

Boundedness of the Riesz Potential in Local Morrey-Type Spaces

Victor I. Burenkov · Amiran Gogatishvili ·
Vagif S. Guliyev · Rza Ch. Mustafayev

Received: 24 August 2009 / Accepted: 8 September 2010 / Published online: 28 September 2010
© Springer Science+Business Media B.V. 2010

Abstract The problem of boundedness of the Riesz potential in local Morrey-type spaces is reduced to the problem of boundedness of the Hardy operator in weighted L_p -spaces on the cone of non-negative non-increasing functions. This allows obtaining sharp sufficient conditions for boundedness for all admissible values

The research of V. Burenkov was partially supported by the grants of the RFBR (project 09-01-00093a) and DGF-RFBR (project 10-01-91331). The research of A. Gogatishvili was partially supported by the grants no. 201/05/2033 and 201/08/0383 of the Grant Agency of the Czech Republic and by the Institutional Research Plan no. AV0Z10190503 of AS CR. The research of V. Guliyev was partially supported by the grant of 2010-Ahi Evran University Scientific Research Projects (BAP FBA-10-05). The research of V. Guliyev and R. Mustafayev was partially supported by the grant of BGP II (project ANSF Award/AZM1-3110-BA-08). The research of R. Mustafayev was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan no. AV0Z10190503 and by a Post Doctoral Fellowship of INTAS (Grant 06-1000015-6385).

V. I. Burenkov
Dipartimento di matematica pura ed applicata, Padova University, 63 via Trieste,
Padova, 35121, Italy
e-mail: burenkov@math.unipd.it

A. Gogatishvili
Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitna 25, 115 67
Prague 1, Czech Republic
e-mail: gogatish@math.cas.cz

V. S. Guliyev
Department of Mathematics, Ahi Evran University, Kirsehir, Turkey

V. S. Guliyev (✉) · R. Ch. Mustafayev
Institute of Mathematics and Mechanics, Academy of Sciences of Azerbaijan, F. Agayev St. 9,
Baku, AZ 1141, Azerbaijan
e-mail: vagif@guliyev.com

R. Ch. Mustafayev
e-mail: rzamustafayev@mail.az

of the parameters, which, for a certain range of the parameters wider than known before, coincide with the necessary ones.

Keywords Riesz potential · Local and global Morrey-type spaces · Hardy operator on the cone of monotonic functions

Mathematics Subject Classifications (2010) Primary 42B20 · 42B25 · 42B35

1 Introduction

If E is a nonempty measurable subset on \mathbb{R}^n and f is a measurable function on E , then we put

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p dy \right)^{\frac{1}{p}}, \quad 0 < p < +\infty,$$

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \geq \alpha\}| > 0\}.$$

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x of radius r and ${}^c B(x, r)$ denote the set $\mathbb{R}^n \setminus B(x, r)$.

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The Riesz potential I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

The operator I_α plays an important role in real and harmonic analysis (see, for example [22, 23]).

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ are widely used. They were introduced by C. Morrey in 1938 [20] and defined as follows: For $0 \leq \lambda \leq n$, $1 \leq p \leq \infty$, $f \in \mathcal{M}_{p,\lambda}$ if $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))} < \infty.$$

(If $\lambda = 0$, then $\mathcal{M}_{p,0} = L_p(\mathbb{R}^n)$; if $\lambda = n$, then $\mathcal{M}_{p,n} = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on \mathbb{R}^n .)

These spaces appeared to be quite useful in the study of a number of problems in the theory of partial differential equations, in particular in the study of local behavior of solutions to elliptic and parabolic differential equations.

Also by $W\mathcal{M}_{p,\lambda}$ we denote the weak Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\lambda/p} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x, r))$ denotes the weak L_p -space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t |\{y \in B(x, r) : |f(y)| > t\}|^{1/p} \\ &= \sup_{0<t \leq |B(x,r)|} t^{1/p} (f\chi_{B(x,r)})^*(t) < \infty, \end{aligned}$$

where g^* denotes the non-increasing rearrangement of g .

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p_1 < p_2 < \infty$, then I_α is bounded from $L_{p_1}(\mathbb{R}^n)$ to $L_{p_2}(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ and for $p_1 = 1 < p_2 < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_{p_2}(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{p_2}\right)$. Adams [1] studied boundedness of the Riesz potential in Morrey spaces and obtained the following statement.

Theorem 1.1

- (1) Let $1 < p_1 < p_2 < \infty$ and $0 < \alpha < n$. Then I_α is bounded from $\mathcal{M}_{p_1,\lambda}$ to $\mathcal{M}_{p_2,\lambda}$ if and only if

$$\alpha \leq n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ and } \lambda = \left(n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \alpha\right)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)^{-1}.$$

- (2) Let $1 < p_2 < \infty$ and $0 < \alpha < n$. Then I_α is bounded from $\mathcal{M}_{1,\lambda}$ to $WM_{p_2,\lambda}$ if and only if

$$\alpha \leq n\left(1 - \frac{1}{p_2}\right) \text{ and } \lambda = \left(n\left(1 - \frac{1}{p_2}\right) - \alpha\right)\left(1 - \frac{1}{p_2}\right)^{-1}.$$

If $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then $\lambda = 0$ and the statement of Theorem 1.1 reduces to the aforementioned result by Hardy-Littlewood-Sobolev.

If in the place of the power function $r^{-\lambda/p}$ in the definitions of $\mathcal{M}_{p,\lambda}$ and $WM_{p,\lambda}$ we consider any positive measurable function w defined on $(0, \infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,w}$, $WM_{p,w}$ respectively. Mizuhara [19], Nakai [21] and Guliyev [15] generalized Theorem 1.1 and obtained sufficient conditions on functions w_1 and w_2 ensuring boundedness of the Riesz potential I_α for the limiting case $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ from \mathcal{M}_{p_1,w_1} to \mathcal{M}_{p_2,w_2} .

The following statement, containing weaker results obtained earlier in [19, 21], was proved in [15] (see also [16–18]).

Theorem 1.2 Let $1 \leq p_1 \leq p_2 < \infty$ and $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Moreover, let w_1, w_2 be positive measurable functions satisfying the following condition:

$$\sup_{t>0} w_2(t)t^{\frac{n}{p_2}} \int_t^\infty \frac{s^{-\frac{n}{p_2}-1}}{w_1(s)} ds < \infty. \tag{1.1}$$

Then for $p_1 > 1$ I_α is bounded from \mathcal{M}_{p_1,w_1} to \mathcal{M}_{p_2,w_2} and for $p_1 = 1$ I_α is bounded from \mathcal{M}_{1,w_1} to WM_{p_2,w_2} .

In [2–10] and [15–18] boundedness of maximal operator, fractional maximal operator, Riesz potential and singular integral operators from one local Morrey-type space $LM_{p_1\theta_1, w_1}$ to another one $LM_{p_2\theta_2, w_2}$ has been investigated and, in particular, in [6, 9] for a certain range of the parameters necessary and sufficient conditions for the operator I_α to be bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$ were obtained. (The definition and basic properties of these spaces are given in Section 2. In particular it is noted there that local Morrey-type spaces are non-trivial only if w_1, w_2 belong to classes $\Omega_{\theta_1}, \Omega_{\theta_2}$ respectively, defined in that section.)

Theorem 1.3

1. If $1 \leq p_1 \leq \infty, 0 < p_2 \leq \infty, 0 < \alpha < \frac{n}{p_1}, 0 < \theta_1, \theta_2 \leq \infty, w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition

$$t^{\alpha-n/p_1+\min\{n-\alpha, n/p_2\}} \left\| w_2(r) \frac{r^{n/p_2}}{(t+r)^{\min\{n-\alpha, n/p_2\}}} \right\|_{L_{\theta_2}(0, \infty)} \leq c \|w_1\|_{L_{\theta_1}(t, \infty)} \tag{1.2}$$

for all $t > 0$, where $c > 0$ is independent of t , is necessary for boundedness of I_α from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.

