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Boundedness of Fractional Maximal Operator and its Commutators on Generalized Orlicz–Morrey Spaces

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Abstract We consider generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ including their weak versions $WM_{\Phi,\varphi}(\mathbb{R}^n)$. We find the sufficient conditions on the pairs (φ_1, φ_2) and (Φ, Ψ) which ensures the boundedness of the fractional maximal operator M_α from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$. As applications of those results, the boundedness of the commutators of the fractional maximal operator $M_{b,\alpha}$ with $b \in BMO(\mathbb{R}^n)$ on the spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ is also obtained. In all the cases the conditions for the boundedness are given in terms of supremal-type inequalities on weights $\varphi(x, r)$, which do not assume any assumption on monotonicity of $\varphi(x, r)$ on r.

Keywords Generalized Orlicz–Morrey space · Fractional maximal operator · Commutator · BMO

Mathematics Subject Classification 42B20 · 42B25 · 42B35 · 46E30

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1 Introduction

Boundedness of classical operators of the Real Analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [2,34,35]. These boundedness extended to several function spaces which are generalizations of L_p-spaces, for example, Orlicz spaces, Morrey spaces, Lorentz spaces, Herz spaces, etc.

Orlicz spaces, introduced in [29,30], are generalizations of Lebesgue spaces L_p. They are useful tools in harmonic analysis and its applications. For example, the Hardy–Littlewood maximal operator is bounded on L_p for $1 < p < \infty$, but not on L_1. Using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ more precisely (see [4,18,19]).

On the other hand, Morrey spaces were introduced in [25] to estimate solutions of partial differential equations, and studied by many authors.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \quad 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator.

The operator M_α is of weak type $(p, np/(n-\alpha p))$ if $1 \le p \le n/\alpha$ and of strong type $(p, np/(n-\alpha p))$ if $1 < p \le n/\alpha$. Also the operator I_α is of weak type $(p, np/(n-\alpha p))$ if $1 \le p < n/\alpha$ and of strong type $(p, np/(n-\alpha p))$ if $1 < p < n/\alpha$.

The boundedness of M_α and I_α from Orlicz space $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ was studied by Cianchi [4]. For boundedness of M_α and I_α on Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$, see Peetre (Spanne) [31], Adams [1].

The definition of generalized Orlicz–Morrey spaces introduced in [6] and used here is different from that of Sawano et al. [33] and Nakai [27,28].

In [6], the boundedness of the maximal operator M and the Calderón–Zygmund operator T from one generalized Orlicz–Morrey space $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ was proved (see, also [17]). Also in [16] the authors prove the boundedness of the Riesz potential operator I_α and its commutator $[b, I_\alpha]$ from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

The main purpose of this paper is to find sufficient conditions on the general Young functions Φ, Ψ and the functions φ_1, φ_2 which ensure the boundedness of M_α from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$, from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$ and in the case $b \in BMO$ the boundedness of the commutator of the fractional maximal operator $M_{b,\alpha}$ from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$.

In the next section we recall the definitions of Orlicz and Morrey spaces and give the definition of generalized Orlicz–Morrey spaces in Sect. 3. These boundedness results extended of M_α and its commutator operator $M_{b,\alpha}$ is obtained in Sects. 4 and 5.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Some Preliminaries on Orlicz and Morrey Spaces

In the study of local properties of solutions of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [9]. They are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \leq \lambda \leq n$, $1 \leq p < \infty$. Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space.

We refer in particular to [22] for the classical Morrey spaces.

We recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0,+\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

Definition 2.2 (*Orlicz Space*) For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. The space $L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$. We refer to the books [20,21,32] for the theory of Orlicz Spaces.

