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BOUNDEDNESS OF INTRINSIC SQUARE FUNCTIONS AND THEIR COMMUTATORS ON GENERALIZED WEIGHTED ORLICZ–MORREY SPACES

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Abstract. We shall investigate the boundedness of the intrinsic square functions and their commutators on generalized weighted Orlicz–Morrey spaces $M_w^{\Phi,\varphi}(\mathbb{R}^n)$. In all the cases, the conditions for the boundedness are given in terms of Zygmund-type integral inequalities on weights φ without assuming any monotonicity property of $\varphi(x,\cdot)$ with x fixed.

1. Introduction

In the present paper, we are concerned with the intrinsic square functions, which Wilson introduced initially [24, 25]. For $0 < \alpha \le 1$, let C_α be the family of Lipschitz functions $\phi : \mathbb{R}^n \to \mathbb{R}$ of order α with the homogeneous norm 1 such that the support of ϕ is contained in the closed ball $\{x : |x| \le 1\}$, and that $\int_{\mathbb{R}^n} \phi(x)dx = 0$. For $(y,\ t) \in \mathbb{R}^{n+1}_+$ and $f \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$, set

$$A_\alpha f(t,\ y) \equiv \sup_{\phi \in C_\alpha} |f * \phi_t(y)|,$$

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where $\phi_t \equiv t^{-n}\phi\left(\frac{\cdot}{t}\right)$. Let β be an auxiliary parameter. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula;

$$G_{\alpha,\beta}(f)(x) \equiv \left(\iint_{\Gamma_\beta(x)} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma_\beta(x) \equiv \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}$. Write $G_\alpha(f) = G_{\alpha,1}(f)$.

Everywhere in the sequel, $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x and we let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$; $|B(x,r)| = v_n r^n$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. We recall generalized weighted Orlicz–Morrey spaces, on which we work in the present paper.

Definition 1.1 (Generalized weighted Orlicz–Morrey Space)**.** Let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, let w be non-negative measurable function on $\mathbb{R}^n$ and Φ any Young function. Denote by $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ the generalized weighted Orlicz–Morrey space, the space of all functions $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$ such that

$$\|f\|_{M^{\Phi,\varphi}_w} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\big(w(B(x,r))^{-1}\big)\, \|f\|_{L^{\Phi}_w(B(x,r))},$$

where $\|f\|_{L^{\Phi}_w(B(x,r))} \equiv \inf\left\{\lambda > 0 \,:\, \int_{B(x,r)} \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx \le 1\right\}$.

According to this definition, we recover the generalized weighted Morrey space $M^{p,\varphi}_w(\mathbb{R}^n)$ by the choice $\Phi(r) = r^p$, $1 \le p < \infty$. If $\Phi(r) = r^p$, $1 \le p < \infty$ and $\varphi(x,r) = r^{-\frac{\lambda}{p}}$, $0 \le \lambda \le n$, then $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ coincides with the weighted Morrey space $M^{p,\varphi}_w(\mathbb{R}^n)$ and if $\varphi(x,r) = \Phi^{-1}(w(B(x,r)^{-1}))$, then $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ coincides with the weighted Orlicz space $L^{\Phi}_w(\mathbb{R}^n)$. When $w = 1$, then $L^{\Phi}_w(\mathbb{R}^n)$ is abbreviated to $L^{\Phi}(\mathbb{R}^n)$. The space $L^{\Phi}(\mathbb{R}^n)$ is the classical Orlicz space.

Our first theorem of the present paper is the following one:

Theorem 1.2. *Let $\alpha \in (0,1]$ and $1 < p_0 \le p_1 < \infty$. Let Φ be a Young function which is lower type p_0 and upper type p_1. Namely,*

$$\Phi(st_0) \le C{t_0}^{p_0}\Phi(s), \quad \Phi(st_1) \le C{t_1}^{p_1}\Phi(s)$$

for all $s > 0$ and $0 < t_0 \le 1 \le t_1 < \infty$. Assume that $w \in A_{p_0}$ and that the measurable functions $\varphi_1, \varphi_2 : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and Φ satisfy the condition;

$$\int_r^\infty \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(w(B(x_0,s))^{-1}\big)} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.1}$$

where C does not depend on x and r. Then G_α is bounded from $M^{\Phi,\varphi_1}_w(\mathbb{R}^n)$ to $M^{\Phi,\varphi_2}_w(\mathbb{R}^n)$.

Theorem 1.2 extends the result below due to Liang, Nakai, Yang and Zhou.

Theorem 1.3. [13] *Let $\alpha \in (0,1]$ and $1 < p_0 \le p_1 < \infty$. Let Φ be a Young function which is lower type p_0 and upper type p_1. Then G_α is bounded from $L^{\Phi}(\mathbb{R}^n)$ to itself.*

The function $G_{\alpha,\beta}(f)$ is independent of any particular kernel, such as the Poisson kernel. It dominates pointwise the classical square function (Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ depends on kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different β:

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x) \ .$$

See [24] for details.