2. If the condition

$$1 < p_1 < p_2 < \infty \text{ and } \alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$$

or the condition¹

$$1 \leq p_1 < \infty, 0 < p_2 < \infty \text{ and } n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)_+ < \alpha < \frac{n}{p_1}$$

is satisfied, $0 < \theta_1 \leq \theta_2 \leq \infty, \theta_1 \leq 1, w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition

$$\left\| w_2(r) \frac{r^{n/p_2}}{(t+r)^{n/p_1-\alpha}} \right\|_{L_{\theta_2}(0, \infty)} \leq c \|w_1\|_{L_{\theta_1}(t, \infty)} \tag{1.3}$$

for all $t > 0$, where $c > 0$ is independent of t , is sufficient for boundedness of I_α from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.

3. In particular, if $1 < p_1 < p_2 < \infty, \alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right), 0 < \theta_1 \leq \theta_2 \leq \infty, \theta_1 \leq 1, w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then the condition

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0, \infty)} \leq c \|w_1\|_{L_{\theta_1}(t, \infty)} \tag{1.4}$$

for all $t > 0$, where $c > 0$ is independent of t , is necessary and sufficient for boundedness of I_α from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.

The proof in [9] was based on a certain estimate for L_p -quasinorms of $I_\alpha f$ over balls $B(x, r)$, which allowed reducing the problem of boundedness of I_α in local Morrey-type spaces to the problem of boundedness of the Hardy operator on the cone of non-negative non-increasing functions.

¹ Here and in the sequel $t_+ = t$ if $t \geq 0$ and $t_+ = 0$ if $t < 0$.

In this paper we improve the estimate for L_p -quasinorms of $I_\alpha f$ over balls $B(x, r)$ mentioned above and we give an equivalent description of these quasinorms in terms not involving the operator I_α (Theorem 3.9). This allows obtaining sharp sufficient conditions for boundedness for all admissible values of the parameters; if $p_1 = 1$ then these sufficient conditions coincide with necessary ones (Theorem 5.1). Moreover, we prove that Statement 3 of Theorem 1.3 holds without the assumption $\theta_1 \leq 1$ (Theorem 5.5). In these theorems the case of weak local Morrey-type spaces is also considered.

A preliminary version of this paper was published as preprint [7].

2 Definitions and Basic Properties of Morrey-Type Spaces

Definition 2.1 Let $0 < p, \theta \leq \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p\theta, w}$, $GM_{p\theta, w}$ respectively, the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p\theta, w}} \equiv \|f\|_{LM_{p\theta, w}(\mathbb{R}^n)} = \|w(r)\|_{L_\theta(0, \infty)} \|f\|_{L_p(B(0, r))},$$

$$\|f\|_{GM_{p\theta, w}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM_{p\theta, w}}$$

respectively.

Note that

$$\|f\|_{LM_{p\infty, 1}} = \|f\|_{GM_{p\infty, 1}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty, r^{-\lambda/p}} \equiv \mathcal{M}_{p, \lambda}$, $0 \leq \lambda \leq n$, and $GM_{p\infty, w} \equiv \mathcal{M}_{p, w}$.

Definition 2.2 Let $0 < p, \theta \leq \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w\|_{L_\theta(t, \infty)} < \infty.$$

Moreover, we denote by $\Omega_{p, \theta}$ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t_1, t_2 > 0$

$$\|w(r)\|_{L_\theta(t_1, \infty)} < \infty, \quad \|w(r)r^{n/p}\|_{L_\theta(0, t_2)} < \infty.$$

In [3] it was proved that if $\|w\|_{L_\theta(t, \infty)} = \infty$ for all $t > 0$, then $GM_{p\theta, w} = LM_{p\theta, w} = \Theta$ and if $\|w(r)r^{n/p}\|_{L_\theta(0, t_2)} = \infty$ for all $t > 0$, then $GM_{p\theta, w} = \Theta$. For this reason when considering spaces $LM_{p\theta, w}$ we always assume that $w \in \Omega_\theta$ and when considering spaces $GM_{p\theta, w}$ we always assume that $w \in \Omega_{p, \theta}$.

In [6] the following statements were proved.

Lemma 2.3 *Let $1 < p_1 \leq \infty, 0 < p_2 \leq \infty, 0 < \alpha < n, 0 < \theta_1, \theta_2 \leq \infty, w_1 \in \Omega_{\theta_1},$ and $w_2 \in \Omega_{\theta_2}.$ Then the condition*

$$\alpha < \frac{n}{p_1}$$

is necessary for boundedness of I_α from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}.$

Throughout this paper $a \lesssim b$ ($b \gtrsim a$) means that $a \leq \lambda b$, where $\lambda > 0$ depends on unessential parameters. If $b \lesssim a \lesssim b$, then we write $a \approx b$.

3 L_p -Estimates of the Riesz Potential over Balls

We consider the following ‘‘partial’’ Riesz potentials

$$\underline{I}_{\alpha, r} f(x) \equiv I_\alpha(f\chi_{B(x, r)})(x) = \int_{B(x, r)} \frac{|f(y)|}{|x - y|^{n-\alpha}} dy,$$

$$\bar{I}_{\alpha, r} f(x) \equiv I_\alpha(f\chi_{\mathfrak{c}_{B(x, r)}})(x) = \int_{\mathfrak{c}_{B(x, r)}} \frac{|f(y)|}{|x - y|^{n-\alpha}} dy.$$

Lemma 3.1 *Let $0 < p < \infty, 0 < \alpha < n$ and $f \in L_1^{\text{loc}}(\mathbb{R}^n).$ Then for any ball $B(x, r)$ in \mathbb{R}^n*

$$\|I_\alpha(|f|)\|_{W_{L_p}(B(x, r))} \gtrsim r^{\frac{n}{p}} \bar{I}_{\alpha, r}(|f|)(x).$$

Proof If $y \in B(x, r)$ and $z \in \mathfrak{c}_{B(x, r)}$, then $|y - z| \leq 2|x - z|$ and

$$\begin{aligned} I_\alpha(|f|)(y) &\geq \int_{\mathfrak{c}_{B(x, r)}} \frac{|f(z)|}{|y - z|^{n-\alpha}} dz \\ &\geq 2^{\alpha-n} \int_{\mathfrak{c}_{B(x, r)}} \frac{|f(z)|}{|x - z|^{n-\alpha}} dz = 2^{\alpha-n} \bar{I}_{\alpha, r}(|f|)(x). \end{aligned}$$

Hence²

$$\|I_\alpha(|f|)\|_{W_{L_p}(B(x, r))} \geq (v_n r^n)^{\frac{1}{p}} 2^{\alpha-n} \bar{I}_{\alpha, r}(|f|)(x),$$

where v_n is the volume of the unit ball in \mathbb{R}^n . □

Lemma 3.2 *Let $0 < p < \infty, 0 < \alpha < n$ and $f \in L_1^{\text{loc}}(\mathbb{R}^n).$ Then for any ball $B(x, r)$ in \mathbb{R}^n*

$$\|I_\alpha(|f|)\|_{L_p(B(x, r))} \approx \|I_\alpha(|f|\chi_{B(x, 2r)})\|_{L_p(B(x, r))} + r^{\frac{n}{p}} \bar{I}_{\alpha, 2r}(|f|)(x) \tag{3.1}$$

and

$$\|I_\alpha(|f|)\|_{W_{L_p}(B(x, r))} \approx \|I_\alpha(|f|\chi_{B(x, 2r)})\|_{W_{L_p}(B(x, r))} + r^{\frac{n}{p}} \bar{I}_{\alpha, 2r}(|f|)(x). \tag{3.2}$$

²We apply the following simple statement. If Ω is a measurable set in $\mathbb{R}^n, M > 0$ and for almost all $y \in \Omega$ $g(y) \geq M$, then for any $0 < p \leq \infty$ $\|g\|_{W_{L_p}(\Omega)} \geq M|\Omega|^{\frac{1}{p}}.$

