

Soft Intersection Bi-quasi-interior Ideals of Semigroups

Aslıhan Sezgin, Zeynep Hare Baş, Thiti Gakatem, Akın Osman Atagün

Abstract—In this paper, we introduce the concept of the soft intersection (S-int) bi-quasi-interior ($\mathfrak{BQI}$) ideal of semigroups and provide an equivalent definition. The relationships between S-int ideals and S-int $\mathfrak{BQI}$ ideals are established. We prove that every S-int bi-ideal, left ideal, right ideal, interior ideal, quasi-ideal, bi-interior ideal, left/right bi-quasi-ideal, and left/right quasi-interior ideal is also an S-int $\mathfrak{BQI}$ ideal. Counterexamples are given to show that the converses do not hold, and we demonstrate that additional conditions, such as regularity or right/left simplicity, are required for the converses. We also show that if a subsemigroup of a semigroup is a $\mathfrak{BQI}$ ideal, then its soft characteristic function is an S-int $\mathfrak{BQI}$ ideal, and the converse holds as well. Thus, this work establishes an important connection between classical semigroup theory and soft set theory. Furthermore, we show that finite soft AND-products, Cartesian products, and intersections of S-int $\mathfrak{BQI}$ ideals remain S-int $\mathfrak{BQI}$ ideals, whereas finite soft OR-products and unions do not. This study provides a broad conceptual characterization and analysis of S-int $\mathfrak{BQI}$ ideals.

Index Terms—(soft intersection) bi-quasi-interior ideals, (regular) semigroup, soft set

I. INTRODUCTION

Semigroups play a crucial role in various areas of mathematics. In applied mathematics, semigroups, first studied in the early 20th century, serve as essential tools for analyzing linear time-invariant processes. Moreover, because finite semigroups are closely related to finite automata, their study is fundamental in theoretical computer science. In probability theory, semigroups are also connected to Markov processes. The concept of ideals plays a key role in understanding mathematical structures and their applications. Consequently, many mathematicians have focused their research on generalizing ideals within algebraic structures. Indeed, further exploration of algebraic structures necessitates the generalization of ideals. By employing the notions and properties of these generalized

ideals, many mathematicians have made significant contributions, providing new insights and characterizations

of algebraic structures. Dedekind introduced the concept of ideals in the context of algebraic number theory, and Noether later extended this idea to associative rings. The notion of a one-sided ideal in any algebraic structure can be seen as a generalization of the original ideal concept, and both one-sided and two-sided ideals continue to be central to the study of ring theory.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals in semigroups. Steinfeld [2] was the first to define the concept of quasi-ideals in semigroups and later extended this notion to rings. Quasi-ideals generalize $\mathbb{R}$ and $\mathbb{L}$ ideals, while bi-ideals represent a further generalization of quasi-ideals. The notion of interior ideals was first introduced by Lajos [3] and later studied in more detail by Szasz [4,5]. The concept of interior ideals arose from the generalization of the ideal concept. Rao [6-11] introduced several novel types of semigroup ideals, which expand upon existing ones, such as bi-interior ideals, $\mathfrak{BQI}$ ideals, bi-quasi ideals, quasi-interior ideals, weak-interior ideals, tri-ideals and tri-quasi ideals. Rao [6-11] introduced several novel types of semigroup ideals, extending existing ones, including bi-interior ideals, $\mathfrak{BQI}$ ideals, bi-quasi-ideals, quasi-interior ideals, weak-interior ideals, tri-ideals, and tri-quasi ideals. Baupradist et al. [12] also introduced the concept of essential ideals in semigroups. As a broader extension of various types of ideals, the notion of “almost” ideals was proposed, and their characteristics, together with their relationships to other ideals, were thoroughly investigated. In this context, the concept of almost ideals in semigroups was first introduced in [13]. Subsequently, [14] expanded the notion of bi-ideals to almost bi-ideals in semigroups. The notion of almost interior ideals was introduced in [15], while almost quasi-ideals in semigroups were first studied in [16]. The authors in [17-19] proposed certain types of almost ideals. Moreover, in [15, 17-22], various fuzzy almost ideal types for semigroups were examined. Many of these new structures have been employed to further characterize semigroups and their generalizations.