The intrinsic Littlewood–Paley g-function is defined by

$$g_\alpha f(x) \equiv \left(\int_0^\infty (A_\alpha f(t,x))^2 \frac{dt}{t} \right)^{\frac{1}{2}} .$$

Also, the intrinsic $g^*_{\lambda,\alpha}$ function is defined by

$$g^*_{\lambda,\alpha} f(x) \equiv \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} .$$

About this intrinsic Littlewood–Paley g-function, we shall prove the following boundedness property:

Theorem 1.4. *Let $\alpha \in (0,1]$, $1 < p_0 \le p_1 < \infty$ and $\lambda \in \left(3 + \dfrac{2\alpha}{n}, \infty\right)$. Let also Φ be a Young function which is lower type p_0 and upper type p_1. Assume that $w \in A_{p_0}$ and that the functions $\varphi_1, \varphi_2 : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and Φ satisfy the condition (1.1). Then $g^*_{\lambda,\alpha}$ is bounded from $M^{\Phi,\varphi_1}_w(\mathbb{R}^n)$ to $M^{\Phi,\varphi_2}_w(\mathbb{R}^n)$.*

In [24], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 1.2, we have the following result:

Corollary 1.5. *Let $\alpha \in (0,1]$ and $1 < p_0 \le p_1 < \infty$. Let also Φ be a Young function which is lower type p_0 and upper type p_1. Assume in addition that $w \in A_{p_0}$ and that the functions $\varphi_1, \varphi_2 : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and Φ satisfy the condition (1.1). Then g_α is bounded from $M^{\Phi,\varphi_1}_w(\mathbb{R}^n)$ to $M^{\Phi,\varphi_2}_w(\mathbb{R}^n)$.*

Let b be a locally integrable function on $\mathbb{R}^n$. Setting

$$A_{\alpha,b} f(t,y) \equiv \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} [b(y) - b(z)] \phi_t(y-z) f(z) dz \right|,$$

we can define the commutators $[b, G_\alpha]$, $[b, g_\alpha]$ and $[b, g^*_{\lambda,\alpha}]$ by;

$$[b, G_\alpha] f(x) \equiv \left(\iint_{\Gamma(x)} (A_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

$$[b, g_\alpha] f(x) \equiv \left(\int_0^\infty (A_{\alpha,b} f((t,x))^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$[b, g^*_{\lambda,\alpha}] f(x) \equiv \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} (A_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

respectively. A function $f \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$ is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ [9] if

$$\|f\|_* \equiv \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty,$$

where $f_{B(x,r)} \equiv \dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} f(y)dy.$

About the boundedness of $[b, G_\alpha]$ on Orlicz spaces, we shall invoke the following result:

Theorem 1.6. [13] *Let $\alpha \in (0,1]$, $1 < p_0 \le p_1 < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ be a Young function which is lower type p_0 and upper type p_1 and $w \in A_{p_0}$. Then $[b, G_\alpha]$ is bounded on $L^\Phi_w(\mathbb{R}^n)$.*

About the commutator above, we shall prove the following boundedness property in the present paper:

Theorem 1.7. *Suppose that we are given parameters α, p_0, p_1 and functions b, w, φ, φ_2 with the following properties:*

(1) $\alpha \in (0,1], 1 < p_0 \le p_1 < \infty$,
(2) $b \in \mathrm{BMO}(\mathbb{R}^n)$
(3) *Φ is a Young function which is lower type p_0 and upper type p_1.*
(4) *$w \in A_{p_0}$,*
(5) *φ_1, φ_2 and Φ satisfy the condition;*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,s))^{-1}\big)}\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.2}$$

where C does not depend on x and r.

Then the operator $[b, G_\alpha]$ is bounded from $M^{\Phi,\varphi_1}_w(\mathbb{R}^n)$ to $M^{\Phi,\varphi_2}_w(\mathbb{R}^n)$.

In [24], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. From the definition of the commutators, the same can be said for $[b, G_\alpha]$ and $[b, g_\alpha]$. Thus, as a consequence of Theorem 1.2, we have the following result:

Corollary 1.8. *Let $\alpha \in (0,1]$, $1 < p_0 \le p_1 < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ be a Young function which is lower type p_0 and upper type p_1. Assume $w \in A_{p_0}$ and that the functions φ_1, φ_2 and Φ satisfy the condition (1.2), then $[b, g_\alpha]$ is bounded from $M^{\Phi,\varphi_1}_w(\mathbb{R}^n)$ to $M^{\Phi,\varphi_2}_w(\mathbb{R}^n)$.*

Remark 1.9. By going through an argument similar to the above proofs and that of Theorem 1.4, we can also show the boundedness of $[b, g^*_{\lambda,\alpha}]$. We omit the details.

Here let us make a historical remark. Wilson [24] proved that G_α is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 < \alpha \le 1$. After that, Huang and Liu [7] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [22] and [23], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [21], Wang considered intrinsic functions and commutators generated by BMO functions on weighted Morrey spaces. In [26], Wu proved the boundedness of intrinsic square functions and their commutators inspired by the ideas of Guliyev [3, 4]. In [13], Liang et al. studied the boundedness of these operators on Musielak–Orlicz Morrey spaces. Orlicz–Morrey spaces were initially introduced and studied by Nakai in [16]. Also for the boundedness of the operators of harmonic analysis on Orlicz–Morrey spaces, see also [1, 16, 20]. Our definition of Orlicz–Morrey spaces (see [1]) is different from those by Nakai [16] and Sawano et al. [20] used recently in [2].

Here and below, we use the following notations: By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of relevant quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

Finally, we describe how we organize the present paper. In Section 2 we recall some preliminary facts such as Young functions and John–Nirenberg inequality. Section 3 is devoted to the proof of Theorems 1.2 and 1.4. We prove Theorem 1.7 in Section 4.

2. Preliminaries

As is well known, classical Morrey spaces stemmed from Morrey's observation for the local behavior of solutions to second order elliptic partial differential equations [15]. We recall its definition:

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^{p,\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty \right\},$$

where $0 \le \lambda \le n$, $1 \le p < \infty$. The scale $M_{p,\lambda}(\mathbb{R}^n)$ covers the $L^p(\mathbb{R}^n)$ in the sense that $M_{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

We are thus oriented to a generalization of the parameters p and λ.

