$, $b\in\Lambda^{\theta}_{\nu}(\rho)$, $\theta>0$, $0<\nu<1$. *Let* $0<\beta+\nu<n$ *and let* $1<\vec{p}<\vec{q}<\infty$ *satisfy* $\sum_{i=1}^{n}1/p_i-\sum_{i=1}^{n}1/q_i=\beta+\nu$, *then, the inequality*

$$\|[b,\mathcal{I}^L_{\beta}]f\|_{L_{\vec{q}}(B(x_0,r))}\lesssim\|I_{\beta+\nu}(|f|)\|_{L_{\vec{q}}(B(x_0,r))}\lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}}\int_{2r}^{\infty}\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}$$

holds for any $f\in L^{\mathrm{loc}}_{\vec{p}}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0\in\mathbb{R}^n$, set $B=B(x_0,r)$ and $\lambda B=B(x_0,\lambda r)$ for any $\lambda>0$. We write f as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then,

$$\|[b,\mathcal{I}^L_{\beta}]f\|_{L_{\vec{q}}(B(x_0,r))}\lesssim\|I_{\beta+\nu}(|f|)\|_{L_{\vec{q}}(B(x_0,r))}\leq\|I_{\beta+\nu}f_1\|_{L_{\vec{q}}(B(x_0,r))}+\|I_{\beta+\nu}f_2\|_{L_{\vec{q}}(B(x_0,r))}.$$

Since $f_1\in L_{\vec{p}}(\mathbb{R}^n)$ and from the boundedness of $I_{\beta+\nu}$ from $L_{\vec{p}}(\mathbb{R}^n)$ to $L_{\vec{q}}(\mathbb{R}^n)$ (see [35]), it follows that

$$\|I_{\beta+\nu}f_1\|_{L_{\vec{q}}(B(x_0,r))}\lesssim\|f\|_{L_{\vec{p}}(B(x_0,2r))}\lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}}\|f\|_{L_{\vec{p}}(B(x_0,2r))}\int_{2r}^{\infty}\frac{dt}{t^{\sum_{i=1}^{n}\frac{1}{q_i}+1}}\lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}}\int_{2r}^{\infty}\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}. \tag{3.1}$$

To estimate $\|I_{\beta+\nu}f_2\|_{L_{\vec{p}}(B(x_0,r))}$, observe that $x\in B$, $y\in(2B)^c$ implies $|x-y|\approx|x_0-y|$. Then, by (2.3), we have

$$\sup_{x\in B}|I_{\beta+\nu}f_2(x)|\lesssim\int_{(2B)^c}\frac{|f(y)|}{|x_0-y|^{n-\beta-\nu}}dy\lesssim\sum_{k=1}^{\infty}(2^{k+1}r)^{-n+\beta}\int_{2^{k+1}B}|f(y)|dy.$$

By Hölder's inequality, we obtain

$$\begin{aligned}\sup_{x\in B}|I_{\beta+\nu}f_2(x)|&\lesssim\sum_{k=1}^{\infty}\|f\|_{L_{\vec{p}}(2^{k+1}B)}(2^{k+1}r)^{-1-\sum_{i=1}^{n}\frac{1}{p_i}+\beta}\int_{2^kr}^{2^{k+1}r}dt\\&\lesssim\sum_{k=1}^{\infty}\int_{2^kr}^{2^{k+1}r}\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}\lesssim\int_{2r}^{\infty}\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}.\end{aligned} \tag{3.2}$$

Then,

$$\|I_{\beta+v}f_2\|_{L_{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{\mathrm{d}t}{t} \tag{3.3}$$

holds for $1 < \vec{p} < n/\beta$. Therefore, by (3.1) and (3.3), we obtain

$$\|I_{\beta+v}(|f|)\|_{L_{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{\mathrm{d}t}{t} \tag{3.4}$$

holds for $1 < \vec{p} < n/\beta$. □

Proof of Theorem 1.2. From Lemma 2.6, we have

$$\frac{1}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}}.$$

Note the fact that $\|f\|_{L_{\vec{p}}(B(x_0,t))}$ is a nondecreasing function of t, and $f \in LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}$, then

$$\begin{aligned}
\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_{\vec{p}}(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}} &\lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_{\vec{p}}(B(x_0,t))}}{\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \\
&\lesssim \sup_{0<s<\infty} \frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_{\vec{p}}(B(x_0,s))}}{\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}}.
\end{aligned}$$

Since $\alpha \geq 0$, and (φ_1, φ_2) satisfy condition (1.3),

$$\begin{aligned}
\int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{\mathrm{d}t}{t} &= \int_{2r}^{\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_{\vec{p}}(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}} \int_{2r}^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^{n}\frac{1}{p_i}}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r).
\end{aligned} \tag{3.5}$$