Proof Clearly

$$\|I_\alpha(|f|)\|_{L_p(B(x,r))} \lesssim \|I_\alpha(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))} + \|I_\alpha(|f|\chi_{\mathbb{C}_{B(x,2r)}})\|_{L_p(B(x,r))}$$

and

$$\|I_\alpha(|f|)\|_{WL_p(B(x,r))} \lesssim \|I_\alpha(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))} + \|I_\alpha(|f|\chi_{\mathbb{C}_{B(x,2r)}})\|_{WL_p(B(x,r))}.$$

If $y \in B(x, r)$, $z \in \mathbb{C}B(x, 2r)$, then $|x - z|/2 \leq |y - z| \leq 3|x - z|/2$. Therefore

$$\begin{aligned} \|I_\alpha(|f|\chi_{\mathbb{C}_{B(x,2r)}})\|_{WL_p(B(x,r))} &\leq \|I_\alpha(|f|\chi_{\mathbb{C}_{B(x,2r)}})\|_{L_p(B(x,r))} \\ &= \left(\int_{B(x,r)} \left(\int_{\mathbb{C}B(x,2r)} \frac{f(z)}{|y-z|^{n-\alpha}} dz \right)^p dy \right)^{\frac{1}{p}} \\ &\approx r^{\frac{n}{p}} \int_{\mathbb{C}B(x,2r)} \frac{|f(z)|}{|x-z|^{n-\alpha}} dz \\ &= r^{\frac{n}{p}} \bar{I}_{\alpha,2r}(|f|)(x) \end{aligned}$$

and the right-hand side inequalities in Eqs. 3.1 and 3.2 follow.

The left-hand side inequalities in Eqs. 3.1 and 3.2 follow by Lemma 3.1 and obvious inequalities

$$\|I_\alpha(|f|)\|_{L_p(B(x,r))} \geq \|I_\alpha(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))}$$

and

$$\|I_\alpha(|f|)\|_{WL_p(B(x,r))} \geq \|I_\alpha(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))}.$$

□

Lemma 3.3 *Let $1 \leq p_1 < p_2 < \infty$ and $0 < \alpha < n$. The inequality*

$$\|I_\alpha(f\chi_{B(x,2r)})\|_{L_{p_2}(B(x,r))} \lesssim r^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(x,2r))} \tag{3.3}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$ if and only if in the case $p_1 > 1$

$$\alpha \geq n \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \tag{3.4}$$

and in the case $p_1 = 1$

$$\alpha > n \left(1 - \frac{1}{p_2} \right).$$

Moreover for $1 < p_2 < \infty$ and $\alpha = n \left(1 - \frac{1}{p_2} \right)$ the inequality

$$\|I_\alpha(f\chi_{B(x,2r)})\|_{WL_{p_2}(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))} \tag{3.5}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof Recall the well-known inequalities for the Riesz potential [22]. If $1 < p < q < \infty$, then

$$\|I_n(\frac{1}{p} - \frac{1}{q})f\|_{L_q(\mathbb{R}^n)} \lesssim \|f\|_{L_p(\mathbb{R}^n)}. \tag{3.6}$$

Also if $1 < q < \infty$, then

$$\|I_n(1 - \frac{1}{q})f\|_{W L_q(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}. \tag{3.7}$$

If $1 < p_1 < p_2 < \infty$, inequality (3.4) holds and $z \in B(x, r)$, then

$$I_\alpha(|f|\chi_{B(x,2r)})(z) \lesssim r^{\alpha-n(\frac{1}{p_1} - \frac{1}{p_2})} I_n(\frac{1}{p_1} - \frac{1}{p_2})(|f|\chi_{B(x,2r)})(z)$$

and by Eq. 3.6

$$\|I_\alpha(|f|\chi_{B(x,2r)})\|_{L_{p_2}(B(x,r))} \lesssim r^{\alpha-n(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{L_{p_1}(B(x,2r))}.$$

If $1 < p_2 < \infty$ and inequality (3.5) holds then by Eq. 3.7

$$\begin{aligned} \|I_\alpha(|f|\chi_{B(x,2r)})\|_{L_{p_2}(B(x,r))} &\leq \| (I_\alpha(|f|\chi_{B(x,2r)}))^* \|_{L_{p_2}(0,|B(x,r)|)} \\ &\leq \sup_{0 < t \leq |B(x,r)|} t^{1-\frac{\alpha}{n}} (I_\alpha(|f|\chi_{B(x,2r)}))^*(t) \|t^{\frac{\alpha}{n}-1}\|_{L_{p_2}(0,|B(x,r)|)} \\ &\approx r^{\alpha-n(1-\frac{1}{p_2})} \|I_\alpha(|f|\chi_{B(x,2r)})\|_{W L_{\frac{n}{n-\alpha}}(B(x,r))} \\ &\lesssim r^{\alpha-n(1-\frac{1}{p_2})} \|f\|_{L_1(B(x,2r))}. \end{aligned}$$

If $p_1 \geq 1$ and $\alpha < n(\frac{1}{p_1} - \frac{1}{p_2})$, then inequality (3.3) cannot hold for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$. Indeed if $f \in L_{p_1}(\mathbb{R}^n)$ and $f \approx 0$ then by passing in Eq. 3.3 to the limit as $r \rightarrow +\infty$ we arrive at a contradiction.

Assume that $p_1 = 1$, $1 < p_2 < \infty$, $\alpha = n(1 - \frac{1}{p_2})$ and $f \in L_1(\mathbb{R}^n)$. Then by passing to the limit in Eq. 3.3 as $r \rightarrow +\infty$ we get

$$\|I_\alpha f\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}$$

which, according to known results [22], is not possible. □

Corollary 3.4 *Let*

$$1 < p_1 \leq \infty, 0 < p_2 \leq \infty \text{ or } p_1 = 1, 0 < p_2 < \infty, \text{ and } n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ < \alpha < n, \tag{3.8}$$

or

$$1 < p_1 < p_2 < \infty \text{ and } \alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right). \tag{3.9}$$

Then the inequality

$$\|I_\alpha(f\chi_{B(x,2r)})\|_{L_{p_2}(B(x,r))} \lesssim r^{\alpha-n(\frac{1}{p_1} - \frac{1}{p_2})} \|f\|_{L_{p_1}(B(x,2r))}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$.

Moreover for $1 < p_2 < \infty$ and $\alpha = n \left(1 - \frac{1}{p_2}\right)$, then the inequality

$$\|I_\alpha(f\chi_{B(x,2r)})\|_{WL_{p_2}(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof If $p_2 > p_1$, the statement follows by Lemma 3.3.

If $p_2 = p_1$, then by applying Minkowski’s inequality for integrals we have

$$\begin{aligned} \|I_\alpha(f\chi_{B(x,2r)})\|_{L_{p_1}(B(x,r))} &\leq \left\| \int_{B(x,2r)} \frac{|(f\chi_{B(x,2r)})(y)|}{|\cdot - y|^{n-\alpha}} dy \right\|_{L_{p_1}(B(x,r))} \\ &\leq \left\| \int_{B(0,3r)} \frac{|(f\chi_{B(x,2r)})(\cdot - u)|}{|u|^{n-\alpha}} du \right\|_{L_{p_1}(\mathbb{R}^n)} \\ &\leq \int_{B(0,3r)} \frac{du}{|u|^{n-\alpha}} \|f\chi_{B(x,2r)}\|_{L_{p_1}(\mathbb{R}^n)} \lesssim r^\alpha \|f\|_{L_{p_1}(B(x,2r))}. \end{aligned}$$

If $p_2 < p_1$, then by applying Hölder’s inequality and this inequality we get

$$\begin{aligned} \|I_\alpha(f\chi_{B(x,2r)})\|_{L_{p_2}(B(x,r))} &\lesssim r^{\frac{n}{p_2} - \frac{n}{p_1}} \|I_\alpha(f\chi_{B(x,2r)})\|_{L_{p_1}(B(x,r))} \\ &\lesssim r^{\alpha - n \left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(x,2r))}. \end{aligned}$$

□

Lemma 3.2 and Corollary 3.4 imply the following statement.