Molodtsov [23] introduced the “Soft Set Theory” ($\mathfrak{Ss}$ Theory) to address problems involving uncertainty and to find appropriate solutions for them. Since then, numerous important studies have focused on various concepts related to $\mathfrak{Ss}$, particularly on the operations performed on them. Maji et al. [24] proposed several definitions related to $\mathfrak{Ss}$ and established specific operations for them. Pei and Mia [25] as well as Ali et al. [26] introduced different operations on $\mathfrak{Ss}$. Sezgin and Atagün [27] also contributed to the study of $\mathfrak{Ss}$ operations. For further insights into $\mathfrak{Ss}$ operations, which have gained popularity since their inception, we refer to [28-39]. The concept and operations of $\mathfrak{Ss}$ were further modified by Çağman and Enginoğlu [40]. Later, Çağman et

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al. [41] developed the idea of S-int groups, which led to the exploration of various soft algebraic systems. Sezer et al. [42-44], by applying Ss to semigroup theory, introduced soft intersection (S-int) semigroups, ideals, interior ideals, quasi-ideal ideals, (generalized) bi-ideals of semigroups, providing an in-depth analysis of their fundamental properties. In the context of S-int substructures of semigroups, Sezgin and Orbay [44] defined and classified several types of semigroups. Soft intersection (int) almost ideals, as a generalization of various types of S-int ideals, were introduced and examined in [45-56]. The soft versions of different algebraic structures were explored in [57-68].

Rao [7] introduced the concept of the $\mathcal{BQI}$ ideal as a generalization of the bi-ideal, quasi-ideal, interior ideal, bi-quasi ideal, and bi-interior ideal of a semigroup, and examined the properties of these ideals and their relationships. In Rao [69], the concept of $\mathcal{BQI}$ ideal is studied as a generalization of bi-ideal, quasi-ideal, interior ideal, bi-quasi ideal, and bi-interior ideal in Γ -semirings. The properties of these ideals and their relationships with other ideals are explored. The regular ($\mathcal{R}$) and simple Γ -semirings are characterized, and the conditions necessary for a Γ -semiring to be $\mathcal{R}$ or simple are examined. Additionally, Rao [70,71] investigates the properties of $\mathcal{BQI}$ ideals in Γ -semigroups and semirings and explores their relationships with other ideals. The regularity and simplicity properties of Γ -semigroups and semirings are characterized, and the conditions for Γ -semigroups and semirings to be $\mathcal{R}$ or simple are identified.

The motivation for this study stems from the fact that generalizing classical ideals within the framework of soft set theory opens new directions for algebraic research. In this study, the concept of “Soft intersection bi-quasi ideal” (S-int $\mathcal{BQI}$ Ideal) of a semigroup is defined for Ss theory, illustrated with examples, and its properties and relationships with other S-int ideals are examined in detail. It is concluded that if a subsemigroup of a semigroup is a $\mathcal{BQI}$ ideal, then its soft characteristic function is also an S-int $\mathcal{BQI}$ ideal, and the converse is also true. This is an crucial theorem establishing a significant connection between classical semigroup theory and Ss theory. Furthermore, by investigating the relationships of S-int $\mathcal{BQI}$ ideal of a semigroup with other S-int ideals, it is observed that every S-int bi-ideal, S-int ideal, S-int interior ideal, S-int quasi-ideal, S-int bi-interior ideal, S-int bi-quasi ideal, and S-int quasi-interior ideal of a semigroup is an S-int $\mathcal{BQI}$ ideal, and counterexample sare provided to show that the converses do not hold. The conditions for the converses to hold are also obtained. Additionally, the relationships between S-int $\mathcal{BQI}$ ideals and soft set operations as well as the concepts like soft image and soft inverse image are investigated. The paper is structured into four sections. Section 1 offers a general introduction to the topic, while Section 2 explores the fundamental concepts of semigroups and Ss ideals, along with their associated definitions and implications. We introduce the concept of S-int $\mathcal{BQI}$ ideals in Section 3 and examine their properties, as well as how they relate to other types of S-int ideals, through concrete examples. Section 4 provides a summary of our findings and considers potential directions for future research.