Then, by Theorem 3.1, we obtain

$$\begin{aligned}
\|[b,\mathcal{I}^L_\beta]f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{q},\varphi_2}} &\lesssim \|I_{\beta+v}(|f|)\|_{LM^{\alpha,V,\{x_0\}}_{\vec{q},\varphi_2}} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|I_{\beta+v}(|f|)\|_{L_{\vec{p}}(B(x_0,r))} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{\vec{p},\varphi_1}}.
\end{aligned}$$

□

4 Proof of Theorem 1.3

The statement is derived from estimate (3.4). The estimation of the norm of the operator, i.e., the boundedness in the non-vanishing space, immediately follows from by Theorem 1.2. So we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{\vec{p},\varphi_1}(f;\,x,r)=0\Rightarrow\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{q,\varphi_2}([b,\mathcal{I}^L_\beta]f;\,x,r)=0. \tag{4.1}$$

To show that $\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^\alpha\varphi_2(x,r)^{-1}r^{-\sum_{i=1}^n\frac{1}{p_i}}\|[b,\mathcal{I}^L_\beta]f\|_{L_q(B(x,r))}<\varepsilon$ for small r, we split the right-hand side of (3.4):

$$\left(1+\frac{r}{\rho(x)}\right)^\alpha\varphi_2(x,r)^{-1}r^{-\sum_{i=1}^n\frac{1}{p_i}}\|[b,\mathcal{I}^L_\beta]f\|_{L_{\vec{q}}(B(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{4.2}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(x,r):=\frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_r^{\delta_0}t^{-\sum_{i=1}^n\frac{1}{q_i}-1}\|f\|_{L_{\vec{p}}(B(x,t))}\mathrm{d}t$$

and

$$J_{\delta_0}(x,r):=\frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_{\delta_0}^{\infty}t^{-\sum_{i=1}^n\frac{1}{q_i}-1}\|f\|_{L_{\vec{p}}(B(x,t))}\mathrm{d}t,$$

and it is supposed that $r<\delta_0$. We use the fact that $f\in VM^{\alpha,V}_{\vec{p},\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^\alpha\varphi_1(x,t)^{-1}t^{-\sum_{i=1}^n\frac{1}{p_i}}\|f\|_{L_{\vec{p}}(B(x,t))}<\frac{\varepsilon}{2CC_0},$$

where C and C_0 are the constants from (1.5) and (4.2). This allows us to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\quad 0<r<\delta_0.$$

The estimation of the second term now may be made already by the choice of r sufficiently small. Indeed, thanks to condition (2.7), we have

$$J_{\delta_0}(x,r)\le c_{\sigma_0}\frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_1(x,r)}\|f\|_{VM^{\alpha,V}_{\vec{p},\varphi_1}},$$

where c_{σ_0} is the constant from (1.2). Then, by (2.7), it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n}\frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\le\frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{\vec{p},\varphi_1}}},$$

which completes the proof of (4.1).

5 Conclusions

In this article, we study the boundedness of the fractional integral operator $\mathcal{I}_\beta^L$ associated with Schrödinger operator and its commutators $[b, \mathcal{I}_\beta^L]$ with $b \in \Lambda_\nu^\theta(\rho)$ on local generalized mixed Morrey spaces $LM_{\vec{p},\varphi}^{\alpha,V,\{x_0\}}$ associated with Schrödinger operator, generalized mixed Morrey spaces $M_{\vec{p},\varphi}^{\alpha,V}$ associated with Schrödinger operator, and vanishing generalized mixed Morrey spaces $VM_{\vec{p},\varphi}^{\alpha,V}$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1, φ_2), which ensures the boundedness of the operator $\mathcal{I}_\beta^L$ from $LM_{\vec{p},\varphi_1}^{\alpha,V,\{x_0\}}$ to $LM_{\vec{q},\varphi_2}^{\alpha,V,\{x_0\}}$, from $M_{\vec{p},\varphi_1}^{\alpha,V}$ to $M_{\vec{q},\varphi_2}^{\alpha,V}$, and from $VM_{\vec{p},\varphi_1}^{\alpha,V}$ to $VM_{\vec{q},\varphi_2}^{\alpha,V}$, $\sum_{i=1}^n \frac{1}{q_i} = \sum_{i=1}^n \frac{1}{p_i} - \beta$. When b belongs to $\mathrm{BMO}_\theta(\rho)$ and (φ_1, φ_2) satisfies some conditions, we also show that the commutator operator $[b, \mathcal{I}_\beta^L]$ is bounded from $LM_{\vec{p},\varphi_1}^{\alpha,V,\{x_0\}}$ to $LM_{\vec{q},\varphi_2}^{\alpha,V,\{x_0\}}$, from $M_{\vec{p},\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$, and from $VM_{\vec{p},\varphi_1}^{\alpha,V}$ to $VM_{\vec{q},\varphi_2}^{\alpha,V}$, $\sum_{i=1}^n \frac{1}{q_i} = \sum_{i=1}^n \frac{1}{p_i} - \beta$.

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