Lemma 3.5 *Let condition (3.8) or condition (3.9) be satisfied. Then the inequality*

$$\|I_\alpha f\|_{L_{p_2}(B(x,r))} \lesssim r^{\alpha - n \left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(x,2r))} + r^{\frac{n}{p_2}} \bar{I}_{\alpha,2r}(|f|)(x) \tag{3.10}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$.

Moreover, for $1 < p_2 < \infty$ and $\alpha = n \left(1 - \frac{1}{p_2}\right)$ the inequality

$$\|I_\alpha f\|_{WL_{p_2}(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))} + r^{\frac{n}{p_2}} \bar{I}_{\alpha,2r}(|f|)(x) \tag{3.11}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Lemma 3.6 *Let the condition (3.8) or condition (3.9) be satisfied. Then the inequality*

$$\|I_\alpha f\|_{L_{p_2}(B(x,r))} \lesssim r^{\frac{n}{p_2}} \int_r^\infty \|f\|_{L_{p_1}(B(x,t))} \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}} \tag{3.12}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$.

Proof Note that if $\alpha \geq \frac{n}{p_1}$ and f is not equivalent to 0 on \mathbb{R}^n , then right hand side of Eq. 3.12 is infinite, and in this case inequality (3.12) is trivial.

Let $\alpha < \frac{n}{p_1}$. By Lemma 6 in [5] and Hölder’s inequality

$$\begin{aligned} r^{\frac{n}{p_2}} \bar{I}_{\alpha, 2r}(|f|)(x) &= r^{\frac{n}{p_2}} \int_{B(x, 2r)} \frac{|f(y)|}{|x - y|^{n-\alpha}} dy \\ &= (n - \alpha) r^{\frac{n}{p_2}} \int_{2r}^{\infty} \left(\int_{2r \leq |x-y| \leq t} |f(y)| dy \right) \frac{dt}{t^{n-\alpha+1}} \\ &\leq (n - \alpha) r^{\frac{n}{p_2}} \int_{2r}^{\infty} \|f\|_{L_1(B(x, t))} \frac{dt}{t^{n-\alpha+1}} \\ &\lesssim r^{\frac{n}{p_2}} \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x, t))} \frac{dt}{t^{\frac{n}{p_1}-\alpha+1}}. \end{aligned}$$

In the other hand

$$\begin{aligned} r^{\alpha-n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(x, 2r))} &= \left(\frac{n}{p_1} - \alpha\right) 2^{\alpha-\frac{n}{p_1}} r^{\frac{n}{p_2}} \|f\|_{L_{p_1}(B(x, 2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p_1}-\alpha+1}} \\ &\lesssim r^{\frac{n}{p_2}} \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x, t))} \frac{dt}{t^{\frac{n}{p_1}-\alpha+1}}. \end{aligned}$$

Hence the statement of the lemma follows by inequalities (3.10) and (3.11). □

Remark 3.7 Note that inequality (37) in [6]

$$\|I_{\alpha} f\|_{L_{p_2}(B(x, r))} \lesssim r^{\frac{n}{p_2}-\delta} \left(\int_r^{\infty} \left(\int_{B(x, t)} |f(y)|^{p_1} dy \right) \frac{dt}{t^{n-(\alpha+\delta)p_1+1}} \right)^{\frac{1}{p_1}}$$

follows from the inequality (3.12) by applying Hölder’s inequality.

Indeed for any $\delta > 0$ by Eq. 3.12

$$\begin{aligned} \|I_{\alpha} f\|_{L_{p_2}(B(x, r))} &\lesssim r^{\frac{n}{p_2}} \int_r^{\infty} \left(\int_{B(x, t)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1}-(\alpha+\delta)+\frac{1}{p_1}+\delta+\frac{1}{p_1}}} \\ &\lesssim r^{\frac{n}{p_2}} \left(\int_r^{\infty} \left(\int_{B(x, t)} |f(y)|^{p_1} dy \right) \frac{dt}{t^{n-(\alpha+\delta)p_1+1}} \right)^{\frac{1}{p_1}} \left(\int_r^{\infty} \frac{dt}{t^{p_1'\delta+1}} \right)^{\frac{1}{p_1'}} \\ &\lesssim r^{\frac{n}{p_2}-\delta} \left(\int_r^{\infty} \left(\int_{B(x, t)} |f(y)|^{p_1} dy \right) \frac{dt}{t^{n-(\alpha+\delta)p_1+1}} \right)^{\frac{1}{p_1}}. \end{aligned}$$

Lemma 3.8 *Let $0 < p < \infty$, $0 < \alpha < n$. Then the inequality*

$$\begin{aligned} \|I_{\alpha}(|f|)\|_{W_{L_p}(B(x, r))} &\gtrsim r^{\frac{n}{p}} \int_r^{\infty} \|f\|_{L_1(B(x, t))} \frac{dt}{t^{n-\alpha+1}} \\ &\gtrsim r^{\alpha-n\left(1-\frac{1}{p}\right)} \|f\|_{L_1(B(x, 2r))} \end{aligned}$$

holds for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof For all $y \in B(x, r)$ $|y - z| \leq 2r$ if $z \in B(x, r)$ and $|y - z| \leq 2|x - z|$ if $z \in \complement B(x, r)$, therefore

$$\begin{aligned} I_\alpha(|f|)(y) &= \int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-\alpha}} dz + \int_{\complement B(x,r)} \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\ &\geq (2r)^{\alpha-n} \int_{B(x,r)} |f(z)| dz + 2^{\alpha-n} \int_{\complement B(x,r)} \frac{|f(z)|}{|x-z|^{n-\alpha}} dz \\ &= (n-\alpha)2^{\alpha-n} \int_r^\infty \left(\int_{B(x,r)} |f(z)| dz \right) \frac{dt}{t^{n-\alpha+1}} \\ &\quad + (n-\alpha)2^{\alpha-n} \int_{\complement B(x,r)} \left(\int_{|x-z|}^\infty \frac{dt}{t^{n-\alpha+1}} \right) |f(z)| dz \\ &= (n-\alpha)2^{\alpha-n} \left(\int_r^\infty \left(\int_{B(x,r)} |f(z)| dz \right) \frac{dt}{t^{n-\alpha+1}} \right. \\ &\quad \left. + \int_r^\infty \left(\int_{B(x,t) \setminus B(x,r)} |f(z)| dz \right) \frac{dt}{t^{n-\alpha+1}} \right) \\ &= (n-\alpha)2^{\alpha-n} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n-\alpha+1}}. \end{aligned}$$

Hence the first of the desired inequalities follows.³

The second one follows since

$$\begin{aligned} r^{\frac{n}{p}} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n-\alpha+1}} &\geq r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n-\alpha+1}} \\ &\gtrsim r^{\alpha-n(1-\frac{1}{p})} \|f\|_{L_1(B(x,2r))}. \end{aligned}$$

□

Theorem 3.9

1. Let $0 < p < \infty$ and $n \left(1 - \frac{1}{p}\right)_+ < \alpha < n$. Then the equivalences

$$\begin{aligned} \|I_\alpha(|f|)\|_{WL_p(B(x,r))} &\approx \|I_\alpha(|f|)\|_{L_p(B(x,r))} \\ &\approx r^{\frac{n}{p}} \bar{I}_{\alpha,r}(|f|)(x) + r^{\alpha-n(1-\frac{1}{p})} \|f\|_{L_1(B(x,2r))} \\ &\approx r^{\frac{n}{p}} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n-\alpha+1}} \end{aligned} \tag{3.13}$$

hold for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

³See footnote 2.