2.1. **Young functions and Orlicz spaces.** We next recall the definition of Young functions.

Definition 2.1. A function $\Phi : [0,+\infty) \to [0,\infty]$ is called a Young function, if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \infty$.

The convexity and the condition $\Phi(0) = 0$ force any Young function to be increasing. In particular, if there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then it follows that $\Phi(r) = +\infty$ for $r \ge s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Orlicz spaces, introduced in [17, 18], also generalize Lebesgue spaces. They are useful tools in harmonic analysis and these spaces are applied to many other problems in harmonic analysis. For example, the Hardy–Littlewood maximal operator is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, but not on $L^1(\mathbb{R}^n)$. Using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ more precisely.

In the present paper we are concerned with the weighted setting.

Definition 2.2 (Weighted Orlicz Space)**.** For a Young function Φ and a non-negative measurable function w on $\mathbb{R}^n$, the set

$$L^{\Phi}_w(\mathbb{R}^n) \equiv \left\{ f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)w(x)dx < +\infty \text{ for some } k > 0 \right\}$$

is called the weighted Orlicz space. The local weighted Orlicz space $L_w^{\Phi,\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L_w^{\Phi}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$ and this space is endowed with the natural topology.

Note that $L_w^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_w^\Phi} \equiv \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)dx \le 1\right\}.$$

See [19, Section 3, Theorem 10] for example. In particular, we have

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_w^\Phi}}\Big) w(x)dx \le 1.$$

If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L_w^\Phi = L_w^p(\mathbb{R}^n)$ with norm coincidence. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_w^\Phi = L_w^\infty(\mathbb{R}^n)$.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) \equiv \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We also note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty. \tag{2.1}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition and it fails the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but it fails the Δ_2-condition.

Definition 2.3. A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$ (resp. $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \le Ct^p\Phi(s).$$

Remark 2.4. If Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$; see [11] for example.

About the norm $\|f\|_{M_w^{\Phi,\varphi}}$, we have the following equivalent expression: If Φ satisfies the Δ_2-condition, then the norm $\|f\|_{M_w^{\Phi,\varphi}}$ is equivalent to the norm

$$\|f\|_{\overline{M}^{\Phi,\varphi}(w)} \equiv \inf\Big\{\lambda > 0 : \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\big(w(B(x,r))^{-1}\big) \times \int_{B(x,r)} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)dx \le 1\Big\}.$$

See [14, p. 416]. The latter was used in [14, 16, 20], see also references therein. For Φ and $\widetilde{\Phi}$, we have the following estimate, whose proof is similar to [12, Lemmas 4.2]. So, we omit the details.

Lemma 2.5. *Let $0 < p_0 \le p_1 < \infty$ and let $\widetilde{C}$ be a positive constant. Suppose that we are given a non-negative measurable function w on $\mathbb{R}^n$ and a Young function Φ which is lower type p_0 and upper type p_1. Then there exists a positive constant C such that for any ball B of $\mathbb{R}^n$ and $\mu \in (0,\infty)$*

$$\int_B \Phi\left(\frac{|f(x)|}{\mu}\right) w(x)dx \le \widetilde{C}$$

implies that $\|f\|_{L^\Phi_w(B)} \le C\mu$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) \equiv \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & \text{if} \quad r \in [0,\infty), \\ +\infty & \text{if} \quad r = +\infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and it satisfies $\widetilde{\widetilde{\Phi}} = \Phi$. Here we recall three examples.

Example 2.6.

(1) If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$.
(2) If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$.
(3) If $\Phi(r) = e^r - r - 1$, then a calculation shows $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$.

Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is also known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.2}$$

Note that Young functions satisfy the properties;

$$\Phi(\alpha t) \le \alpha\Phi(t)$$

for all $0 \le \alpha \le 1$ and $0 \le t < \infty$, and

$$\Phi(\beta t) \ge \beta\Phi(t)$$

for all $\beta > 1$ and $0 \le t < \infty$.

The following analogue of the Hölder inequality is known, see [11, 19].

Theorem 2.7. *For a non-negative measurable function w on $\mathbb{R}^n$, a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid for all measurable functions f and g: $\|fg\|_{L^1(\mathbb{R}^n)} \le 2\|f\|_{L^\Phi_w}\|w^{-1}g\|_{L^{\widetilde{\Phi}}_w}$.*

An analogy of Theorem 2.7 for weak type spaces is available. If we define

$$\|f\|_{WL^\Phi_w} \equiv \sup_{\lambda>0} \lambda\|\chi_{\{|f|>\lambda\}}\|_{L^\Phi_w},$$

we can prove the following by a direct calculation:

Corollary 2.8. *Let Φ be a Young function and let B be a measurable set in $\mathbb{R}^n$. Then $\|\chi_B\|_{WL^\Phi_w} = \|\chi_B\|_{L^\Phi_w} = \frac{1}{\Phi^{-1}(w(B)^{-1})}$.*

In the next sections where we prove our main estimates, we need the following lemma, which follows from Theorem 2.7.

