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Full Paper

On a variation of co-coatomically supplemented modules

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Abstract: Co-coatomically ss-supplemented modules are identified by extending the concept of ss-supplemented modules, and certain characteristics of co-coatomically ss-supplemented modules are demonstrated. It is established that the sum of a finite set of co-coatomically ss-supplemented modules remains co-coatomically ss-supplemented. Moreover, it is proven that any quotient module of a co-coatomically ss-supplemented module is also co-coatomically ss-supplemented. It is observed that if G is a co-coatomically ss-supplemented submodule of a module E with the condition that E/G lacks a maximal submodule, then E is co-coatomically ss-supplemented. It is demonstrated that the ring S is semi-perfect and $Rad(S) \subseteq Soc({}_SS)$ if and only if S is semi-local and $Rad(S) \subseteq Soc({}_SS)$ if and only if each left S-module is co-coatomically ss-supplemented. Furthermore, specific characteristics of co-coatomically ss-supplemented modules over Dedekind rings are examined.

Keywords: co-coatomically ss-supplemented modules, co-coatomic submodules, semi-perfect rings, semi-local rings, left ss-perfect rings

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INTRODUCTION

All along this text, S is considered an associative ring with a unit, and all modules are assumed to be unital left S-modules unless explicitly mentioned otherwise. Let E be an S-module. The notation ${}_SS$ is referred to the left S-module structure of the ring S. A submodule G of E is called *small* in E, denoted as $G \ll E$ if $E \neq G + G'$ for each proper submodule G' of E. $Rad(E)$ represents the intersection of all maximal submodules of E, equivalently the sum of all small submodules of E. A module E is called *radical* if E does not have any maximal submodule, i.e. $E = Rad(E)$. Moreover, $Soc(E)$ indicates the socle of a module E, i.e. the sum of all simple submodules of E. As discussed in Corollary 9.1.3 [1], it is widely known that $Soc(E)$ is the largest semi-simple submodule of a module E. Zhou and Zhang [2] introduced the concept of $Soc_s(E)$ for a module E as an extension of the idea of the socle of a module E by considering all simple submodules that are small within E instead of considering all simple submodules of E, i.e.

$$Soc_s(E) = \sum\{G \ll E \mid G \text{ is simple}\}.$$

A module is called *coatomic* when for each proper submodule, there exists a maximal submodule including it [3]. Assume that G is a submodule of a module E. G is called *(cofinite) co-coatomic* in E in the case the quotient module E/G is (finitely generated) coatomic [4, 5]. Examples of coatomic modules include local, semi-simple and finitely generated modules. It is well known that the property of being a coatomic module is transferred by quotient modules. So it can be concluded that each submodule of local modules and each submodule of finitely generated semi-simple modules are also co-coatomic [5]. Throughout this paper, the notations $G \leq E$ and $G \leq_{cc} E$ notify that G is a submodule of E and G is a co-coatomic submodule of E respectively.

A non-zero module E is called *local* when the sum of all proper submodules of E is also a proper submodule of E. A ring S is called *local* when ${}_SS$ is a local module. Kaynar et al. [6] defined strongly local modules and rings as follows. A module E is called *strongly local* when E is local and $Rad(E)$ is semi-simple. A ring S is called *left strongly local* when ${}_SS$ is a strongly local module.

Let E be a module and $G \leq E$. A submodule W is called an *ss-supplement* of G in E when $E = G + W$ and $G \cap W \leq Soc_s(W)$ [6]. It is shown in Lemma 3 [6] that W is an *ss*-supplement of G in E if and only if $E = G + W, G \cap W$ being semi-simple, and $G \cap W \ll W$ if and only if $E = G + W, G \cap W$ being semi-simple, and $G \cap W \leq Rad(W)$. Moreover, a module E is called *ss-supplemented* when each submodule of E has an *ss*-supplement in E [6]. A submodule W is called a *(weak) supplement* of G in E when $E = G + W$ and $G \cap W \ll W$ $(G \cap W \ll E)$. A module E is called *(weak) supplemented* when each submodule of E has a (weak) supplement in E. Semi-simple, artinian and local modules are supplemented [7, 8]. A submodule W is called a *Rad-supplement* of G in E when $E = G + W$ and $G \cap W \leq Rad(W)$. A module E is called *Rad-supplemented* when each submodule of E has a Rad-supplement in E [9]. Based on the provided definitions, the following implication regarding submodules of a module is observed:

direct summand $\Longrightarrow$ *ss*-supplement $\Longrightarrow$ supplement.

A module E is called *cofinitely supplemented* when each cofinite submodule has a supplement in E [4]. A module E is called *cofinitely ss-supplemented* in the case each cofinite submodule has an *ss*-supplement within E [10]. The same paper introduced different characteristics of cofinitely *ss*-supplemented modules.

A module E is called *co-coatomically supplemented* in the case each $G \leq_{cc} E$ has a supplement in E [5]. Moreover, a module E is called *co-coatomically weak supplemented* in the case each $G \leq_{cc} E$ has a weak supplement in E, in which case there exists a submodule W of E such that $E = G + W$ and $G \cap W \ll E$ [5]. Explicitly, the class of modules that are co-coatomically weakly supplemented includes modules that are co-coatomically supplemented, and also the class of modules that are cofinitely supplemented includes the modules that are co-coatomically supplemented. Sozen et al. [11] generalised co-coatomically supplemented modules to Rad-cc-supplemented modules and studied the module class S_{Rad-cc}. If for every co-coatomic submodule G of E, there is $W \leq E$ such that W is a Rad-supplement of G, then the module E is called belonging to the class S_{Rad-cc} [11]. Furthermore, in recent years more generalisations of co-coatomically supplemented modules have been studied [12, 13].