Note that, $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

Definition 2.3 The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) := \left\{f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\right\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t)|\{x \in \mathbb{R}^n : |f(x)| > \lambda t\}| \ \le 1\right\}.$$

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty.$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \qquad \text{for } r \ge 0$$

for some $k > 1$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr) \qquad \text{for } r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ +\infty, & r = +\infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1 + r)\log(1 + r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.1}$$

The following analogue of the Hölder's inequality is known, see [36].

Theorem 2.4 [36] *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\|fg\|_{L_1(\mathbb{R}^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}.$$

The following lemma is valid.

Lemma 2.5 [2,23] *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL_\Phi(\mathbb{R}^n)} = \|\chi_B\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.4, Lemma 2.5 and the inequality (2.1).

Lemma 2.6 *For a Young function Φ and $B = B(x,r)$, the following inequality is valid*

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}.$$

Necessary and sufficient conditions on (Φ, Ψ) for the boundedness of M_α and I_α from Orlicz spaces $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ and $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ have been obtained in [4, Theorem 1 and 2]. In the statement of the theorems, Ψ_p is the Young function associated with the Young function Ψ and $p \in (1,\infty]$ whose Young conjugate is given by

$$\widetilde{\Psi_p}(s) = \int_0^s r^{p'-1}(\mathcal{B}_p^{-1}(r^{p'}))^{p'}dr, \tag{2.2}$$

where

$$\mathcal{B}_p(s) = \int_0^s \frac{\Psi(t)}{t^{1+p'}}dt$$

and p', the Hölder conjugate of p, equals either $p/(p-1)$ or 1, according to whether $p < \infty$ or $p = \infty$ and Φ_p denotes the Young function defined by

$$\Phi_p(s) = \int_0^s r^{p'-1}(\mathcal{A}_p^{-1}(r^{p'}))^{p'}dr, \tag{2.3}$$

where

$$\mathcal{A}_p(s) = \int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+p'}}dt.$$

Recall that, if Φ and Ψ are functions from $[0,\infty)$ into $[0,\infty]$, then Ψ is said to dominate Φ globally if a positive constant c exists such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.

Theorem 2.7 [4]

(i) The fractional maximal operator M_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ if and only if

$$\Phi \text{ dominates globally the function } Q, \tag{2.4}$$

whose inverse is given by

$$Q^{-1}(r) = r^{\alpha/n}\Psi^{-1}(r).$$

(ii) The fractional maximal operator M_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \frac{\Psi(t)}{t^{1+n/(n-\alpha)}}dt < \infty \text{ and } \Phi \text{ dominates globally the function } \Psi_{n/\alpha}. \tag{2.5}$$

Theorem 2.8 [4] *Let $0 < \alpha < n$. Let Φ and Ψ Young functions and let $\Phi_{n/\alpha}$ and $\Psi_{n/\alpha}$ be the Young functions defined as in (2.3) and (2.2), respectively. Then*

(i) The Riesz potential I_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty \text{ and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.} \tag{2.6}$$

(ii) The Riesz potential I_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ if and only if

$$\begin{gathered}\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}dt < \infty, \quad \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)}dt < \infty,\\ \Phi \text{ dominates } \Psi_{n/\alpha} \text{ globally and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally.}\end{gathered} \tag{2.7}$$

3 Generalized Orlicz–Morrey Spaces

Definition 3.1 (*generalized Orlicz–Morrey Space*) Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and Φ any Young function. We denote by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz–Morrey space, the space of all functions $f \in L_\Phi^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1}\Phi^{-1}(r^{-n})\|f\|_{L_\Phi(B(x,r))}.$$