Corollary 2.9. *For a non-negative measurable function w on $\mathbb{R}^n$, a Young function Φ and a ball $B = B(x,r)$, the following inequality is valid:*

$$\|f\|_{L^1(B)} \le 2\Big\|\frac{1}{w}\Big\|_{L^{\tilde{\Phi}}_w(B)} \|f\|_{L^{\Phi}_w(B)}.$$

Lemma 2.10. *Let $\alpha \in (0,1]$ and $1 < p_0 \le p_1 < \infty$. Let also Φ be a Young function which is lower type p_0 and upper type p_1. Assume in addition $w \in A_{p_0}$. For a ball $B = B(x,r)$, the following inequality is valid:*

$$\|f\|_{L^1(B)} \lesssim |B|\Phi^{-1}\left(w(B)^{-1}\right)\|f\|_{L^{\Phi}_w(B)}.$$

Proof. We know that M is bounded on $L^{\Phi}_w(B)$; see [10]. Thus,

$$\frac{\|f\|_{L^1(B)}}{|B|}\|\chi_B\|_{L^{\Phi}_w(B)} \le \|Mf\|_{L^{\Phi}_w(B)} \lesssim \|f\|_{L^{\Phi}_w(B)}.$$

So, Lemma 2.10 is proved. □

2.2. **Weighted Hardy operator.** We will use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [6]. In (2.3) and (2.4) below, it will be understood that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Theorem 2.11. *Let v_1, v_2 and w be weights on $(0,\infty)$. Assume that v_1 is bounded outside a neighborhood of the origin. Then the inequality*

$$\sup_{t>0} v_2(t)H^*_w g(t) \le C \sup_{t>0} v_1(t)g(t) \tag{2.3}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t)\int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{2.4}$$

Moreover, the value $C = B$ is the best constant for (2.3).

2.3. **John-Nirenberg inequality.** When we deal with commutators generated by BMO functions, we need the following fundamental estimates.

Lemma 2.12. (The John–Nirenberg inequality [9]) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

(1) *There exist constants C_1, $C_2 > 0$ independent of b, such that*

$$|\{x \in B : |b(x) - b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \ \forall B \subset \mathbb{R}^n$$

for all $\beta > 0$.

(2) *The following norm equivalence holds:*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y)-b_{B(x,r)}|^p dy\right)^{\frac{1}{p}}$$

for $1<p<\infty$.

(3) *There exists a constant* $C>0$ *such that*

$$\left|b_{B(x,r)}-b_{B(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad \textit{for} \quad 0<2r<t, \tag{2.5}$$

where C *is independent of* b, x, r *and* t.

3. Intrinsic square functions in $M_w^{\Phi,\varphi}(\mathbb{R}^n)$

The following lemma generalizes Guliyev's lemma [3, 4] for Orlicz spaces:

Lemma 3.1. *Let* $\alpha\in(0,1]$ *and* $1<p_0\le p_1<\infty$. *Let* Φ *be a Young function which is lower type* p_0 *and upper type* p_1. *Assume that the weight belongs to the class* $w\in A_{p_0}$. *Then for the operator* G_α *the following inequality is valid:*

$$\|G_\alpha f\|_{L^\Phi_w(B)} \lesssim \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)}\frac{dt}{t}$$

for all $f\in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$, $B=B(x_0,r)$, $x_0\in\mathbb{R}^n$ *and* $r>0$.

Proof. With the notation $2B=B(x_0,2r)$, we decompose f as

$$f=f_1+f_2, \qquad f_1(y)\equiv f(y)\chi_{2B}(y), \qquad f_2(y)\equiv f(y)\chi_{{}^C(2B)}(y).$$

We have

$$\|G_\alpha f\|_{L^\Phi_w(B)} \le \|G_\alpha f_1\|_{L^\Phi_w(B)} + \|G_\alpha f_2\|_{L^\Phi_w(B)}$$

by the triangle inequality. Since $f_1\in L^\Phi_w(\mathbb{R}^n)$, it follows from Theorem 1.3 that

$$\|G_\alpha f_1\|_{L^\Phi_w(B)} \le \|G_\alpha f_1\|_{L^\Phi_w(\mathbb{R}^n)} \lesssim \|f_1\|_{L^\Phi_w(\mathbb{R}^n)} = \|f\|_{L^\Phi_w(2B)}. \tag{3.1}$$

So, we can control f_1.

Now let us estimate $\|G_\alpha f_2\|_{L^\Phi_w(B)}$. Let $x\in B=B(x_0,r)$ and write out $G_\alpha f_2(x)$ in full:

$$G_\alpha(f)(x) \equiv \left(\iint_{\Gamma(x)}\left(\sup_{\phi\in C_\alpha}|f_2*\phi_t(y)|\right)^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{3.2}$$

Let $(y,t)\in\Gamma(x)$. We next write the convolution $f_2*\phi_t(y)$ out in full:

$$|f_2*\phi_t(y)| = \left|t^{-n}\int_{|y-z|\le t}\phi\left(\frac{y-z}{t}\right)f_2(z)dz\right| \lesssim \frac{1}{t^n}\int_{|y-z|\le t}|f_2(z)|dz.$$

Recall that the support of f is contained in ${}^C(2B)$. Keeping this in mind, let $z\in B(y,t)\cap {}^C(2B)$. Since $(y,t)\in\Gamma(x)$, we have

$$|z-x|\le|z-y|+|y-x|\le 2t. \tag{3.3}$$

Another geometric observation shows

$$r=2r-r\le|z-x_0|-|x_0-x|\le|x-z|.$$

Thus, we obtain

$$2t \ge r \tag{3.4}$$

from (3.3). So, putting together (3.2)–(3.4), we obtain

$$\begin{aligned} G_\alpha f_2(x) &\lesssim \left(\int\int_{\Gamma(x)} \left| t^{-n} \int_{|y-z|\le t} |f_2(z)| dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\ &\le \left(\int_{t>r/2} \int_{|x-y|<t} \left(\int_{|z-x|\le 2t} |f(z)| dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{t>r/2} \left(\int_{|z-x|\le 2t} |f(z)| dz \right)^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}}. \end{aligned}$$