In this paper firstly a module E is called *co-coatomically ss-supplemented* when each $G \leq_{cc} E$ has an *ss*-supplement in E as a proper generalisation of *ss*-supplemented modules. It can be observed that each co-coatomically ss-supplemented module is cofinitely ss-supplemented. An example of a module that is co-coatomically ss-supplemented but not ss-supplemented is given. In continuation of the study it is demonstrated that if E is an ss-semi-local module such that $Soc_s(E) \ll E$, then E being a co-coatomically ss-supplemented module is equivalent to it being an ss-supplemented module. It is established that quotient modules and finite sums of co-coatomically ss-supplemented modules remain unchanged. It is proven that when G is a co-coatomically ss-supplemented submodule of a module E and E/G does not have a maximal submodule, then E is co-coatomically ss-supplemented. Later, it is determined that a necessary and sufficient condition for every left S-module to be co-coatomically ss-supplemented is that the ring S is semi-perfect such that $Rad(S) \subseteq Soc({}_SS$. Equivalently S is a semi-local ring such that $Rad(S) \subseteq Soc({}_SS)$. Moreover, a novel characterisation of left ss-perfect rings via co-coatomically ss-supplemented modules is presented.

In the rest of this paper, a module E is called *co-coatomically ss-semi-local* when each $G \leq_{cc} E$ has a weak supplement W in E such that $G \cap W$ is semi-simple, and the rings whose left modules are co-coatomically ss-semi-local are determined. It is proven that when the ring has a semi-simple radical, each projective cover of a co-coatomically ss-semi-local module is co-coatomically ss-semi-local. It is established that the quotient modules of co-coatomically ss-semi-local modules continue to be co-coatomically ss-semi-local. Furthermore, attention is directed to specific algebraic properties of the modules defined in this paper, particularly when they are over Dedekind domains. Specifially, it is proven that over a non-local Dedekind domain, a torsion co-coatomically ss-semi-local module is a co-coatomically ss-supplemented module. It is also shown that over a Dedekind domain which is not a field, a torsion-free co-coatomically ss-supplemented module is a divisible module.

CO-COATOMICALLY *SS*-SUPPLEMENTED MODULES

It is explicit that each ss-supplemented module is co-coatomically ss-supplemented. Nevertheless, it is crucial to emphasise that the reverse of this statement is generally not valid.

Consider a commutative domain denoted as S with an S-module E. Let $T(E)$ be the set including all elements e within E for which there exists a non-zero element s in S, leading to $se = 0$; in other words, $Ann(e) \neq 0$. As a submodule of E, $T(E)$ is called the *torsion submodule* of E. If E coincides with $T(E)$, E is called a *torsion module*. Additionally, E is called *torsion-free* when $T(E)$ is equal to 0.

Example 1. Consider $\mathbb{Q}$, the set of rational numbers, as a $\mathbb{Z}$-module. It is co-coatomically ss-supplemented, given that the only co-coatomic submodule is $\mathbb{Q}$ itself. Besides that, $\mathbb{Q}$ is not ss-supplemented since it is not supplemented due to its torsion-free property by Theorem 3.1 [14].

A module E is called *ss-semi-local* in the case the quotient module $E/Soc_s(E)$ is semi-simple, or equivalently each submodule of E has (is) a weak ss-supplement in E. That is, for each submodule G of E, E has a submodule W such that $E = G + W$ and $G \cap W \leq Soc_s(E)$, or equivalently $E = G + W$, $G \cap W$ is semi-simple and $G \cap W \ll E$ [15].

Proposition 1. Let E be an ss-semi-local module with $Soc_s(E) \ll E$. Then E is a co-coatomically ss-supplemented module if and only if E is an ss-supplemented module.

Proof. Suppose that E is a co-coatomically ss-supplemented module and $G \leq E$. Due to the ss-semi-locality of the module E, we conclude that $E/Soc_s(E)$ is coatomic as it is semi-simple. Since

$$E/(G + Soc_s(E)) = [E/Soc_s(E)]/[(G + Soc_s(E))/Soc_s(E)]$$

and each quotient module of a coatomic module is coatomic, then $E/(G + Soc_s(E))$ is also coatomic. By the hypothesis, $G + Soc_s(E)$ has an *ss*-supplement in E, say W. Then $E = (G + Soc_s(E)) + W$, $(G + Soc_s(E)) \cap W$ is semi-simple and $(G + Soc_s(E)) \cap W \ll W$. Since $Soc_s(E) \ll E$ and $G \cap W \leq (G + Soc_s(E)) \cap W$, we have $E = G + W$ and $G \cap W \ll W$ by Section 19.3 [8]. Moreover, $G \cap W$ is semi-simple as a submodule of $(G + Soc_s(E)) \cap W$ from Corollary 8.1.5 [1]. Hence E is an *ss*-supplemented module. The rest of the proof is obvious.

A module E is called *semi-local* in the case its quotient module $E/Rad(E)$ is semi-simple [7]. Each *ss*-semi-local module is obviously semi-local.

Corollary 1. For a semi-local module E with small radical, E is a co-coatomically *ss*-supplemented module if and only if E is an *ss*-supplemented module.

Proof. The proof can be made with a similar method for the proof of Proposition 1.

Now we provide an example showing that a cofinitely supplemented module may not necessarily be co-coatomically *ss*-supplemented.

It is recalled that an ideal J of the ring S is called *left t-nilpotent* when for any sequence of elements $a_1, a_2, \ldots$ belonging to J, there is a $k \in \mathbb{Z}^+$ with $a_k a_{k-1} \ldots a_1 = 0$. A ring S is called *left perfect* in the case that S is semi-local and $Rad(S)$ is left *t*-nilpotent [8].