2. Let $1 < p < \infty$ and $\alpha = n \left(1 - \frac{1}{p}\right)$. Then the equivalences

$$\begin{aligned} \|I_{n(1-\frac{1}{p})}(|f|)\|_{WL_p(B(x,r))} &\approx r^{\frac{n}{p}} \bar{I}_{n(1-\frac{1}{p}),r}(|f|)(x) + \|f\|_{L_1(B(x,2r))} \\ &\approx r^{\frac{n}{p}} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{\frac{n}{p}+1}} \end{aligned} \tag{3.14}$$

hold for any ball $B(x, r) \subset \mathbb{R}^n$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

Proof The second equivalence in Eq. 3.13 for both $\|I_\alpha(|f|)\|_{WL_p(B(x,r))}$ and $\|I_\alpha(|f|)\|_{L_p(B(x,r))}$ and the first equivalence in Eq. 3.14 follow by Lemma 3.5 (estimate above) and Lemmas 3.1 and 3.8 (estimate below). The third equivalence in Eq. 3.13 for both $\|I_\alpha(|f|)\|_{WL_p(B(x,r))}$ and $\|I_\alpha(|f|)\|_{L_p(B(x,r))}$ and the second equivalence in Eq. 3.14 follow by Lemmas 3.6 and 3.8. \square

4 Riesz Potential and Hardy Operator

Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset consisting of all non-negative functions on $(0, \infty)$. We denote by $\mathfrak{M}^+(0, \infty; \downarrow)$ the cone of all functions in $\mathfrak{M}^+(0, \infty)$ which are non-increasing on $(0, \infty)$ and we set

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0, \infty; \downarrow) : \lim_{t \rightarrow \infty} \varphi(t) = 0 \right\}.$$

Let H be the Hardy operator

$$(Hg)(t) := \int_0^t g(r)dr, \quad 0 < t < \infty,$$

Lemma 4.1 *Let condition (3.8) or condition (3.9) be satisfied. Moreover, let $0 < \theta_2 \leq \infty$ and $w_2 \in \Omega_{\theta_2}$.*

Then

$$\|I_\alpha f\|_{LM_{p_2\theta_2,w_2}} \lesssim \|Hg_{p_1}\|_{L_{\theta_2,v_2}(0,\infty)} \tag{4.1}$$

for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$, where

$$g_{p_1}(t) = \left(\int_{B(0,t^{-\frac{1}{\sigma}})} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}}, \quad \sigma = \frac{n}{p_1} - \alpha > 0$$

and

$$v_2(r) = w_2(r^{-\frac{1}{\sigma}})r^{-\frac{n}{\sigma p_2} - \frac{1}{\theta_2 \sigma} - \frac{1}{\theta_2}}. \tag{4.2}$$

Moreover, if $p_1 = 1, 0 < p_2 < \infty$ and $n \left(1 - \frac{1}{p_2}\right)_+ < \alpha < n$, then

$$\|I_\alpha f\|_{WLM_{p_2\theta_2,w_2}} \approx \|I_\alpha f\|_{LM_{p_2\theta_2,w_2}} \approx \|Hg_1\|_{L_{\theta_2,v_2}(0,\infty)}$$

for all non-negative functions $f \in L_1^{loc}(\mathbb{R}^n)$.

Also if $1 < p_2 < \infty$ and $\alpha = n \left(1 - \frac{1}{p_2}\right)$, then

$$\|I_\alpha f\|_{WLM_{p_2\theta_2,w_2}} \approx \|Hg_1\|_{L_{\theta_2,v_2}(0,\infty)}$$

for all non-negative functions $f \in L_1^{loc}(\mathbb{R}^n)$.

Proof By Lemma 3.6 we have

$$\begin{aligned} \|I_\alpha f\|_{LM_{p_2\theta_2,w_2}} &\lesssim \left\| w_2(r)r^{\frac{n}{p_2}} \int_r^\infty \|f\|_{L_{p_1}(B(0,t))} \frac{dt}{t^{\sigma+1}} \right\|_{L_{\theta_2}(0,\infty)} \\ &\approx \left\| w_2(r)r^{\frac{n}{p_2}} \int_0^{r^{-\sigma}} \|f\|_{L_{p_1}(B(0,\tau^{-\frac{1}{\sigma}}))} d\tau \right\|_{L_{\theta_2}(0,\infty)} \\ &= \left\| w_2(r)r^{\frac{n}{p_2}} \int_0^{r^{-\sigma}} g_{p_1}(\tau) d\tau \right\|_{L_{\theta_2}(0,\infty)} \\ &= \left\| w_2(\rho^{-\frac{1}{\sigma}})\rho^{-\frac{n}{\sigma p_2} - \frac{1}{\theta_2\sigma} - \frac{1}{\theta_2}} (Hg_{p_1}(\rho)) \right\|_{L_{\theta_2}(0,\infty)} \\ &= \|Hg_{p_1}\|_{L_{\theta_2,v_2}(0,\infty)}. \end{aligned}$$

The second and third statement of the lemma follow by applying also Theorem 3.9. □

Theorem 4.2 *Let condition (3.8) or condition (3.9) be satisfied. Moreover, let $0 < \theta_1, \theta_2 \leq \infty, w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.*

Then I_α is bounded from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ if, and in the case $p_1 = 1, 0 < p_2 < \infty$ and $n \left(1 - \frac{1}{p_2}\right)_+ < \alpha < n$ only if, the operator H is bounded from $L_{\theta_1,v_1}(0, \infty)$ to $L_{\theta_2,v_2}(0, \infty)$ on the cone \mathbb{A} , that is

$$\|Hg\|_{L_{\theta_2,v_2}(0,\infty)} \lesssim \|g\|_{L_{\theta_1,v_1}(0,\infty)} \tag{4.3}$$

for all functions $g \in \mathbb{A}$, where

$$v_1(r) = w_1(r^{-\frac{1}{\sigma}})r^{-\frac{1}{\theta_1\sigma} - \frac{1}{\theta_1}} \tag{4.4}$$

and v_2 is defined by equality (4.2).

Moreover, if $p_1 = 1, 0 < p_2 < \infty$ and $n \left(1 - \frac{1}{p_2}\right)_+ < \alpha < n$ or $1 < p_2 < \infty$ and $\alpha = n \left(1 - \frac{1}{p_2}\right)$, then I_α is bounded from $LM_{1\theta_1,w_1}$ to $WLM_{p_2\theta_2,w_2}$ if and only if, the operator H is bounded from $L_{\theta_1,v_1}(0, \infty)$ to $L_{\theta_2,v_2}(0, \infty)$ on the cone \mathbb{A} .

Proof Assume that the operator H is bounded from $L_{\theta_1,v_1}(0, \infty)$ to $L_{\theta_2,v_2}(0, \infty)$ on the cone \mathbb{A} . Since $g_{p_1} \in \mathbb{A}$, by Lemma 4.1 we have

$$\|I_\alpha f\|_{LM_{p_2\theta_2,w_2}} \lesssim \|Hg_{p_1}\|_{L_{\theta_2,v_2}(0,\infty)} \lesssim \|g_{p_1}\|_{L_{\theta_1,v_1}(0,\infty)}.$$

Note that

$$\begin{aligned} \|g_{p_1}\|_{L_{\theta_1, v_1}(0, \infty)} &= \|v_1(t)\|_{L_{p_1}(B(0, t^{-\frac{1}{\sigma}}))} \|f\|_{L_{\theta_1, v_1}(0, \infty)} \\ &\approx \|v_1(\rho^{-\sigma})\rho^{-\frac{\sigma+1}{\theta_1}}\|_{L_{p_1}(B(0, \rho))} \|f\|_{L_{\theta_1, v_1}(0, \infty)} \\ &= \|w_1(\rho)\|_{L_{p_1}(B(0, \rho))} \|f\|_{L_{\theta_1, v_1}(0, \infty)} \\ &= \|f\|_{LM_{p_1\theta_1, w_1}}. \end{aligned}$$

Hence it follows that I_α is bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.