We make another geometric observation:

$$|z-x| \ge |z-x_0| - |x_0 - x| \ge \frac{1}{2}|z-x_0|. \tag{3.5}$$

By Minkowski's inequality, we obtain

$$G_\alpha f_2(x) \lesssim \int_{\mathbb{R}^n} \left(\int_{t>\frac{|z-x|}{2}} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} |f(z)| dz.$$

Thanks to (3.5), we have

$$\begin{aligned} G_\alpha f_2(x) &\lesssim \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x|^n} dz \\ &\lesssim \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x_0|^n} dz \\ &= \int_{|z-x_0|>2r} |f(z)| \left(\int_{|z-x_0|}^{+\infty} \frac{dt}{t^{n+1}} \right) dz \\ &= \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |f(z)| dz \right) \frac{dt}{t^{n+1}}. \end{aligned}$$

If we invoke Lemma 2.10, then we obtain

$$G_\alpha f_2(x) \lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \frac{dt}{t}. \tag{3.6}$$

Moreover,

$$\|G_\alpha f_2\|_{L^\Phi_w(B)} \lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}.$$

Thus, it follows from (3.1) and (3.6) that

$$\|G_\alpha f\|_{L^\Phi_w(B)} \lesssim \|f\|_{L^\Phi_w(2B)} + \int_{2r}^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}. \tag{3.7}$$

On the other hand, by (2.2) we get

$$\Phi^{-1}\big(w(B(x_0,r))^{-1}\big) \approx \Phi^{-1}\big(w(B(x_0,r))^{-1}\big) r^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \frac{dt}{t}$$

and hence

$$\|f\|_{L^\Phi_w(2B)} \lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}. \tag{3.8}$$

Thus, it follows from (3.7) and (3.8) that

$$\|G_\alpha f\|_{L^\Phi_w(B)} \lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}.$$

So, we are done. □

With this preparation, we can prove Theorem 1.2

Proof. Fix $x \in \mathbb{R}^n$. Write

$$v_1(r) \equiv \varphi_1(x,r)^{-1}, \quad v_2(r) \equiv \frac{1}{\varphi_2(x,r)\Phi^{-1}(w(B(x_0,r))^{-1})},$$
$$g(r) \equiv \|f\|_{L^\Phi_w(B(x_0,r))}, \quad \omega(r) \equiv \frac{\Phi^{-1}(w(B(x_0,r))^{-1})}{r}.$$

We omit a routine produce of truncation to justify the application of Theorem 2.11. By Lemma 3.1 and Theorem 2.11, we have

$$\begin{aligned}
&\|G_\alpha f\|_{M^{\Phi,\varphi_2}_w(\mathbb{R}^n)} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_2(x,r)} \int_r^{\infty} \|f\|_{L^\Phi_w(B(x_0,t))} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi_1(x,r)} \Phi^{-1}\big(w(B(x_0,r))^{-1}\big) \|f\|_{L^\Phi_w(B(x_0,r))} \\
&= \|f\|_{M^{\Phi,\varphi_1}}.
\end{aligned}$$

So we are done. □

The following lemma is an easy consequence of the monotonicity of the norm $\|\cdot\|_{L^\Phi_w}$ and Wilson's estimate;

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x) \quad (x \in \mathbb{R}^n),$$

which was proved in [24].

Lemma 3.2. *For $j \in \mathrm{Z}^+$, denote*

$$G_{\alpha,2^j}(f)(x) \equiv \left(\int_0^{\infty} \int_{|x-y|\le 2^j t} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

Let Φ be a Young function and $0 < \alpha \le 1$. Then we have

$$\|G_{\alpha,2^j}(f)\|_{L^\Phi_w} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \|G_\alpha(f)\|_{L^\Phi_w}$$

for all $f \in L^{\Phi}_w(\mathbb{R}^n)$.

Now we can prove Theorem 1.4. We write $g^*_{\lambda,\alpha}(f)(x)$ out in full:

$$[g^*_{\lambda,\alpha}(f)(x)]^2 = \iint_{\Gamma(x)} + \iint_{\complement\Gamma(x)} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} := I + II.$$

As for I, a crude estimate suffices;

$$I \le \iint_{\Gamma(x)} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} \le (G_\alpha f(x))^2. \tag{3.9}$$

Thus, the heart of the matters is to control II. We decompose the ambient space $\mathbb{R}^n$:

$$\begin{aligned} II &\le \sum_{j=1}^{\infty} \int_0^\infty \int_{2^{j-1}t \le |x-y| \le 2^j t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} \\ &\lesssim \sum_{j=1}^{\infty} \int_0^\infty \int_{2^{j-1}t \le |x-y| \le 2^j t} 2^{-jn\lambda} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} \\ &\lesssim \sum_{j=1}^{\infty} \iint_{\Gamma_{2^j}(x)} \frac{(A_\alpha f(t,y))^2}{2^{jn\lambda}} \frac{dydt}{t^{n+1}} := \sum_{j=1}^{\infty} \frac{(G_{\alpha,2^j}(f)(x))^2}{2^{jn\lambda}}. \end{aligned} \tag{3.10}$$