Example 2. Assuming that $t \in \mathbb{Z}$ is prime, let us consider the local Dedekind domain below:

$$S = \mathbb{Z}_{(t)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, (b, t) = 1\}.$$

Let us say $E = {}_S S^{(\mathbb{N})}$. ${}_S S$ is a supplemented module as S is a local ring. Thus, E is cofinitely supplemented according to Corollary 2.4 [4]. Let us put $G = Rad({}_S S^{(\mathbb{N})})$. The ring S is semi-local; however it is not a left perfect ring, as $Rad(S)$ is not left *t*-nilpotent by Section 43.9 [8]. It should be noted that $G \leq_{cc} E$ and G does not have an *ss*-supplement in E because it neither has a supplement in E. This is due to the fact that S is not a left perfect ring, as stated in Theorem 1 [16]. Therefore, it follows that E is not a co-coatomically *ss*-supplemented module.

In Section 19.4 [8], it is defined that a *projective cover* of a module X as a module E is equipped with a homomorphism $h : E \to X$, where E itself is a projective module and h is a small epimorphism, meaning that the kernel of h (denoted as *Ker*(h)) is a small submodule of E.

A ring S is called *semi-perfect* in the case each finitely generated left S-module has a projective cover in Section 42.6 [8]. Example 2 also illustrates that co-coatomically *ss*-supplemented modules and cofinitely supplemented modules do not necessarily have the same characteristics over semi-perfect rings and discrete valuation rings.

Proposition 2. Co-coatomically *ss*-supplemented modules exhibit transfer properties through their quotient modules.

Proof. Suppose that E is a co-coatomically *ss*-supplemented module and $G \leq E$. If we take any co-coatomic submodule in the quotient module E/G, it can be represented as a submodule of the form H/G, where $H \leq_{cc} E$. By the assumption, H has an *ss*-supplement W in E. Therefore, $E = H + W$, $H \cap W$ is semi-simple and $H \cap W \ll W$. Then we have $E/G = H/G + (W + G)/G$. Now consider the canonical projection $\pi : E \to E/G$. Therefore, $\pi(H \cap W) = (H \cap (W + G))/G$ is semi-simple from Corollary 8.1.5 [1], since $H \cap W$ is semi-simple. Moreover, $(H \cap (W + G))/G = \pi(H \cap W) \ll \pi(W) = (W + G)/G$ from Section 19.3 [8]. Hence $(W + G)/G$ is an *ss*-supplement of H/G in E/G.

Proposition 3. Suppose that E is a co-coatomically *ss*-supplemented module. In that case each co-coatomic submodule of the quotient module $E/Soc_s(E)$ is a direct summand.

Proof. The quotient module $E/Soc_s(E)$ has co-coatomic submodules of the form $G/Soc_s(E)$, where $G \leq_{cc} E$ and $Soc_s(E) \leq G$. Then by the assumption, there is a submodule W of E such that $E = G + W$, $G \cap W \leq Soc_s(W)$. This yields $G \cap W \leq Soc_s(E)$. Thus,

$$E/Soc_s(E) = G/Soc_s(E) + (W + Soc_s(E))/Soc_s(E) \text{ and}$$
$$(G/Soc_s(E)) \cap ((W + Soc_s(E))/Soc_s(E)) = ((G \cap W) + Soc_s(E))/Soc_s(E) = 0.$$

Consequently,

$$E/Soc_s(E) = (G/Soc_s(E)) \oplus ((W + Soc_s(E))/Soc_s(E)).$$

Corollary 2. For a co-coatomically *ss*-supplemented module *E*, the conditions below are verified.

(1) For each $G/Rad(E) \leq_{cc} E/Rad(E), G/Rad(E)$ is a direct summand of $E/Rad(E)$.

(2) For each $G/Soc(E) \leq_{cc} E/Soc(E), G/Soc(E)$ is a direct summand of $E/Soc(E)$.

Proof. The proofs for these statements can be carried out in a similar manner as that of Proposition 3.

In the next step it is aimed to demonstrate that any finite sum of modules that are co-coatomically *ss*-supplemented is also co-coatomically *ss*-supplemented. To begin, the validity of the following commonly used lemma is established.

Lemma 1. Suppose that E is a module, $H \leq E$ and $G \leq_{cc} E$. When H is co-coatomically *ss*-supplemented and $G + H$ has an *ss*-supplement in *E*, G has an *ss*-supplement in *E*.

Proof. Let W be an *ss*-supplement of $G + H$ in *E*. Thus, $E = W + G + H$, $W \cap (G + H)$ is semi-simple and $W \cap (G + H) \ll W$. It is denoted that $H/H \cap (G + W) \cong E/(G + W)$. Since $E/(G + W) \cong (E/G)/((G + W)/G)$ and E/G is coatomic, then $E/(G + W)$ is coatomic. Thus, $H \cap (G + W)$ has an *ss*-supplement X in H by assumption, i.e. $H = X + H \cap (G + W)$, $X \cap (G + W)$ is semi-simple and $X \cap (G + W) \ll X$. Therefore, $E = W + G + H = W + G + (X + H \cap (G + W)) = G + W + X$. Also, we obtain

$$\begin{aligned} G \cap (X + W) &\leq X \cap (G + W) + W \cap (G + X) \\ &\leq X \cap (G + W) + W \cap (G + H) \ll X + W \end{aligned}$$

by Section 19.3 [8]. Moreover, since $X \cap (G + W)$ and $W \cap (G + H)$ are semi-simple, then $G \cap (X + W)$ is semi-simple by Corollary 8.1.5 [1]. Hence $X + W$ is an *ss*-supplement of G in E.