II. PRELIMINARIES

In this section, we recall some basic definitions and results that will be used throughout this paper. S denotes a semigroup throughout this paper. A subsemigroup K of S is called a bi-quasi-interior ($\mathcal{BQI}$) ideal of S if $KSKSK \subseteq K$ [7]. If there exists an element $y \in S$ such that $x = xyx$ for all $x \in S$, then S is called a regular ($\mathcal{R}$) semigroup,

Theorem 1 [72] *Let S be a semigroup. Then*

- (1) *S is L (R) simple if and only if (iff) $Sa = S$ ($aS = S$) for all $a \in S$. That is, for every $a, b \in S$, there exists $c \in S$ such that $b = ca$ ($b = ac$)*
- (2) *S is simple iff S is a group. (both L and R simple)*

Definition 1 [23,40] *Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U , and $K \subseteq E$. The soft set $(Ss)_{f_K}$ over U is a function such that $f_K: E \rightarrow P(U)$, where for all $y \notin K$, $f_K(y) = \emptyset$. That is,*

$$f_K = \{(y, f_K(y)) : y \in E, f_K(y) \in P(U)\}$$

The set of all Ss over U is designated by $S_E(U)$.

Definition 2 [40] *Let $f_K \in S_E(U)$. If $f_K(y) = \emptyset$ for all $y \in E$, then f_K is called a null Ss and indicated by $\emptyset_E$.*

Definition 3 [40] *Let $f_A, f_B \in S_E(U)$. If $f_A(y) \subseteq f_B(y)$, for all $y \in E$, then f_A is a soft subset of f_B and indicated by $f_A \subseteq f_B$. If $f_A(y) = f_B(y)$, for all $y \in E$, then f_A is called soft equal to f_B and denoted by $f_A = f_B$.*

Definition 4 [40] *Let $f_A, f_B \in S_E(U)$. The union (intersection) of f_A and f_B is the Ss $f_A \cup f_B$ ($f_A \cap f_B$), where $(f_A \cup f_B)(b) = f_A(b) \cup f_B(b)$ ($(f_A \cap f_B)(b) = f_A(b) \cap f_B(b)$), for all $b \in E$, respectively.*

Definition 5 [40] *Let $f_A, f_B \in S_E(U)$. Then, $\wedge$ -product ($\vee$ -product) of f_A and f_B , denoted by $f_A \wedge f_B$ ($f_A \vee f_B$) is defined by $(f_A \wedge f_B)(x, y) = f_A(x) \cap f_B(y)$ ($(f_A \vee f_B)(x, y) = f_A(x) \cup f_B(y)$) for all $(x, y) \in E \times E$, respectively.*

Definition 6 [41] *Let $f_A, f_B \in S_E(U)$ and φ be a function from A to B . Then, the soft image of f_A under φ , and the soft pre-image (soft inverse image) of f_B under φ are the Ss $\varphi(f_A)$ and $\varphi^{-1}(f_B)$ such that*

$$\begin{aligned} & (\varphi(f_A))(x) \\ &= \begin{cases} \bigcup \{f_A(k) | k \in A \text{ and } \varphi(k) = x\}, & \text{if } \varphi^{-1}(x) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases} \end{aligned}$$

for all $b \in B$ and $(\varphi^{-1}(f_B))(k) = f_B(\varphi(k))$ for all $k \in A$.