Thus, putting together (3.9) and (3.10), we obtain

$$\|g^*_{\lambda,\alpha}(f)\|_{M^{\Phi,\varphi_2}_w} \lesssim \|G_\alpha f\|_{M^{\Phi,\varphi_2}_w} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(f)\|_{M^{\Phi,\varphi_2}_w}. \tag{3.11}$$

By Theorem 1.2, we have

$$\|G_\alpha f\|_{M^{\Phi,\varphi_2}_w(\mathbb{R}^n)} \lesssim \|f\|_{M^{\Phi,\varphi_1}_w(\mathbb{R}^n)}. \tag{3.12}$$

In the sequel, we will estimate $\|G_{\alpha,2^j}(f)\|_{M^{\Phi,\varphi_2}_w}$. We divide $\|G_{\alpha,2^j}(f)\|_{L^{\Phi}_w(B)}$ into two parts:

$$\|G_{\alpha,2^j}(f)\|_{L^{\Phi}_w(B)} \le \|G_{\alpha,2^j}(f_1)\|_{L^{\Phi}_w(B)} + \|G_{\alpha,2^j}(f_2)\|_{L^{\Phi}_w(B)}, \tag{3.13}$$

where $f_1(y) \equiv f(y)\chi_{2B}(y)$ and $f_2(y) \equiv f(y) - f_1(y)$. For $\|G_{\alpha,2^j}(f_1)\|_{L^{\Phi}_w(B)}$, by Lemma 3.2 and (3.8), we have (see also, [5, p. 47, (5.4)])

$$\begin{aligned} \|G_{\alpha,2^j}(f_1)\|_{L^{\Phi}_w(B)} &\lesssim 2^{j(\frac{3n}{2}+\alpha)} \|G_\alpha(f_1)\|_{L^{\Phi}_w(\mathbb{R}^n)} \\ &\lesssim 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{L^{\Phi}_w(2B)} \\ &\lesssim 2^{j(\frac{3n}{2}+\alpha)} \int_{2r}^\infty \|f\|_{L^{\Phi}_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}. \end{aligned} \tag{3.14}$$

For $\|G_{\alpha,2^j}(f_2)\|_{L^\Phi_w(B)}$, we first write the quantity out in full:

$$\begin{aligned} G_{\alpha,2^j}(f_2)(x) &= \left(\iint_{\Gamma_{2^j}(x)} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\ &= \left(\iint_{\Gamma_{2^j}(x)} \left(\sup_{\phi\in C_\alpha} |f*\phi_t(y)|\right)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \end{aligned}$$

A geometric observation shows that

$$G_{\alpha,2^j}(f_2)(x) \lesssim \left(\iint_{\Gamma_{2^j}(x)} \left(\int_{|z-y|\le t} |f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}.$$

Since $|z-x|\le|z-y|+|y-x|\le 2^{j+1}t$, we get

$$\begin{aligned} G_{\alpha,2^j}(f_2)(x) &\lesssim \left(\iint_{\Gamma_{2^j}(x)} \left(\int_{|z-x|\le 2^{j+1}t} |f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^\infty \left(\int_{|z-x|\le 2^{j+1}t} |f_2(z)|dz\right)^2 \frac{2^{jn}dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{jn}{2}} \int_{\mathbb{R}^n} \left(\int_{\frac{|z-x|}{2^{j+1}}}^\infty \frac{|f_2(z)|^2}{t^{2n+1}}dt\right)^{\frac{1}{2}} dz \lesssim 2^{\frac{3jn}{2}} \int_{{}^C B(x_0,2r)} \frac{|f(z)|dz}{|z-x|^n}. \end{aligned}$$

A geometric observation shows

$$|z-x|\ge|z-x_0|-|x_0-x|\ge|z-x_0|-\frac{1}{2}|z-x_0|=\frac{1}{2}|z-x_0|.$$

Thus, we have

$$G_{\alpha,2^j}(f_2)(x) \lesssim 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x_0|^n}dz.$$

By Fubini's theorem and Lemma 2.10, we obtain

$$\begin{aligned} G_{\alpha,2^j}(f_2)(x) &\lesssim 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} |f(z)| \left(\int_{|z-x_0|}^\infty \frac{dt}{t^{n+1}}\right) dz \\ &\lesssim 2^{\frac{3jn}{2}} \int_{2r}^\infty \left(\int_{|z-x_0|<t} |f(z)| \frac{dt}{t^{n+1}}\right) dz \\ &\lesssim 2^{\frac{3jn}{2}} \int_{2r}^\infty \|f\|_{L^\Phi_w(B(x_0,t))} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\frac{dt}{t}. \end{aligned}$$

So,

$$\|G_{\alpha,2^j}(f_2)\|_{L^\Phi_w(B)} \lesssim 2^{\frac{3jn}{2}} \int_{2r}^\infty \|f\|_{L^\Phi_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)}\frac{dt}{t}. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we have

$$\|G_{\alpha,2^j}(f)\|_{L^\Phi_w(B)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \int_{2r}^\infty \|f\|_{L^\Phi_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)}\frac{dt}{t}.$$

Consequently, we obtain

$$\|G_{\alpha,2^j} f\|_{M_w^{\Phi,\varphi_2}(\mathbb{R}^n)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{\substack{x_0\in\mathbb{R}^n\\ r>0}} \int_r^\infty \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \frac{\|f\|_{L_w^\Phi(B(x,t))}}{\varphi_2(x_0,r)} \frac{dt}{t}.$$

Thus by Theorem 2.11 we have

$$\begin{aligned}\|G_{\alpha,2^j} f\|_{M_w^{\Phi,\varphi_2}(\mathbb{R}^n)} &\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{\substack{x_0\in\mathbb{R}^n\\ r>0}} \frac{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)}{\varphi_1(x_0,r)} \|f\|_{L_w^\Phi(B(x,r))} \\ &= 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{M_w^{\Phi,\varphi_1}(\mathbb{R}^n)}. \end{aligned} \tag{3.16}$$

Since $\lambda > 3 + \dfrac{2\alpha}{n}$, by (3.11), (3.12) and (3.16), we can conclude the proof of the theorem.