Also by $WM_{\Phi,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Orlicz–Morrey space of all functions $f \in WL_\Phi^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1}\Phi^{-1}(r^{-n})\|f\|_{WL_\Phi(B(x,r))} < \infty,$$

where $WL_\Phi(B(x,r))$ denotes the weak L_Φ-space of measurable functions f for which

$$\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}.$$

According to this definition, we recover the Orlicz–Morrey spaces $M_{\Phi,\lambda}$ and weak Orlicz–Morrey spaces $WM_{\Phi,\lambda}$ under the choice $\varphi(x,r) = \frac{\Phi^{-1}\left(r^{-n}\right)}{\Phi^{-1}\left(r^{-\lambda}\right)}$:

$$M_{\Phi,\lambda} = M_{\Phi,\varphi}\Big|_{\varphi(x,r)=\frac{\Phi^{-1}\left(r^{-n}\right)}{\Phi^{-1}\left(r^{-\lambda}\right)}}, \quad WM_{\Phi,\lambda} = WM_{\Phi,\varphi}\Big|_{\frac{\Phi^{-1}\left(r^{-n}\right)}{\Phi^{-1}\left(r^{-\lambda}\right)}}.$$

According to this definition, we recover the generalized Morrey spaces $M_{p,\varphi}$ and weak generalized Morrey spaces $WM_{p,\varphi}$ under the choice $\Phi(r) = r^p$, $1 \le p < \infty$:

$$M_{p,\varphi} = M_{\Phi,\varphi}\Big|_{\Phi(r)=r^p}, \quad WM_{p,\varphi} = WM_{\Phi,\varphi}\Big|_{\Phi(r)=r^p}.$$

Sufficient conditions on φ for the boundedness of M_α and I_α in generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ have been obtained in [3,10–15,24,26].

4 Boundedness of the Fractional Maximal Operator in the Spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$

In this section sufficient conditions on the pairs (φ_1, φ_2) and (Φ, Ψ) for the boundedness of M_α from one generalized Orlicz–Morrey spaces $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$ have been obtained. At first we recall some supremal inequalities which we use at the proof of our main theorem.

Let v be a weight. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)|g(t)|$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [3].

Theorem 4.1 *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0, \infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0, \infty)$ to $L_{\infty,v_2}(0, \infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{4.1}$$

For the fractional maximal operator the following local estimate is valid.

Lemma 4.2 *Let Φ and Ψ Young functions, $f \in L^{\text{loc}}_\Phi(\mathbb{R}^n)$ and $B = B(x, r)$.*
If (Φ, Ψ) satisfy the conditions (2.4), then

$$\|M_\alpha f\|_{WL_\Psi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} t^{-n+\alpha} \|f\|_{L_1(B(x,t))}. \tag{4.2}$$

If (Φ, Ψ) satisfy the conditions (2.5), then

$$\|M_\alpha f\|_{L_\Psi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} t^{-n+\alpha} \|f\|_{L_1(B(x,t))}. \tag{4.3}$$

Proof Let (Φ, Ψ) satisfy the conditions (2.5). We put $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement_{B(x,2r)}}$. Then we get

$$\|M_\alpha f\|_{L_\Psi(B)} \leq \|M_\alpha f_1\|_{L_\Psi(B)} + \|M_\alpha f_2\|_{L_\Psi(B)}.$$

By the boundedness of the operator M_α from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ (see Theorem 2.7) we have

$$\|M_\alpha f_1\|_{L_\Psi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))}.$$

Let y be an arbitrary point in B. If $B(y, t) \cap {}^\complement(B(x, 2r)) \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y, t) \cap {}^\complement(B(x, 2r))$, then $t > |y - z| \geq |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y, t) \cap {}^\complement(B(x, 2r)) \subset B(x, 2t)$. Indeed, if $z \in B(y, t) \cap {}^\complement(B(x, 2r))$, then we get $|x - z| \leq |y - z| + |x - y| < t + r < 2t$.