Assume that I_α is bounded from $LM_{1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$. Then for all non-negative $f \in L_1^{loc}(\mathbb{R}^n)$

$$\|Hg_{p_1}\|_{L_{\theta_2, v_2}(0, \infty)} \approx \|I_\alpha f\|_{LM_{p_2\theta_2, w_2}} \lesssim \|f\|_{LM_{1\theta_1, w_1}} \approx \|g_{p_1}\|_{L_{\theta_1, v_1}(0, \infty)}. \tag{4.5}$$

Let $g \in \mathbb{A}$ be locally absolutely continuous on $(0, \infty)$. Consider the non-negative measurable function h on $(0, \infty)$ defined uniquely up to equivalence by the equality

$$g(t) = \|h(|\cdot|)\|_{L_{p_1}(B(0, t^{-\frac{1}{\sigma}}))} = (nv_n)^{\frac{1}{p_1}} \left(\int_0^{t^{-\frac{1}{\sigma}}} h(\rho)^{p_1} \rho^{n-1} d\rho \right)^{\frac{1}{p_1}}.$$

If we take in Eq. 4.5 $f(x) = h(|x|)$ then $g_{p_1} = g$ and Eq. 4.5 implies that

$$\|Hg\|_{L_{\theta_2, v_2}(0, \infty)} \lesssim \|g\|_{L_{\theta_1, v_1}(0, \infty)}. \tag{4.6}$$

Finally if g is an arbitrary function in \mathbb{A} , then there exist functions $g_n \in \mathbb{A}$ which are locally absolutely continuous on $(0, \infty)$ and $g_n \nearrow g$ on $(0, \infty)$ as $n \rightarrow \infty$. Therefore by passing to the limit it follows that inequality (4.6) holds for all $g \in \mathbb{A}$. \square

5 Necessary and Sufficient Conditions

In order to obtain sufficient conditions on the weight functions ensuring boundedness of I_α , we shall apply Theorem 4.2 and the known necessary and sufficient conditions ensuring boundedness of the Hardy operator H from one weighted Lebesgue space to another one on the cone \mathbb{A} (see, for example, [11, 12]).

Theorem 5.1 *Let condition (3.8) or condition (3.9) be satisfied. Moreover, let $0 < \theta_1, \theta_2 \leq \infty, w_1 \in \Omega_{\theta_1}, w_2 \in \Omega_{\theta_2}$.*

Then the operator I_α is bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$ if, and in the case $p_1 = 1$ only if,

(a) *If $1 < \theta_1 \leq \theta_2 < \infty$, then*

$$B_1^1 := \sup_{t>0} \left(\int_t^\infty w_2^{\theta_2}(r)r^{\theta_2(\alpha-n(\frac{1}{p_1}-\frac{1}{p_2}))} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(r) dr \right)^{-\frac{1}{\theta_1}} < \infty, \tag{5.1}$$

and

$$B_2^1 := \sup_{t>0} \left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r)r^{\theta_1'(\alpha - \frac{n}{p_1})}}{\left(\int_r^\infty w_1^{\theta_1}(\rho)d\rho \right)^{\theta_1'}} dr \right)^{\frac{1}{\theta_1'}} < \infty. \tag{5.2}$$

(b) If $0 < \theta_1 \leq 1, 0 < \theta_1 \leq \theta_2 < \infty$, then $B_1^1 < \infty$ and

$$B_2^2 := \sup_{t>0} t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(r)dr \right)^{-\frac{1}{\theta_1}} < \infty. \tag{5.3}$$

(c) If $1 < \theta_1 < \infty, 0 < \theta_2 < \theta_1 < \infty, \theta_2 \neq 1$, then

$$B_1^3 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2^{\theta_2}(r)r^{\theta_2(\alpha - n(\frac{1}{p_1} - \frac{1}{p_2}))} dr}{\int_t^\infty w_1^{\theta_1}(r)dr} \right)^{\frac{\theta_2}{\theta_1 - \theta_2}} w_2^{\theta_2}(t)t^{\theta_2(\alpha - n(\frac{1}{p_1} - \frac{1}{p_2}))} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty,$$

and

$$B_2^3 := \left(\int_0^\infty \left[\left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r)r^{\theta_1'(\alpha - \frac{n}{p_1})}}{\left(\int_r^\infty w_1^{\theta_1}(\rho)d\rho \right)^{\theta_1'}} dr \right)^{\frac{\theta_2 - 1}{\theta_2}} \right]^{\frac{\theta_1 \theta_2}{\theta_1 - \theta_2}} \times \frac{w_1^{\theta_1}(t)t^{\theta_1'(\alpha - \frac{n}{p_1})}}{\left(\int_t^\infty w_1^{\theta_1}(\rho)d\rho \right)^{\theta_1'}} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty.$$

(d) If $1 = \theta_2 < \theta_1 < \infty$, then

$$B_1^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2(r)r^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})} dr}{\int_t^\infty w_1^{\theta_1}(r)dr} \right)^{\frac{1}{\theta_1 - 1}} w_2(t)t^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})} dt \right)^{\frac{\theta_1 - 1}{\theta_1}} < \infty,$$

and

$$B_2^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2(r)r^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})} dr + t^{\alpha - \frac{n}{p_1}} \int_0^t w_2(r)r^{\frac{n}{p_2}} dr}{\int_t^\infty w_1^{\theta_1}(r)dr} \right)^{\theta_1' - 1} \times t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2(r)r^{\frac{n}{p_2}} dr \right) \frac{dt}{t} \right)^{\theta_1'} < \infty.$$

(e) If $0 < \theta_2 < \theta_1 = 1$, then

$$B_1^5 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2 \left(\alpha - n \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right)} dr}{\int_t^\infty w_1(r) dr} \right)^{\frac{\theta_2}{1-\theta_2}} w_2^{\theta_2}(t) t^{\theta_2 \left(\alpha - n \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right)} dt \right)^{\frac{1-\theta_2}{\theta_2}} < \infty,$$

and

$$B_2^5 := \left(\int_0^\infty \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{1-\theta_2}} \left(\inf_{t < s < \infty} s^{\frac{n}{p_1} - \alpha} \int_s^\infty w_1(\rho) d\rho \right)^{\frac{\theta_2}{\theta_2 - 1}} \times w_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{1-\theta_2}{\theta_2}} < \infty.$$

(f) If $0 < \theta_2 < \theta_1 < 1$, then $B_1^3 < \infty$ and

$$B_2^6 := \left(\int_0^\infty \sup_{t \leq s < \infty} \frac{s^{\left(\alpha - \frac{n}{p_1} \right) \frac{\theta_1 \theta_2}{\theta_1 - \theta_2}}}{\left(\int_s^\infty w_1(\rho) d\rho \right)^{\frac{\theta_2}{\theta_1 - \theta_2}}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{\theta_1 - \theta_2}} \times w_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty.$$

(g) If $0 < \theta_1 \leq 1, \theta_2 = \infty$, then

$$B^7 := \operatorname{ess\,sup}_{0 < t \leq s < \infty} \frac{w_2(t) t^{\frac{n}{p_2}}}{s^{\frac{n}{p_1} - \alpha} \left(\int_s^\infty w_1^{\theta_1}(r) dr \right)^{\frac{1}{\theta_1}}} < \infty.$$

(h) If $1 < \theta_1 < \infty, \theta_2 = \infty$, then

$$B^8 := \operatorname{ess\,sup}_{t > 0} w_2(t) t^{\frac{n}{p_2}} \left(\int_t^\infty \frac{r^{\theta_1 \left(\alpha - \frac{n}{p_1} \right)}}{\left(\int_r^\infty w_1^{\theta_1}(s) ds \right)^{\theta_1 - 1} r} dr \right)^{\frac{1}{\theta_1}} < \infty.$$

(i) If $\theta_1 = \infty, 0 < \theta_2 < \infty$, then

$$B^{10} := \left(\int_0^\infty \left(t^{\frac{n}{p_1} - \alpha} \int_t^\infty \frac{s^{\alpha - \frac{n}{p_1} - 1} ds}{\operatorname{ess\,sup}_{s < y < \infty} w_1(y)} \right)^{\theta_2} \times w_2^{\theta_2}(t) t^{\theta_2 \left(\alpha - n \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right)} dt \right)^{\frac{1}{\theta_2}} < \infty.$$

(j) If $\theta_1 = \theta_2 = \infty$, then

$$B^9 := \operatorname{ess\,sup}_{t>0} w_2(t)t^{\frac{n}{p_2}} \int_t^\infty \frac{s^{\alpha-\frac{n}{p_1}-1}}{\operatorname{ess\,sup}_{s<y<\infty} w_1(y)} ds < \infty.$$

Moreover, if $p_1 = 1$, $0 < p_2 < \infty$ and $n\left(1 - \frac{1}{p_2}\right)_+ < \alpha < n$ or $1 < p_2 < \infty$ and $\alpha = n\left(1 - \frac{1}{p_2}\right)$, then I_α is bounded from $LM_{1\theta_1, w_1}$ to $WLM_{p_2\theta_2, w_2}$ if and only if conditions (a) – (j) are satisfied.