Theorem 1. The sum of finitely many co-coatomically *ss*-supplemented modules is also co-coatomically *ss*-supplemented.

Proof. Consider a finite collection of co-coatomically *ss*-supplemented modules, denoted as $E_1, E_2, \dots, E_n$, and put $E = E_1 + E_2 + \cdots + E_n$. To demonstrate the claim, we can limit our proof to the case when there are only two modules, namely E_1 and E_2, both of which are co-coatomically *ss*-supplemented. We show that if $E = E_1 + E_2$, then the result holds for any finite collection. Suppose that $G \leq_{cc} E$. Then $E = E_1 + E_2 + G$. Note that $E/(E_2 + G)$ is coatomic as a quotient module of the coatomic module E/G, and hence $E_2 + G \leq_{cc} E$. Since E_1 is co-coatomically *ss*-supplemented, $E_2 + G \leq_{cc} E$ and E has an *ss*-supplement 0 in E. Then $E_2 + G$ has an *ss*-supplement in E by Lemma 1. By using Lemma 1 once more, we conclude that G has an *ss*-supplement in E since E_2 is co-coatomically *ss*-supplemented, $G \leq_{cc} E$ and $E_2 + G$ has an *ss*-supplement in E. Hence $E_1 + E_2$ is a co-coatomically *ss*-supplemented module.

A module X is called *finitely E-generated* in the case there is an epimorphism $h : E^{(\Lambda)} \to X$ where Λ is a finite set.

Corollary 3. For a co-coatomically ss-supplemented module E, any finitely E-generated module is co-coatomically ss-supplemented.

Proof. The proof can be seen by Proposition 2 and Theorem 1.

The left $\mathbb{Z}$-module $E = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is a co-coatomically ss-supplemented module, although it is not semi-simple. When each simple left S-module is injective, the ring S is called *left V −ring*. It is widely known that $Rad(E) = 0$ for each left S-module E if and only if the ring S is left V −ring [8].

Proposition 4. Suppose that S is a left V −ring and E is a left S-module. Then E is a semi-simple module if and only if E is a co-coatomically ss-supplemented module.

Proof. (⟹) This is explicit.
(⟸) Let E be co-coatomically ss-supplemented S-module. Then each $G \leq_{cc} E$ has an ss-supplement W in E and thus, $G \cap W \leq Rad(W)$. Since S is a left V −ring, then $Rad(W) = 0$. Therefore, we conclude that $E = G \oplus W$. Thus, $E/Soc(E)$ does not have a maximal submodule from Theorem 2.1 [5]. By Section 23.1 [8], we reach the conclusion that $E/Soc(E) = Rad(E/Soc(E)) = 0$ since S is a left V −ring. Hence E is a semi-simple module.

Corollary 4. Over a left V −ring, a module that is a direct sum of co-coatomically ss-supplemented modules is co-coatomically ss-supplemented.

Proof. According to Proposition 4, we arrive at the conclusion that over left V −rings, semi-simple modules and co-coatomically ss-supplemented modules coincide. This completes the proof.

While a quotient module of a module is co-coatomically ss-supplemented, the module itself does not necessarily have to be co-coatomically ss-supplemented.

Example 3. Let us assume that S signifies the ring $Q[[x]]$ of all power series $\sum_{\lambda=0}^{\infty} k_\lambda x^\lambda$ where x is an indeterminate and coefficients belong to a field Q. The ring S is local [1]. Hence the module ${}_SS$ is supplemented, and so S is a semi-perfect ring by Section 42.6 [8]. Note that

$$Rad(S) = \{\sum_{\lambda=1}^{\infty} k_\lambda x^\lambda \mid k_\lambda \in Q\} = Sx$$

is not left t-nilpotent [1]. Thus, S is not a left perfect ring by Section 43.9 [8]. Since S is semi-perfect, $S/Rad(S)$ is semi-simple. If $E = {}_SS^{(\mathbb{N})}$ and $G = Rad({}_SS^{(\mathbb{N})})$, E/G is a co-coatomically ss-supplemented module as it is semi-simple. It is noted that $G \leq_{cc} E$ as E/G is semi-simple. According to Theorem 1 [16], G does not have a supplement in E and thus, it does not have an ss-supplement in E. Thus, E is not a co-coatomically ss-supplemented module.

Theorem 2. Suppose that $G \leq E$. When G is a co-coatomically ss-supplemented module and E/G does not have any maximal submodule, E is a co-coatomically ss-supplemented module.

Proof. Let $H \leq_{cc} E$. Then $E/(G + H) = (E/H)/((G + H)/H)$ is coatomic since E/H is coatomic. Since E/G gets no maximal submodule, then $E/(G + H)$ also gets no maximal submodule. Therefore, we conclude that $E = G + H$. Since G is a co-coatomically ss-supplemented module and $H \leq_{cc} E$, then H has an ss-supplement in E according to Lemma 1. Hence E is a co-coatomically ss-supplemented module.

Corollary 5. For a module E, when $E/Soc(E)$ does not include a maximal submodule, E is a co-coatomically ss-supplemented module.

Proof. Explicitly, $Soc(E)$ is a co-coatomically ss-supplemented submodule of E. By the hypothesis, E is a co-coatomically ss-supplemented module according to Theorem 2.

Proposition 5. When a co-coatomically ss-supplemented module E includes a maximal submodule, E also includes a strongly local submodule.

Proof. Suppose that $H \leq E$ is maximal. Then $H \leq_{cc} E$. By assumption, there is a submodule W of E such that $E = H + W, H \cap W$ is semi-simple and $H \cap W \ll W.$ Hence W is a strongly local submodule of E by Proposition 12 [6].