Hence

$$\begin{aligned} M_\alpha f_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t)\cap {}^\complement(B(x,2r))} |f(z)|dz \\ &\leq 2^{n-\alpha} \sup_{t>r} \frac{1}{|B(x,2t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} |f(z)|dz \\ &= 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)|dz. \end{aligned}$$

Therefore, for all $y \in B$ we have

$$M_\alpha f_2(y) \le 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)|dz. \tag{4.4}$$

Thus

$$\|M_\alpha f\|_{L_\Psi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))} + \frac{1}{\Psi^{-1}\left(r^{-n}\right)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)|dz\right).$$

Let now Φ dominate globally the function Q. It is obvious that

$$\|M_\alpha f\|_{WL_\Psi(B)} \lesssim \|M_\alpha f_1\|_{WL_\Psi(B)} + \|M_\alpha f_2\|_{WL_\Psi(B)}$$

for every ball $B = B(x, r)$.

By the boundedness of the operator M_α from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$, provided by Theorem 2.7, we have

$$\|M_\alpha f_1\|_{WL_\Psi(B)} \lesssim \|f\|_{L_\Phi(B(x,2r))}.$$

Then by (4.4) we get the inequality (4.2). □

Lemma 4.3 *Let Φ and Ψ Young functions, $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ and $B = B(x, r)$.*
If (Φ, Ψ) satisfy the conditions (2.4), then

$$\|M_\alpha f\|_{WL_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\left(r^{-n}\right)} \sup_{t>2r} \Psi^{-1}\left(t^{-n}\right) \|f\|_{L_\Phi(B(x,t))}. \tag{4.5}$$

If (Φ, Ψ) satisfy the conditions (2.5), then

$$\|M_\alpha f\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\left(r^{-n}\right)} \sup_{t>2r} \Psi^{-1}\left(t^{-n}\right) \|f\|_{L_\Phi(B(x,t))}. \tag{4.6}$$

Proof Suppose that the condition (2.5) satisfied. Denote

$$\mathcal{M}_1 := \frac{1}{\Psi^{-1}\left(r^{-n}\right)} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)|dz\right),$$
$$\mathcal{M}_2 := \|f\|_{L_\Phi(B(x,2r))}.$$

By Lemma 2.6, we get

$$\mathcal{M}_1 \lesssim \frac{1}{\Psi^{-1}\left(r^{-n}\right)} \sup_{t>2r} t^\alpha \Phi^{-1}\left(t^{-n}\right) \|f\|_{L_\Phi(B(x,t))}.$$

On the other hand, the conditions (2.5) implies the condition (2.4). Since from Theorem 2.7

$$(2.5) \Rightarrow M_\alpha \text{ strong type } (\Phi, \Psi) \Rightarrow M_\alpha \text{ weak type } (\Phi, \Psi) \Rightarrow (2.4).$$

The condition (2.4) is equivalent the condition $\Phi^{-1}(t) \lesssim t^{\frac{\alpha}{n}} \Psi^{-1}(t)$. Indeed,

$$\begin{aligned} Q^{-1}(t) &= \inf\{r \geq 0 : Q(r) > t\} \\ &\geq \inf\{r \geq 0 : \Phi(Cr) > t\} \\ &= \frac{1}{C} \inf\{Cr \geq 0 : \Phi(Cr) > t\} \\ &= \frac{1}{C} \Phi^{-1}(t). \end{aligned}$$

So we arrive

$$\mathcal{M}_1 \lesssim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}.$$

On the other hand

$$\begin{aligned} &\frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \\ &\gtrsim \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,2r))} \approx \mathcal{M}_2. \end{aligned} \tag{4.7}$$

Since $\|M_\alpha f\|_{L_\Psi(B)} \leq \mathcal{M}_1 + \mathcal{M}_2$ by Lemma 4.2, we arrive at (4.6).