Proof From results in [11, 12] it follows that conditions (a)–(j) are necessary and sufficient for inequality (4.3) to hold, where v_1 and v_2 are defined by Eqs. 4.2, 4.4 respectively.

For example, let $1 < \theta_1 \leq \theta_2 < \infty$, then by [11, 12] inequality (4.3) holds if and only if

$$A_1^1 := \sup_{t>0} \left(\int_0^t v_2^{\theta_2}(s) ds \right)^{\frac{1}{\theta_2}} \left(\int_0^t v_1^{\theta_1}(s) ds \right)^{-\frac{1}{\theta_1}} < \infty$$

and

$$A_2^1 := \sup_{t>0} \left(\int_t^\infty v_2^{\theta_2}(s) ds \right)^{\frac{1}{\theta_2}} \left(\int_0^t \frac{v_1^{\theta_1}(s)s^{\theta_1'}}{\left(\int_0^s v_2^{\theta_2}(\tau) ds\right)^{\theta_1}} ds \right)^{\frac{1}{\theta_1}} < \infty.$$

If v_1 and v_2 are defined by Eqs. 4.2, 4.4 respectively, then by using the substitute $r = s^{-\frac{1}{\sigma}}$ we get that

$$A_1^1 := \sup_{t>0} \left(\int_0^t w_2^{\theta_2} \left(s^{-\frac{1}{\sigma}}\right) s^{-\frac{n\theta_2}{\sigma p_2} - \frac{1}{\sigma} - 1} ds \right)^{\frac{1}{\theta_2}} \left(\int_0^t w_1^{\theta_1} \left(s^{-\frac{1}{\sigma}}\right) r^{-\frac{1}{\sigma} - 1} ds \right)^{-\frac{1}{\theta_1}} \approx B_1^1$$

and similarly $A_2^1 \approx B_2^1$.

Hence the statement follows by Theorem 4.2. □

Remark 5.2 Note that two conditions (5.1) and (5.3) are equivalent to one condition (1.3).

Corollary 5.3 *Let condition (3.8) or condition (3.9) be satisfied. Moreover, let functions $w_1 \in \Omega_{p_1, \infty}$ and $w_2 \in \Omega_{p_2, \infty}$ satisfy the following condition:*

$$\sup_{t>0} w_2(t)t^{\frac{n}{p_2}} \int_t^\infty \frac{s^{\alpha-\frac{n}{p_1}-1}}{\operatorname{ess\,sup}_{s<\tau<\infty} w_1(\tau)} ds < \infty. \tag{5.4}$$

Then I_α is bounded from \mathcal{M}_{p_1, w_1} to \mathcal{M}_{p_2, w_2} .

Proof Clearly boundedness of I_α from $LM_{p_1\infty, w_1}$ to $LM_{p_2\infty, w_2}$ implies boundedness of I_α from $GM_{p_1\infty, w_1} = \mathcal{M}_{p_1, w_1}$ to $GM_{p_2\infty, w_2} = \mathcal{M}_{p_2, w_2}$. □

Remark 5.4 Let $1 < p_1 < p_2 < \infty$, $\alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$. It is obvious that if condition (1.1) holds, then condition (5.4) holds too. Moreover for non-increasing continuous functions w_1 conditions (1.1) and (5.4) coincide. However, in general, condition (5.4) does not imply condition (1.1). For example, the functions

$$w_1(r) = \chi_{(1,\infty)}(r)r^{-\beta}, \quad w_2(t) = \frac{1}{t^\beta + 1}, \quad 0 < \beta < \frac{n}{p_1} - \alpha$$

satisfy condition (5.4) but do not satisfy condition (1.1).

Theorem 5.5

- 1) Let $1 < p_1 < p_2 < \infty$, $\alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$, $0 < \theta_1 < \infty$ and $\theta_1 \leq \theta_2 \leq \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then condition (1.4) is necessary and sufficient for boundedness of I_α from LM_{p_1, θ_1, w_1} to LM_{p_2, θ_2, w_2} .
- 2) Let $1 \leq p_1 < p_2 < \infty$, $\alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$, $0 < \theta_1 < \infty$ and $\theta_1 \leq \theta_2 \leq \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, then condition (1.4) is necessary and sufficient for boundedness of I_α from LM_{p_1, θ_1, w_1} to WLM_{p_2, θ_2, w_2} .

Proof If $0 < \theta_1 \leq 1$, $0 < \theta_1 \leq \theta_2 \leq \infty$, then the statement of the theorem was proved in [6, 9]. Let $1 < \theta_1 \leq \theta_2 < \infty$. Since

$$\begin{aligned} B_2^1 &= \sup_{t>0} \left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r)r^{\theta_1(\alpha - \frac{n}{p_1})}}{\left(\int_r^\infty w_1^{\theta_1}(\rho)d\rho \right)^{\theta_1'}} dr \right)^{\frac{1}{\theta_1'}} \\ &\leq \sup_{t>0} t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r)}{\left(\int_r^\infty w_1^{\theta_1}(\rho)d\rho \right)^{\theta_1'}} dr \right)^{\frac{1}{\theta_1'}} \\ &\approx \sup_{t>0} t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(- \int_t^\infty d \left(\int_r^\infty w_1^{\theta_1}(\rho)d\rho \right)^{1-\theta_1'} \right)^{\frac{1}{\theta_1'}} \\ &= \sup_{t>0} t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2^{\theta_2}(r)r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(r)dr \right)^{-\frac{1}{\theta_1'}} = B_2^2, \end{aligned}$$

sufficiency of Eqs. 5.1 and 5.3 follows by Theorem 5.1, part (a). Hence condition (1.4) is sufficient (by Remark 5.2) and necessary (by Theorem 1.3, part 1) for boundedness of I_α from LM_{p_1, θ_1, w_1} to LM_{p_2, θ_2, w_2} . The case $1 < \theta_1 < \infty$, $\theta_2 = \infty$ is similar, because in this case $B^8 \leq B_2^2$ by this same argument as above.

The proof of sufficiency for the second statement is similar. As for necessity one should note that boundedness of I_α from LM_{p_1, θ_1, w_1} to WLM_{p_2, θ_2, w_2} implies boundedness of the fractional maximal operator M_α from LM_{p_1, θ_1, w_1} to WLM_{p_2, θ_2, w_2} and that condition (1.2) is necessary for boundedness of M_α from LM_{p_1, θ_1, w_1} to WLM_{p_2, θ_2, w_2} (see [9]). □

Corollary 5.6 *Let $1 < p_1 \leq p_2 < \infty$, $\alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$, $0 < \theta_1 < \infty$ and $\theta_1 \leq \theta_2 \leq \infty$, $w_2 \in \Omega_{\theta_2}$ and*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} < \infty$$

for all $t > 0$. Moreover, if $\theta_2 = \infty$ and $\theta_1 < \infty$ it is also assumed that

$$\lim_{t \rightarrow \infty} \left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_\infty(0,\infty)} = 0.$$

Then

- 1) I_α is bounded from $LM_{p_1\theta_1, w_1^*}$ to $LM_{p_2\theta_2, w_2}$, where w_1^* is a non-increasing continuous function on $(0, \infty)$ defined by

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} = \left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)}, \quad t \in (0, \infty).$$

- 2) If $w_1 \in \Omega_{\theta_1}$ and I_α is bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$, then

$$LM_{p_1\theta_1, w_1} \subset LM_{p_1\theta_1, w_1^*}.$$

(Hence $LM_{p_1\theta_1, w_1^*}$ is the maximal among spaces $LM_{p_1\theta_1, w_1}$ for which I_α is bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.)