RINGS WHOSE MODULES ARE CO-COATOMICALLY *SS*-SUPPLEMENTED

In this section it is firstly aimed to provide a characterisation of rings whose modules are co-coatomically ss-supplemented. However, initially, we need to articulate a lemma. The remainder of the section will examine the behaviour of co-coatomically ss-supplemented modules over various rings.

Lemma 2. Given a coatomic module E, the statements below are equivalent:

(1) E is the sum of all strongly local submodules.
(2) E is an ss-supplemented module.
(3) E is a co-coatomically ss-supplemented module.
(4) Each co-coatomic (cofinite, maximal) submodule of E has an ss-supplement in E.

Proof. (1) $\Leftrightarrow$ (2): By Corollary 31 [6].
The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are explicit.
(4) $\Rightarrow$ (1): Suppose that G is the sum of all strongly local submodules of E and $G \neq E$. Because of coatomic E, there is a maximal submodule H of E such that $G \leq H$. Note that $H \leq_{cc} E$. By (4), H has an ss-supplement W in E. Thus, according to Proposition 12 [6], we conclude that W is a strongly local module. Therefore, the inclusions $W \leq G \leq H$ arise, and this is a contradiction.

G is called *co-closed submodule* in a module E when G does not have any proper submodule K for which $G/K \ll E/K$ [9]. It is remembered that a co-closed submodule of a coatomic module is coatomic as given in Lemma 4.1 [17]. Thus, we obtain the direct consequence below.

Corollary 6. Suppose that E is a coatomic module and G is a co-closed submodule of E. In that case G is a co-coatomically ss-supplemented module if and only if G is an ss-supplemented module.

Theorem 3. For any ring S, the conditions below are equivalent:

(1) ${}_SS$ is a co-coatomically ss-supplemented module.
(2) S is a semi-perfect ring and $Rad(S) \subseteq Soc({}_SS)$.
(3) S is a semi-local ring and $Rad(S) \subseteq Soc({}_SS)$.
(4) Each projective left S-module is co-coatomically ss-supplemented.
(5) Each left S-module is co-coatomically ss-supplemented.
(6) Each left S-module is cofinitely ss-supplemented.
(7) Each left S-module is the sum of all strongly local submodules.
(8) ${}_SS$ is a finite sum of strongly local submodules.
(9) Each maximal left ideal of S has an ss-supplement in S.

Proof. (1) $\Longrightarrow$ (2): Since ${}_S S$ is a co-coatomically *ss*-supplemented module, then each left ideal J of S has an *ss*-supplement in S. Thus, ${}_S S$ is a supplemented module and so S is a semi-perfect ring by Section 42.6 [8]. Moreover, since S is an *ss*-supplement of $Rad(S)$ in S, then we conclude that $Rad(S) \subseteq Soc({}_S S)$.
(2) $\Longrightarrow$ (3): By Section 42.6 [8].
(3) $\Longrightarrow$ (4): By Theorem 41 [6], the result holds.
(4) $\Longrightarrow$ (5): The claim holds from Section 18.6 [8] and Proposition 2.
The implications (5) $\Longrightarrow$ (6) and (7) $\Longrightarrow$ (8) are explicit.
(6) $\Longrightarrow$ (7): By Lemma 5 [10].
(8) $\Longrightarrow$ (9): By Corollary 31 [6].
(9) $\Longrightarrow$ (1): The assertion can be seen from Lemma 2.

Now an example of a module that is co-coatomically supplemented but not co-coatomically *ss*-supplemented is provided. The notation $P(E)$ is used to represent the sum of all radical submodules of a module E, defined as $P(E) = \sum\{G \leq E \mid Rad(G) = G\}$. It is obvious that $P(E)$ is the largest radical submodule of E. A module E is called *reduced* when $P(E) = 0$.

Example 4 [18]**.** Consider the polynomial ring $Q[x_1, x_2, \ldots]$ with countably many indeterminates x_k where Q is a field and $k \in \mathbb{Z}^+$. Let $J = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ be the ideal generated by x_1^2 and $x_{k+1}^2 - x_k$ for each $k \in \mathbb{Z}^+$. Then the quotient ring $S = Q[x_1, x_2, \ldots]/J$ is local with the unique maximal ideal $M = (x_1, x_2, \ldots)/J$ generated by all $\overline{x_k} = x_k + J$, $k \in \mathbb{Z}^+$. Note that $Rad(M) = M \neq 0$ by Example 6.2 [18]. If $E = {}_S S$, then E is a co-coatomically supplemented module because local modules are supplemented. However, since $P(E) = Rad(M) = M \neq 0$, then E is not reduced. Thus, E is not strongly local, which means that $Rad(E)$ is not semi-simple by Proposition 6 [6]. Consequently, E is not a co-coatomically *ss*-supplemented module by Theorem 3.

As an extension of the concept of left $V-$rings, a ring S is called a *left weakly* $V-$*ring* (abbreviated $WV-$*ring*) when each simple S-module is S/J-injective for any left ideal J of S such that S/J is a proper quotient ring [19].

Proposition 6. Suppose that E is an S-module over the left WV-ring S. E is a co-coatomically *ss*-supplemented module if and only if E is a co-coatomically supplemented module.

Proof. Assume that E is a co-coatomically supplemented module and $G \leq_{cc} E$. Then by the hypothesis, G has a supplement W in E, i.e. $E = G + W$ and $G \cap W \ll W$. Note that $G \cap W \leq Rad(W)$. Now two cases arise:

Case 1: When S is a left $V-$ring, according to Section 23.1 [8] we conclude that $Rad(W) = 0$, and so $E = G \oplus W$. Hence E is a co-coatomically *ss*-supplemented module.