Suppose that the condition (2.4) satisfied. The inequality (4.5) directly follows from (4.2). □

Theorem 4.4 *Let* $0 \leq \alpha < n$ *and the functions* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy the condition*

$$\sup_{r<t<\infty} \Psi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \leq C\, \varphi_2(x,r), \tag{4.8}$$

where C *does not depend on* x *and* r. *Then for the conditions* (2.5)*, the fractional maximal operator* M_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ *and for the conditions* (2.4)*, it is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof By Lemma 4.3 and Theorem 4.1 we get

$$\begin{aligned} \|M_\alpha f\|_{M_{\Psi,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \Phi^{-1}\big(r^{-n}\big) \|f\|_{L_\Phi(B(x,r))} \\ &= \|f\|_{M_{\Phi,\varphi_1}}, \end{aligned}$$

if (2.5) satisfied and

$$\|M_\alpha f\|_{WM_{\Psi,\varphi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x,t))}$$

$$\begin{aligned}&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\Phi^{-1}\big(r^{-n}\big)\|f\|_{L_\Phi(B(x,r))}\\&=\|f\|_{M_{\Phi,\varphi_1}},\end{aligned}$$

if (2.4) satisfied. □

Note that analogue of the Theorem 4.4 for the Riesz potential proved in [16] as follows.

Theorem 4.5 *Let* $0<\alpha<n$ *and the functions* (φ_1,φ_2) *and* (Φ,Ψ) *satisfy the condition*

$$\int_r^\infty \operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Psi^{-1}\big(t^{-n}\big)\frac{dt}{t}\le C\,\varphi_2(x,r), \tag{4.9}$$

where C *does not depend on* x *and* r. *Then for the conditions* (2.7), I_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ *and for the conditions* (2.6), I_α *is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Remark 4.6 The condition (4.8) is weaker than (4.9). Indeed, (4.9) implies (4.8):

$$\begin{aligned}\varphi_2(x,r)&\gtrsim\int_r^\infty \operatorname*{ess\,inf}_{t<\tau<\infty}\frac{\varphi_1(x,\tau)}{\Phi^{-1}\big(\tau^{-n}\big)}\Psi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\&\gtrsim\int_s^\infty \operatorname*{ess\,inf}_{t<\tau<\infty}\frac{\varphi_1(x,\tau)}{\Phi^{-1}\big(\tau^{-n}\big)}\Psi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\&\gtrsim\operatorname*{ess\,inf}_{s<\tau<\infty}\frac{\varphi_1(x,\tau)}{\Phi^{-1}\big(\tau^{-n}\big)}\int_s^\infty\Psi^{-1}\big(t^{-n}\big)\frac{dt}{t}\\&\approx\operatorname*{ess\,inf}_{s<\tau<\infty}\frac{\varphi_1(x,\tau)}{\Phi^{-1}\big(\tau^{-n}\big)}\Psi^{-1}\big(s^{-n}\big),\end{aligned}$$

where we took $s\in(r,\infty)$, so that

$$\sup_{s>r}\operatorname*{ess\,inf}_{s<\tau<\infty}\frac{\varphi_1(x,\tau)}{\Phi^{-1}\big(\tau^{-n}\big)}\Psi^{-1}\big(s^{-n}\big)\lesssim\varphi_2(x,r).$$

On the other hand the functions $\varphi_1(x,t)=\frac{\Phi^{-1}\big(t^{-n}\big)}{\Psi^{-1}\big(t^{-n}\big)}$ and $\varphi_2(x,t)=1$ satisfy the condition (4.8), but do not satisfy the condition (4.9).

Consider the case $\alpha=0$ and $\Phi=\Psi$. In this case condition (2.4) satisfied by any Young function and condition (2.5) satisfied if and only if $\Phi\in\nabla_2$ (see [4,19] for details). Therefore we get the following corollary for Hardy-Littlewood maximal operator which was proved in [6].

Corollary 4.7 *Let the functions* φ_1,φ_2 *and* Φ *satisfy the condition*

$$\sup_{r<t<\infty}\Phi^{-1}\big(t^{-n}\big)\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\le C\,\varphi_2(x,r), \tag{4.10}$$

where C does not depend on x and r. Then for $\Phi \in \nabla_2$*, the maximal operator M is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ *and for every Young function* Φ*, it is bounded from* $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$*.*

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 \le p, q < \infty$ at Theorem 4.4 we get the Spanne–Guliyev type result which was proved in [15].