Proof Since condition (1.4) is also necessary and sufficient for boundedness of the fractional maximal operator M_α (see [10]), the proof of Corollary 5.6 is also the same as for the case of M_α . □

An analogue of Corollary 5.6 also holds for the case in which $LM_{p_2\theta_2, w_2}$ is replaced by $WLM_{p_2\theta_2, w_2}$.

Corollary 5.7 *Let $1 < p_1 \leq p_2 < \infty$, $\alpha = n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$, $w_1 \in \Omega_{p_1}$ and $w_2 \in \Omega_{p_2}$, then condition (1.4) is necessary and sufficient for boundedness of I_α from L_{p_1, W_1} to L_{p_2, W_2} , where $W_1(x) = \|w_1\|_{L_{p_1}(|x|,\infty)}$, $W_2(x) = \|w_2\|_{L_{p_2}(|x|,\infty)}$.*

Proof It suffices to take into account that for $0 < p \leq \infty$

$$\|f\|_{LM_{pp, w}} = \|f\|_{L_{p, w}},$$

where for all $x \in \mathbb{R}^n$ $W(x) = \|w\|_{L_p(|x|,\infty)}$ (see [3]). □

It is interesting to note that condition (1.4) have the form that differs from the known necessary and sufficient conditions discussed in detail, for example, in [14].

Example 5.8 Let the condition (3.8) or condition (3.9) be satisfied. Moreover, let $1 < \theta_1 \leq \theta_2 < \infty$, and β be such that

$$\beta + \frac{1}{\theta_2} < 0, \quad \beta + \frac{n}{p_2} + \frac{1}{\theta_2} > 0, \quad \beta + \frac{n}{p_2} + \frac{1}{\theta_2} + \alpha - \frac{n}{p_1} < 0,$$

then it is easy calculate that the functions $w_1(t) = t^{\beta + \frac{n}{p_2} + \frac{1}{\theta_2} + \alpha - \frac{n}{p_1} - \frac{1}{\theta_1}}$, $w_2(t) = t^\beta$ satisfy the condition (a) of Theorem 5.1. Thus I_α is bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.

6 Some Applications

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{6.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t .

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\Delta$ is the Laplacian on \mathbb{R}^n , then $L^{-\alpha/2}$ is the Riesz potential I_α . See, for example, Chapter 5 in [22].

Theorem 6.1 *Let condition (6.1) be satisfied. Moreover, let $\alpha, p_1, p_2, \theta_1, \theta_2, w_1$ and w_2 satisfy the sufficiency conditions of Theorems 5.1 or 5.5. Then $L^{-\alpha/2}$ is bounded from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.*

Proof Since the semigroup e^{-tL} has the kernel $p_t(x, y)$ which satisfies condition (6.1), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [13]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{LM_{p_2\theta_2, w_2}} \lesssim \|I_\alpha(|f|)\|_{LM_{p_2\theta_2, w_2}} \lesssim \|f\|_{LM_{p_1\theta_1, w_1}}.$$

□

Property (6.1) is satisfied for large classes of differential operators (see, for example [9]). In [9] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorems 5.1 and 5.5 are also applicable for proving boundedness of those operators from $LM_{p_1\theta_1, w_1}$ to $LM_{p_2\theta_2, w_2}$.

Acknowledgement The authors thank the referees for careful reading the paper and useful comments.

References

1. Adams, D.R.: A note on Riesz potentials. *Duke Math. J.* **42**, 765–778 (1975)
2. Burenkov, V.I., Guliyev, H.V.: Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. *Dokl. Ross. Akad. Nauk* **391**, 591–594 (2003)

3. Burenkov, V.I., Guliyev, H.V.: Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. *Stud. Math.* **163**(2), 157–176 (2004)
4. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: Necessary and sufficient conditions for the boundedness of the fractional maximal operator in the local Morrey-type spaces. *Dokl. Ross. Akad. Nauk* **409**(4), 443–447 (2006)
5. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces. *J. Comput. Appl. Math.* **208**, 280–301 (2007)
6. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: Necessary and sufficient conditions for the boundedness of the Riesz operator in the local Morrey-type spaces. *Dokl. Ross. Akad. Nauk* **412**(5), 585–589 (2007)
7. Burenkov, V., Gogatishvili, A., Guliyev, V., Mustafayev, R.: Sufficient conditions for boundedness of the Riesz potential in local Morrey-type spaces. Preprint, Institute of Mathematics, p. 14. AS CR, Prague, 21 Dec 2007
8. Burenkov, V.I., Guliyev, V.S., Serbetci, A., Tararykova, T. V.: Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces. *Dokl. Ross. Akad. Nauk* **422**(1), 11–14 (2008)
9. Burenkov, V.I., Guliyev, V.S.: Necessary and sufficient conditions for the boundedness of the Riesz operator in local Morrey-type spaces. *Potential Anal.* **30**(3), 211–249 (2009)
10. Burenkov, V., Gogatishvili, A., Guliyev, V., Mustafayev, R.: Boundedness of the fractional maximal operator in local Morrey-type spaces. *Complex Var. Elliptic Equ.* **55**(8), 739–758 (2010)
11. Carro, M., Pick, L., Soria, J., Stepanov, V.D.: On embeddings between classical Lorentz spaces. *Math. Inequal. Appl.* **4**(3), 397–428 (2001)
12. Carro, M., Gogatishvili, A., Martin, J., Pick, L.: Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces. *J. Oper. Theory* **59**(2), 309–332 (2008)
13. Duong, X.T., Yan, L.X.: On commutators of fractional integrals. *Proc. Am. Math. Soc.* **132**(12), 3549–3557 (2004)
14. Genebashvili, I., Gogatishvili, A., Kokilashvili, V., Krbec, M.: Weight theory for integral transforms on spaces of homogeneous type. In: Pitman Monogr. Surveys in Pure Applied Math., vol. 92. Longman, Harlow (1998)
15. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in \mathbb{R}^n . Doctor of Sciences, Moscow, Mat. Inst. Steklova, pp. 1–329 (1994, Russian)
16. Guliyev, V.S.: Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications. Baku, 1–332 (1999, Russian)
17. Guliyev, V.S., Mustafayev, R.Ch.: Integral operators of potential type in spaces of homogeneous type. *Dokl. Ross. Akad. Nauk* **354**(6), 730–732 (1997, Russian)
18. Guliyev, V.S., Mustafayev, R.Ch.: Fractional integrals in spaces of functions defined on spaces of homogeneous type. *Anal. Math.* **24**(3), 181–200 (1998, Russian)
19. Mizuhara, T.: Boundedness of some classical operators on generalized Morrey spaces. In: Igari, S. (ed.) *Harmonic Analysis, ICM 90 Satellite Proceedings*, pp. 183–189. Springer, Tokyo (1991)
20. Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. *Trans. Am. Math. Soc.* **43**, 126–166 (1938)
21. Nakai, E.: Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. *Math. Nachr.* **166**, 95–103 (1994)
22. Stein, E.M.: *Singular Integrals and Differentiability of Functions*. Princeton University Press, Princeton (1970)
23. Stein, E.M., Weiss, G.: *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton (1971)