Case 2: When S is not a left $V-$ring, $Rad(S)$ is simple and $S/Rad(S)$ is a left $V-$ring [19]. Therefore, we infer that S is a left good ring from Section 23.7 [8]. Thus, we have $G \cap W \leq Rad(W) = Rad(S)W \leq Soc({}_S S)W \leq Soc(W)$ from Section 23.7 [8]. Hence E is a co-coatomically *ss*-supplemented module. The other part of the proof is explicit.

The following theorem belongs to Zöschinger [20]. Using this theorem, the aim is to illustrate an alternative characterisation of left perfect rings through co-coatomically *ss*-supplemented modules. A module E is called $\sum$-*self-projective* when for each index set Λ, the direct sum module $E^{(\Lambda)}$ is self-projective.

Theorem 4 [20]. When the module E is $\sum$-self-projective and $G \leq Rad(E)$, G has a supplement in E, so G is a small submodule of E.

Theorem 5. Each left S-module is co-coatomically ss-supplemented if and only if the ring S is a left perfect ring with $Rad(S) \subseteq Soc({}_SS)$.

Proof. The sufficiency is obviously seen by Theorem 41 [6]. To demonstrate the necessity, assume that each left S-module is co-coatomically ss-supplemented. Then each left S-module is cofinitely ss-supplemented. Therefore, according to Theorem 3 [10], S is a semi-perfect ring with $Rad(S) \subseteq Soc({}_SS)$. By Section 42.6 [8] we conclude that the quotient ring $S/Rad(S)$ is semi-simple, and hence ${}_SS^{(\mathbb{N})}/Rad({}_SS^{(\mathbb{N})})$ is semi-simple as an $S/Rad(S)$-module. Thus, we obtain $Rad({}_SS^{(\mathbb{N})}) \leq_{cc} {}_SS^{(\mathbb{N})}$. By the hypothesis, $Rad({}_SS^{(\mathbb{N})})$ has an ss-supplement in ${}_SS^{(\mathbb{N})}$. By applying Theorem 4 we conclude that $Rad({}_SS^{(\mathbb{N})}) \ll {}_SS^{(\mathbb{N})}$. Therefore, by deducing that $S/Rad(S)$ is a left semi-simple ring and $Rad({}_SS^{(\mathbb{N})}) \ll {}_SS^{(\mathbb{N})}$, ${}_SS$ is left perfect by Section 43.9 [8]. As a result, S is a left perfect ring.

A specific class of the left perfect rings is introduced via ss-semi-local modules as follows. A ring S is called *left ss-perfect*, provided that ${}_SS$ is an ss-semi-local module, or equivalently each left S-module is ss-semi-local [15].

Corollary 7. Each left S-module is co-coatomically ss-supplemented if and only if the ring S is left ss-perfect.

Proof. To prove the necessity, assume that each left S-module is co-coatomically ss-supplemented. Then by Theorem 5, S is a left perfect ring with semi-simple radical. Thus, each left S-module is ss-supplemented by Theorem 41 [6], and so is ss-semi-local. Hence according to Theorem 2.15 [15], S is a left ss-perfect ring. The sufficiency can be seen by Theorem 5 and Proposition 2.17 [15].

An S-module E is called *radical supplemented* when $Rad(E)$ has a supplement in E [20].

Proposition 7. Suppose that S is a discrete valuation ring whose maximal ideal is St where $t \in S$ is the unique prime element and E is an S-module. İt is assumed that the radical of each coatomic S-module is semi-simple. Then the basic submodule of E is coatomic if and only if E is a co-coatomically ss-supplemented module.

Proof. ($\Longrightarrow$) Let $H \leq_{cc} E$ and G be the basic submodule of E. Then $E/(H + G)$ is also coatomic. Thus, $E/(H + G)$ is reduced by Lemma 2.1 [14]. On the other hand, since E/G is divisible, then $E/(H + G)$ is divisible. Therefore, $E/(H + G) = 0$, i.e. $E = H + G$. By the hypothesis, G is coatomic and hence it is supplemented by Lemma 2.1 [14]. Thus, by assumption G is an ss-supplemented module according to Theorem 20 [6]. By applying Lemma 1, H has an ss-supplement in E. Hence E is a co-coatomically ss-supplemented module.
($\Longleftarrow$) Since S is a discrete valuation ring, $E/Rad(E) = E/tE$ is semi-simple, and so it is coatomic. By the hypothesis, since E is a co-coatomically ss-supplemented module, then tE has an ss-supplement in E. Thus, E is a radical supplemented module. Hence according to Theorem 3.1 [20], the basic submodule of E is coatomic.

A module E is called *ss-radical supplemented* when $Rad(E)$ has an ss-supplement in E [21].

Corollary 8. Suppose that E is an S-module over discrete valuation ring S and that the radical of each coatomic S-module is semi-simple. Then the conditions below are equivalent:

(1) E is a co-coatomically ss-supplemented module.

(2) E is a radical supplemented module.
(3) E is an *ss*-radical supplemented module.

Proof. (1) ⟺ (2): This can be seen by Proposition 7 and Theorem 3.1 [20].
(1) ⟹ (3): Since S is a discrete valuation ring, $E/Rad(E)$ is coatomic as it is semi-simple. By the hypothesis, since E is a co-coatomically *ss*-supplemented module, then $Rad(E)$ has an *ss*-supplement in E. Thus, E is an *ss*-radical supplemented module.
(3) ⟹ (2): Obvious.

Corollary 9. Suppose that E is an S-module over discrete valuation ring S and that the radical of each coatomic S-module is semi-simple. Then E is a co-coatomically *ss*-supplemented module if and only if $E = T(E) \oplus X$ where $T(E)$ is the torsion part of E, the reduced part of $T(E)$ is bounded and $X/Rad(X)$ is finitely generated.