Corollary 4.8 *Let* $0 \le \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* (φ_1, φ_2) *satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \le C\varphi_2(x,r), \tag{4.11}$$

where C does not depend on x and r. Then for $p > 1$, M_α *is bounded from* $M_{p,\varphi_1}(\mathbb{R}^n)$ *to* $M_{q,\varphi_2}(\mathbb{R}^n)$ *and for* $p = 1$*, it is bounded from* $M_{1,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{q,\varphi_2}(\mathbb{R}^n)$*.*

5 Commutators of the Fractional Maximal Operator in the Spaces $M_{\Phi,\varphi}$

The theory of commutator was originally studied by Coifman, Rochberg and Weiss in [5]. Since then, many authors have been interested in studying this theory.

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 5.1 Suppose that $f \in L_1^{\rm loc}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{f \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|f\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Remark 5.2 (1) The John–Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that for all $f \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \ : \ |f(x) - f_B| > \beta\}| \le C_1 |B| e^{-C_2\beta/\|f\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John–Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{5.1}$$

for $1 < p < \infty$.

(3) Let $f \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left| f_{B(x,r)} - f_{B(x,t)} \right| \le C \|f\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{5.2}$$

where C is independent of f, x, r and t.

Definition 5.3 A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$(resp. $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \le C t^p \Phi(s).$$

Remark 5.4 We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ (see [20]).

In the following lemma which was proved in [16] we provide a generalization of the property (5.1) from L_p-norms to Orlicz norms.

Lemma 5.5 *Let $f \in BMO(\mathbb{R}^n)$ and Φ be a Young function. Let Φ is lower type p_0 and upper type p_1 with $1 \le p_0 \le p_1 < \infty$, then*

$$\|f\|_* \approx \sup_{x \in \mathbb{R}^n, r>0} \Phi^{-1}\left(r^{-n}\right) \left\| f(\cdot) - f_{B(x,r)} \right\|_{L_\Phi(B(x,r))}.$$

Definition 5.6 Let Φ be a Young function. Let

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 5.7 It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$ (See [21]).

Remark 5.8 Remark 5.7 and Remark 5.4 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $1 < a_\Phi \le b_\Phi < \infty$.

The commutators generated by $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the operators M_α and I_α are defined by

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)|dy, \tag{5.3}$$

$$[b, I_\alpha]f(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} f(y)dy, \tag{5.4}$$

$$|b, I_\alpha|f(x) := \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} f(y)dy, \tag{5.5}$$

respectively.

The known boundedness statements for the commutator operators $[b, I_\alpha]$ and $|b, I_\alpha|$ in Orlicz spaces run as follows.

Theorem 5.9 [8] *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined by its inverse $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. If $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$, then $[b, I_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Remark 5.10 Note that, the operator $|b, I_\alpha|$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ under the conditions of Theorem 5.9. The proof of this fact is similar to proof of Theorem 5.9.

In [7] it was proved that the commutator of the Hardy–Littlewood maximal operator $M_{b,0} \equiv M_b$ with $b \in BMO(\mathbb{R}^n)$, is bounded in $L^\Phi(\mathbb{R}^n)$ for any Young function Φ with $1 < a_\Phi \le b_\Phi < \infty$. This result together with the well known inequality $M_{\alpha,b}(f)(x) \lesssim |b, I_\alpha|(|f|)(x)$ and Remark 5.10, imply the following theorem.

Theorem 5.11 *Let $0 \le \alpha < n$ and b, Φ and Ψ the same as in Theorem 5.9. If $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$, then $M_{b,\alpha}$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

The following lemma is valid.