4. Commutators of the intrinsic square functions in $M_w^{\Phi,\varphi}(\mathbb{R}^n)$

We start with a characterization of the BMO norm.

Lemma 4.1. *Let $0 < p_0 \le p_1 < \infty$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and Φ be a Young function which is lower type p_0 and upper type p_1. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(w(B(x,r))^{-1}\big) \left\|b - b_{B(x,r)}\right\|_{L_w^\Phi(B(x,r))}.$$

Proof. By Hölder's inequality, we have

$$\|b\|_* \lesssim \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(w(B(x,r))^{-1}\big) \left\|b - b_{B(x,r)}\right\|_{L_w^\Phi(B(x,r))}.$$

Now we show that

$$\sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(w(B(x,r))^{-1}\big) \left\|b - b_{B(x,r)}\right\|_{L_w^\Phi(B(x,r))} \lesssim \|b\|_*.$$

Without loss of generality, we may assume that $\|b\|_* = 1$; otherwise, we replace b by $b/\|b\|_*$. By the fact that Φ is lower type p_0 and upper type p_1 and (2.1) it follows that

$$\begin{aligned}&\int_{B(x,r)} \Phi\left(\frac{|b(y) - b_{B(x,r)}|\Phi^{-1}\big(|B(x,r)|^{-1}\big)}{\|b\|_*}\right) dy \\ &= \int_{B(x,r)} \Phi\left(|b(y) - b_{B(x,r)}|\Phi^{-1}\big(|B(x,r)|^{-1}\big)\right) dy \\ &\lesssim \frac{1}{|B(x,r)|} \int_{B(x,r)} \left[|b(y) - b_{B(x,r)}|^{p_0} + |b(y) - b_{B(x,r)}|^{p_1}\right] dy \lesssim 1.\end{aligned}$$

By Lemma 2.5 we get the desired result. $\square$

Remark 4.2. Note that a counterpart to Lemma 4.1 for the variable exponent Lebesgue space $L^{p(\cdot)}$ case was obtained in [8].

Lemma 4.3. *Let $\alpha \in (0,1]$, $1 < p_0 \le p_1 < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ be a Young function which is lower type p_0 and upper type p_1. Then the inequality*

$$\begin{aligned}&\|[b,G_\alpha]f\|_{L^\Phi_w(B(x_0,r))}\\ &\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)}\int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^\Phi_w(B(x_0,t))}\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)\frac{dt}{t}\end{aligned}$$

holds for any ball $B(x_0,r)$ and for any $f \in L^{\Phi,\mathrm{loc}}_w(\mathbb{R}^n)$.

Proof. For an arbitrary $x_0 \in \mathbb{R}^n$, set $B \equiv B(x_0,r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 \equiv f\chi_{2B}$ and $f_2 \equiv f\chi_{C_{(2B)}}$. We have $\|[b,G_\alpha]f\|_{L^\Phi_w(B)} \le \|[b,G_\alpha]f_1\|_{L^\Phi_w(B)} + \|[b,G_\alpha]f_2\|_{L^\Phi_w(B)}$ by the triangle inequality. From Theorem 1.6, the boundedness of $[b,G_\alpha]$ in $L^\Phi_w(\mathbb{R}^n)$ it follows that $\|[b,G_\alpha]f_1\|_{L^\Phi_w(B)} \le \|[b,G_\alpha]f_1\|_{L^\Phi_w(\mathbb{R}^n)} \lesssim \|b\|_*\,\|f_1\|_{L^\Phi_w(\mathbb{R}^n)} = \|b\|_*\,\|f\|_{L^\Phi_w(2B)}$. For $\|[b,G_\alpha]f_2\|_{L^\Phi_w(B)}$, we write it out in full

$$[b,G_\alpha]f_2(x) = \left(\iint_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(y)-b(z)]\phi_t(y-z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}.$$

We then divide it into two parts:

$$\begin{aligned}[b,G_\alpha]f_2(x) &\le \left(\iint_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(y)-b_B]\phi_t(y-z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}\\ &\quad+\left(\iint_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b_B-b(z)]\phi_t(y-z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}\\ &:= \mathfrak{A}+\mathfrak{B}.\end{aligned}$$

First, for the quantity $\mathfrak{A}$, we proceed as follows:

$$\begin{aligned}\mathfrak{A} &= \left(\iint_{\Gamma(x)\cap\mathbb{R}^n\times[r,\infty)} |b(y)-b_B|^2 \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}\phi_t(y-z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}\\ &\lesssim \left(\iint_{\Gamma(x)\cap\mathbb{R}^n\times[r,\infty)} |b(y)-b_B|^2 \left(\frac{1}{t^n}\int_{B(x,t)}|f(z)|\,dz\right)^2\frac{dydt}{t^{n+1}}\right)^{\frac12}.\end{aligned}$$

Note that

$$\int_{B(x,t)}|f(z)|\,dz \lesssim |B(x,t)|\Phi^{-1}(w(B(x,t)^{-1})\|f\|_{L^\Phi_w(B(x,t))}.$$