Proof. This can be proved by Corollary 8 and Theorem 3.1 [20].

In Lemma 3.2 [20], certain properties below were presented for radical supplemented modules over discrete valuation rings. Through the application of Corollary 8, it has been established that radical supplemented modules coincide with co-coatomically *ss*-supplemented modules over discrete valuation rings, provided a specific condition is met. Therefore, co-coatomically *ss*-supplemented modules exhibit these properties under the specified condition over discrete valuation rings.

Corollary 10. Suppose that S is a discrete valuation ring, E is an S-module and that the radical of each coatomic S-module is semi-simple. Then the assertions below hold.

(1) Suppose that E is a co-coatomically *ss*-supplemented module and $G \leq E$ is pure. In that case G is a co-coatomically *ss*-supplemented module.
(2) Suppose that E and E'/E are co-coatomically *ss*-supplemented modules. Then E' is a co-coatomically *ss*-supplemented module.
(3) When E is a co-coatomically *ss*-supplemented module and E/G is reduced for some submodule G of E, then G is also a co-coatomically *ss*-supplemented module.
(4) Each submodule of E is a co-coatomically *ss*-supplemented module if and only if $T(E)$ is a supplemented module and $E/T(E)$ has finite rank where $T(E)$ is the torsion part of E.

Proof. (1) E is a radical supplemented module according to Corollary 8. Thus, G is a radical supplemented module by Lemma 3.2 [20]. Hence G is a co-coatomically *ss*-supplemented module according to Corollary 8.
(2) E and E'/E are radical supplemented modules according to Corollary 8. Therefore, E' is a radical supplemented module by Lemma 3.2 [20]. Hence E' is a co-coatomically *ss*-supplemented module according to Corollary 8.
(3) Since E is a co-coatomically *ss*-supplemented S-module, then according to Corollary 8, E is a radical supplemented module. By the hypothesis, since E/G is reduced, then G is a radical supplemented module by Lemma 3.2 [20]. Hence G is a co-coatomically *ss*-supplemented module according to Corollary 8.
(4) (⟹) Since each submodule of E is co-coatomically *ss*-supplemented, then each submodule of E is a radical supplemented module according to Corollary 8. Therefore, $T(E)$ is a supplemented module and $E/T(E)$ has finite rank where $T(E)$ is the torsion part of E by Lemma 3.2 [20].

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($\Longleftarrow$) By the hypothesis, each submodule of E is a radical supplemented module by Lemma 3.2 [20]. Thus, each submodule of E is a co-coatomically ss-supplemented module according to Corollary 8.

CO-COATOMICALLY *SS*-SEMI-LOCAL MODULES

For a module E in the subsequent discussion, we call it *co-coatomically ss-semi-local* when each $G \leq_{cc} E$ has a weak ss-supplement W within E, i.e. $E = G + W$, $G \cap W$ is semi-simple and $G \cap W \ll E$. Explicitly, each co-coatomically ss-supplemented module is a co-coatomically ss-semi-local module. However, as demonstrated in Example 1, the $\mathbb{Z}$-module $\mathbb{Q}$ is a co-coatomically ss-semi-local module despite not being ss-semi-local as indicated in Example 2.2 [15].

Lemma 3. Suppose that E is a coatomic module. Then E is a co-coatomically ss-semi-local module if and only if E is an ss-semi-local module.

Proof. To demonstrate the necessity, assume that the coatomic module E is co-coatomically ss-semi-local and $G \leq E$. Then the quotient module E/G is coatomic. Therefore, G has a weak ss-supplement in E by assumption. Hence E is an ss-semi-local module.

Theorem 6. The statements below for a ring S are equivalent:

(1) ${}_SS$ is an ss-semi-local module.
(2) Each left S-module is an ss-semi-local module.
(3) Each left S-module is a co-coatomically ss-semi-local module.
(4) S is a semi-local ring and $Rad(S) \subseteq Soc({}_SS)$.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (4)$: By Theorem 2.15 [15].
$(2) \Rightarrow (3)$: Obvious.
$(3) \Rightarrow (4)$: By (3), the coatomic module ${}_SS$ is a co-coatomically ss-semi-local module. Then according to Lemma 3, ${}_SS$ is an ss-semi-local module. Thus, ${}_SS$ is a semi-local module and so S is a semi-local ring. Moreover, as ${}_SS$ is a weak ss-supplement of $Rad(S)$ in ${}_SS$, $Rad(S)$ is semi-simple.

Proposition 8. Suppose that E is a projective module over a ring S with semi-simple radical. When the quotient module E/G is a co-coatomically ss-semi-local module with $G \ll E$, E is a co-coatomically ss-semi-local module.

Proof. Let $L \leq_{cc} E$. Thus, $(L+G)/G \leq_{cc} E/G$ because $L+G \leq_{cc} E$. By the hypothesis, there exists a weak ss-supplement W/G of $(L+G)/G$ in E/G, i.e. $E/G = (L+G)/G + W/G$, $((L+G) \cap W)/G$ is semi-simple and $((L+G) \cap W)/G \ll E/G$. Since $G \ll E$, then we have $(W \cap L) + G = W \cap (L+G) \ll E$ from Section 2.2 [9]. Therefore, we obtain $E = L + W$ and $L \cap W \ll E$. Thus, $L \cap W \leq Rad(E)$. Since E is a projective module, then by assumption, $L \cap W \leq Rad(S)E \leq Soc({}_SS)E = Soc(E)$. Hence W is a weak ss-supplement of L in E, and so E is a co-coatomically ss-semi-local module.