Lemma 5.12 *Let $0 \le \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$, then the inequality*

$$\|M_{b,\alpha} f\|_{L_\Psi(B(x_0,r))} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Big(1 + \ln \frac{t}{r}\Big) \Psi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x_0,t))}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{\mathrm{loc}}_\Phi(\mathbb{R}^n)$.

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence

$$\big\|M_{b,\alpha} f\big\|_{L_\Psi(B)} \le \big\|M_{b,\alpha} f_1\big\|_{L_\Psi(B)} + \big\|M_{b,\alpha} f_2\big\|_{L_\Psi(B)}.$$

From the boundedness of $M_{b,\alpha}$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ it follows that

$$\begin{aligned}\|M_{b,\alpha} f_1\|_{L_\Psi(B)} &\le \|M_{b,\alpha} f_1\|_{L_\Psi(\mathbb{R}^n)}\\ &\lesssim \|b\|_* \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|b\|_* \|f\|_{L_\Phi(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}M_{b,\alpha} f_2(x) &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |b(y) - b(x)||f_2(y)|dy\\ &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)\cap \complement_{(2B)}} |b(y) - b(x)||f(y)|dy.\end{aligned}$$

Let x be an arbitrary point in B. If $B(x, t) \cap \{\complement(2B)\} \ne \emptyset$, then $t > r$. Indeed, if $y \in B(x, t) \cap \{\complement(2B)\}$, then $t > |x - y| \ge |x_0 - y| - |x_0 - x| > 2r - r = r$.

On the other hand, $B(x,t)\cap\{^{\complement}(2B)\}\subset B(x_0,2t)$. Indeed, if $y\in B(x,t)\cap\{^{\complement}(2B)\}$, then we get $|x_0-y|\le|x-y|+|x_0-x|<t+r<2t$.

Hence

$$\begin{aligned}M_{b,\alpha}(f_2)(x)&=\sup_{t>0}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)\cap^{\complement}(2B)}|b(y)-b(x)||f(y)|dy\\&\le 2^{n-\alpha}\sup_{t>r}\frac{1}{|B(x_0,2t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,2t)}|b(y)-b(x)||f(y)|dy\\&=2^{n-\alpha}\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,t)}|b(y)-b(x)||f(y)|dy.\end{aligned}$$

Therefore, for all $x\in B$ we have

$$M_{b,\alpha}(f_2)(x)\le 2^{n-\alpha}\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,t)}|b(y)-b(x)||f(y)|dy. \tag{5.6}$$

Then

$$\begin{aligned}\|M_{b,\alpha}f_2\|_{L_\Psi(B)}&\lesssim\left\|\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,t)}|b(y)-b(\cdot)||f(y)|dy\right\|_{L_\Psi(B)}\\&\lesssim\left\|\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,t)}|b(y)-b_B||f(y)|dy\right\|_{L_\Psi(B)}\\&\quad+\left\|\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,t)}|b(\cdot)-b_B||f(y)|dy\right\|_{L_\Psi(B)}\\&=J_1+J_2.\end{aligned}$$

Let us estimate J_1.

$$\begin{aligned}J_1&\approx\frac{1}{\Psi^{-1}\big(r^{-n}\big)}\sup_{t>2r}\frac{1}{|B(x_0,t)|^{1-\frac{\alpha}{n}}}\int_{B(x_0,t)}|b(y)-b_B||f(y)|dy\\&\approx\frac{1}{\Psi^{-1}\big(r^{-n}\big)}\sup_{t>2r}t^{\alpha-n}\int_{B(x_0,t)}|b(y)-b_B||f(y)|dy.\end{aligned}$$