Thus, by virtue of the embedding $\ell^2(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N})$, we obtain

$$
\begin{aligned}
\mathfrak{A} &\lesssim \left(\iint_{\Gamma(x)\cap\mathbb{R}^n\times[r,\infty)} |b(y)-b_B|^2 \Phi^{-1}(w(B(x,t)^{-1})^2 \|f\|_{L^\Phi_w(B(x,t))}{}^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_r^\infty \Phi^{-1}(w(B(x,t)^{-1})^2 \log\left(2+\frac{t}{r}\right)^2 \|f\|_{L^\Phi_w(B(x,t))}{}^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{j=1}^\infty \Phi^{-1}(w(B(x,2^j r)^{-1})^2 \log\left(2+2^j\right)^2 \|f\|_{L^\Phi_w(B(x,2^j r))}{}^2 \right)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty \Phi^{-1}(w(B(x,2^j r)^{-1}) \log\left(2+2^j\right) \|f\|_{L^\Phi_w(B(x,2^j r))} \\
&\lesssim \int_r^\infty \Phi^{-1}(w(B(x,t)^{-1}) \log\left(2+\frac{t}{r}\right) \|f\|_{L^\Phi_w(B(x,t))} \frac{dt}{t}.
\end{aligned}
$$

For the quantity $\mathfrak{B}$, since $|y-x|<t$, we have $|x-z|<2t$. Thus, by Minkowski's inequality, we have a pointwise estimate:

$$
\begin{aligned}
\mathfrak{B} &\le \left(\iint_{\Gamma(x)} \left| \int_{B(x,2t)} |b_B-b(z)||f_2(z)|dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^\infty \left| \int_{B(x,2t)} |b_B-b(z)||f_2(z)|dz \right|^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} \\
&\lesssim \int_{{}^C B(x_0,2r)} \frac{|b_B-b(z)||f(z)|}{|x-z|^n} dz.
\end{aligned}
$$

Thus, we have

$$
\|\mathfrak{B}\|_{L^\Phi_w(B)} \lesssim \left\| \int_{{}^C(2B)} \frac{|b(z)-b_B|}{|x_0-z|^n} |f(z)|dz \right\|_{L^\Phi_w(B)}.
$$

Since $|z-x| \ge \frac{1}{2}|z-x_0|$, we obtain

$$
\begin{aligned}
\|\mathfrak{B}\|_{L^\Phi_w(B)} &\lesssim \frac{1}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{{}^C(2B)} \frac{|b(z)-b_B|}{|x_0-z|^n} |f(z)|dz \\
&\approx \frac{1}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{{}^C(2B)} |b(z)-b_B||f(z)| \int_{|x_0-z|}^\infty \frac{dt}{t^{n+1}} dz \\
&\approx \frac{1}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{2r}^\infty \left(\int_{2r\le|x_0-z|\le t} |b(z)-b_B||f(z)|dz \right) \frac{dt}{t^{n+1}} \\
&\lesssim \frac{1}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{2r}^\infty \left(\int_{B(x_0,t)} |b(z)-b_B||f(z)|dz \right) \frac{dt}{t^{n+1}}.
\end{aligned}
$$

We decompose the matters by using the triangle inequality:

$$\|\mathfrak{B}\|_{L^{\Phi}_w(B)} \lesssim \frac{1}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |b(z)-b_{B(x_0,t)}||f(z)|dz \right) \frac{dt}{t^{n+1}}$$
$$+ \int_{2r}^{\infty} \frac{|b_B - b_{B(x_0,t)}|}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \left(\int_{B(x_0,t)} |f(z)|dz \right) \frac{dt}{t^{n+1}}$$

Applying Hölder's inequality, by Lemma 4.1 and (2.5) we get

$$\|\mathfrak{B}\|_{L^{\Phi}_w(B)} \lesssim \int_{2r}^{\infty} \big\| |b-b_{B(x_0,t)}| w(\cdot)^{-1} \big\|_{L^{\tilde{\Phi}}_w(B)} \frac{\|f\|_{L^{\Phi}_w(B(x_0,t))} dt}{t^{n+1}\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)}$$
$$+ \int_{2r}^{\infty} |b_B - b_{B(x_0,t)}| \|f\|_{L^{\Phi}_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}$$
$$\lesssim \|b\|_* \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^{\Phi}_w(B(x_0,t))} \frac{\Phi^{-1}\big(w(B(x_0,t))^{-1}\big)}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \frac{dt}{t}.$$

Summing $\|\mathfrak{A}\|_{L^{\Phi}_w(B)}$ and $\|\mathfrak{B}\|_{L^{\Phi}_w(B)}$, we obtain

$$\|[b,G_\alpha]f_2\|_{L^{\Phi}_w(B)}$$
$$\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^{\Phi}_w(B(x_0,t))} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \frac{dt}{t}.$$

Finally,

$$\|[b,G_\alpha]f\|_{L^{\Phi}_w(B)} \lesssim \|b\|_* \, \|f\|_{L^{\Phi}_w(2B)}$$
$$+ \frac{\|b\|_*}{\Phi^{-1}\big(w(B(x_0,r))^{-1}\big)} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^{\Phi}_w(B(x_0,t))} \Phi^{-1}\big(w(B(x_0,t))^{-1}\big) \frac{dt}{t},$$

and the statement of Lemma 4.3 follows by (3.8). □

Finally, Theorem 1.7 follows by Lemma 4.3 and Theorem 2.11 in the same manner as in the proof of Theorem 1.2.

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