Corollary 11. Suppose that S is a ring with semi-simple radical and that E is a co-coatomically ss-semi-local S-module. Then the projective cover of E is a co-coatomically ss-semi-local module.

Proof. By Proposition 8.

Proposition 9. Co-coatomically ss-semi-local modules exhibit transfer properties through their quotient modules.

Proof. Suppose that E is a co-coatomically *ss*-semi-local module and $G \leq E$. Then any co-coatomic submodule of E/G has the form L/G, where $L \leq_{cc} E$. By the hypothesis, L has a weak *ss*-supplement W in E, i.e. $E = L + W$, $L \cap W$ is semi-simple and $L \cap W \ll E$. For this reason, $E/G = L/G + (W + G)/G$. Also by considering the canonical projection $\pi : E \to E/G$, we conclude that $((W + G) \cap L)/G = ((L \cap W) + G)/G = \pi(L \cap W) \ll \pi(E) = E/G$ from Section 2.2 [9]. Moreover, $((W + G) \cap L)/G$ is semi-simple from Corollary 8.1.5 [1]. Hence L/G has a weak *ss*-supplement $(W + G)/G$ in E/G, and so E/G is a co-coatomically *ss*-semi-local module.

Let E be a module and $G, H \leq E$. G is called a *complement submodule* in E of H when it is maximal element in the set of whole submodules L of E such that $H \cap L = 0$ [9]. By Section 1.10 [9] it is known that a submodule of E is a complement if and only if it is closed. Over a Dedekind domain, closed submodules and co-closed submodules coincide as indicated in Lemma 3.3 [14]. Consequently, a torsion submodule $T(E)$ of a module E is a co-closed submodule of E over a Dedekind domain, as it is closed as mentioned in Example 6.34 [22].

Proposition 10. Suppose that *S* is a non-local Dedekind domain and E is a torsion *S*-module. Then E is a co-coatomically *ss*-semi-local module if and only if E is a co-coatomically *ss*-supplemented module.

Proof. ($\Longrightarrow$) Let $G \leq_{cc} E$. By the hypothesis, G has a weak *ss*-supplement W in E. Thus, we have $E = G + W$, $G \cap W$ is semi-simple and $G \cap W \ll E$. Since E is a torsion module, then W is too. Hence $W \leq E$ is co-closed. Therefore, $G \cap W \ll W$ by Section 3.7 [9]. Hence E is a co-coatomically *ss*-supplemented module.
($\Longleftarrow$) Obvious.

Proposition 11. Suppose that E is a reduced *S*-module where *S* is non-local Dedekind domain. When E is a co-coatomically *ss*-supplemented module and $T(E)$ has a weak *ss*-supplement in E, $E/T(E)$ is divisible and $T(E)/(T(E) \cap G)$ is co-coatomically *ss*-semi-local for any submodule *G* of E.

Proof. We claim that $E/T(E)$ is a radical module. To demonstrate this, assume on the contrary that there is a maximal $H \leq E$ including $T(E)$. By the hypothesis, H has an *ss*-supplement W in E. Since H is a maximal submodule, then W is a strongly local module according to Proposition 12 [6], so that W is cyclic, and for some left ideal J of *S*, we deduce that $W \cong S/J$. However, $J \neq 0$ as *S* is not local. Hence W is a torsion module. From this, we reach the contradiction that $W \subseteq T(E)$. So $E/T(E)$ does not have any maximal submodule, i.e. $E/T(E)$ is divisible by Lemma 4.4 [4]. On the other hand, $T(E)$ is closed from Example 6.34 [22]. Thus, $T(E)$ is co-closed according to Lemma 3.3 [14]. Since $T(E)$ has a weak *ss*-supplement, then we can conclude that $T(E)$ is a supplement submodule in E according to Section 20.2 [9]. Hence there is a submodule G of E such that $E = T(E) + G$, $T(E) \cap G \ll T(E)$. Note here that $E/G \cong T(E)/(T(E) \cap G)$. Since E is also co-coatomically *ss*-semi-local module, then $T(E)/(T(E) \cap G)$ is a co-coatomically *ss*-semi-local module by Proposition 9.

Proposition 12. Suppose that *S* is a Dedekind domain and E is an *S*-module. When $T(E) = E_1 \oplus E_2$, where E_1 is semi-simple and E_2 and $E/T(E)$ are divisible, E is a co-coatomically *ss*-supplemented module.

Proof. By the hypothesis, co-coatomic submodules of E are direct summands from Theorem 4.1 [5]. Hence E is a co-coatomically *ss*-supplemented module.

Proposition 13. Suppose that S is a Dedekind domain which is not a field and E is a torsion-free left S-module. If E is a co-coatomically ss-supplemented module, then E is a divisible module.

Proof. Let E be a co-coatomically ss-supplemented module and $H \leq E$ be maximal. Then H has an ss-supplement W in E by assumption. Note that $H \cap W \leq Soc(E) \leq T(E)$ due to S being a domain, not a field. Then $E = H \oplus W$ as $T(E) = 0$. This implies that E is a divisible module according to Lemma 6.10 [17].

CONCLUSIONS

In this note, ss-supplemented modules defined by Kaynar et al. [6] are viewed from the same point of view and co-coatomic submodules defined by Alizade and Güngör [5] having an ss-supplement are considered instead of each submodule of the module. Over a semi-perfect ring whose radical is semi-simple, each module is co-coatomically ss-supplemented. In addition, co-coatomically ss-semi-local modules are defined by weakening ss-semi-local module structure defined by Olgun and Türkmen [15]. A torsion module over a non-local Dedekind domain being co-coatomically ss-supplemented is equivalent to being co-coatomically ss-semi-local.

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