Applying Hölder's inequality, by Lemma 5.5 and (5.2) we get

$$\begin{aligned}J_1&\lesssim\frac{1}{\Psi^{-1}\big(r^{-n}\big)}\sup_{t>2r}t^{\alpha-n}\int_{B(x_0,t)}|b(y)-b_{B(x_0,t)}||f(y)|dy\\&\quad+\frac{1}{\Psi^{-1}\big(r^{-n}\big)}\sup_{t>2r}t^{\alpha-n}|b_{B(x_0,r)}-b_{B(x_0,t)}|\int_{B(x_0,t)}|f(y)|dy\\&\lesssim\frac{1}{\Psi^{-1}\big(r^{-n}\big)}\sup_{t>2r}t^{\alpha-n}\left\|b(\cdot)-b_{B(x_0,t)}\right\|_{L_{\widetilde{\Phi}}(B(x_0,t))}\|f\|_{L_\Phi(B(x_0,t))}\end{aligned}$$

$$+ \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} t^{\alpha-n} |b_{B(x_0,r)} - b_{B(x_0,t)}| t^n \Phi^{-1}\big(t^{-n}\big) \|f\|_{L_\Phi(B(x_0,t))}$$
$$\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big)\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(B(x_0,t))}.$$

In order to estimate J_2 note that

$$J_2 \approx \|b(\cdot) - b_B\|_{L_\Psi(B)} \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |f(y)|dy$$
$$\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big)\|f\|_{L_\Phi(B(x_0,t))}.$$

Summing up J_1 and J_2 we get

$$\|M_{b,\alpha} f_2\|_{L_\Psi(B)} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big)\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(B(x_0,t))}. \quad (5.7)$$

Finally,

$$\|M_{b,\alpha} f\|_{L_\Psi(B)} \lesssim \|b\|_* \|f\|_{L_\Phi(2B)}$$
$$+\|b\|_* \frac{1}{\Psi^{-1}\big(r^{-n}\big)} \sup_{t>2r} \Psi^{-1}\big(t^{-n}\big)\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(B(x_0,t))},$$

and the statement of Lemma 5.12 follows by (4.7). □

Theorem 5.13 *Let $0 \le \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$, (φ_1, φ_2) and (Φ, Ψ) satisfy the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi_2(x,r), \quad (5.8)$$

where C does not depend on x and r. Then the operator $M_{b,\alpha}$ is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$. Moreover

$$\|M_{b,\alpha} f\|_{M_{\Psi,\varphi_2}} \le \|b\|_* \|f\|_{M_{\Phi,\varphi_1}}.$$

Proof The statement of Theorem 5.13 follows by Lemma 5.12 and Theorem 4.1 in the same manner as in the proof of Theorem 4.4. □

If we take $\alpha = 0$ at Theorem 5.13 we get the following new result for the commutator of Hardy–Littlewood maximal operator M_b.

Corollary 5.14 *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function with $1 < a_\Phi \le b_\Phi < \infty$, (φ_1, φ_2) and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\, \varphi_2(x,r), \tag{5.9}$$

where C does not depend on x and r. Then the operator M_b is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$. Moreover

$$\|M_b f\|_{M_{\Phi,\varphi_2}} \le \|b\|_* \|f\|_{M_{\Phi,\varphi_1}}.$$

Note that analogue of the Theorem 5.13 for the commutator of the Riesz potential $[b, I_\alpha]$ proved in [16] as follows.

Theorem 5.15 *Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$. (φ_1, φ_2) and (Φ, Ψ) satisfy the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \Psi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{5.10}$$

where C does not depend on x and r.

Then the operator $[b, I_\alpha]$ is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$. Moreover

$$\|[b, I_\alpha] f\|_{M_{\Psi,\varphi_2}} \le \|b\|_* \|f\|_{M_{\Phi,\varphi_1}}.$$

Remark 5.16 The condition (5.8) is weaker than (5.10). See Remark 4.6 for details.

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 < p, q < \infty$ at Theorem 5.13 we get the Spanne–Guliyev type result which was proved at [15].

Corollary 5.17 *Let $0 \le \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \le C \varphi_2(x,r), \tag{5.11}$$

where C does not depend on x and r. Then $M_{b,\alpha}$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$.

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