Definition 7 [41] *Let $f_A \in S_E(U)$ and $\alpha \subseteq U$. Then, the upper α -inclusion of f_A , denoted by $\mathcal{U}(f_A; \alpha)$, is defined as $\mathcal{U}(f_A; \alpha) = \{x \in A | f_A(x) \supseteq \alpha\}$.*

Definition 8 [42] *Let $f_K, t_K \in S_S(U)$, where K is a semigroup. S-int product $f_K \circ t_K$ is defined by*

$$\begin{aligned} & (f_K \circ t_K)(y) \\ &= \begin{cases} \bigcup_{y=yz} \{f_K(x) \cap t_K(y)\}, & \text{if } \exists x, y \in K \text{ such that } y = xy \\ \emptyset, & \text{otherwise} \end{cases} \end{aligned}$$

Theorem 2 [42] Let $f_K, t_K, p_K \in S_K(U)$, where K is a semigroup. Then,

- i) $(f_K \circ t_K) \circ p_K = f_K \circ (t_K \circ p_K)$
- ii) $f_K \circ t_K \neq f_K \circ t_K$
- iii) $f_K \circ (t_K \cup p_K) = (f_K \circ t_K) \cup (f_K \circ p_K)$ and $(f_K \cup t_K) \circ p_K = (f_K \circ p_K) \cup (t_K \circ p_K)$
- iv) $f_K \circ (t_K \cap p_K) = (f_K \circ t_K) \cap (f_K \circ p_K)$ and $(f_K \cap t_K) \circ p_K = (f_K \circ p_K) \cap (t_K \circ p_K)$
- v) If $f_K \subseteq t_K$, then $f_K \circ p_K \subseteq t_K \circ p_K$ and $p_K \circ f_K \subseteq p_K \circ t_K$
- vi) If $z_K, s_K \in S_K(U)$ such that $z_K \subseteq f_K$ and $s_K \subseteq t_K$, then $z_K \circ s_K \subseteq f_K \circ t_K$.

Definition 9 [42] Let $\emptyset \neq K \subseteq S$. The soft characteristic function (*Schfn*) of K , denoted by S_K , is defined as

$$S_K(x) = \begin{cases} U, & \text{if } x \in K \\ \emptyset, & \text{if } x \in S \setminus K \end{cases}$$

Theorem 3[42,51] Let $G, B \subseteq S$. Then

- i) $G \subseteq B$ iff $S_G \subseteq S_B$
- ii) $S_G \cap S_B = S_{G \cap B}$ and $S_G \cup S_B = S_{G \cup B}$
- iii) $S_G \circ S_B = S_{GB}$

From now on, K denotes a semigroup like S .

Definition 10 [42] An ss f_K over U is called an *S-int subsemigroup of S* if $f_K(l_n) \supseteq f_K(l) \cap f_K(n)$ for all $l, n \in S$.

Note that in [42], the definition of “S-int subsemigroup of S ” is given as “S-int semigroup of S ”; however in this paper, without loss of generality, we prefer to use “S-int subsemigroup of S ”.

From now on, “ideal” is abbreviated by “id”.

Definition 11[42,43] An ss f_K over U is called an *S-int L (R) id of S* if $f_K(l_n) \supseteq f_K(n)(f_K(l_n) \supseteq f_K(l))$ for all $l, n \in S$, and is called an *S-int two-sided id (S-intid) of S* if it is both *S-int L id of S over U* and *S-int R id of S over U*. An *S-int subsemigroup f_K* is called an *S-int bi-id of S* if $f_K(l_n t) \supseteq f_K(l) \cap f_K(t)$ for all $l, n, t \in S$. An ss f_K over U is called an *S-int interior id of S* if $f_K(l_n t) \supseteq f_K(n)$ for all $l, n, t \in S$.

An ss f_K over U is called an *S-int L weak-interior (R) id of S* if $f_K(l_n t) \supseteq f_K(n) \cap f_K(t)(f_K(l_n t) \supseteq f_K(l) \cap f_K(n))$ for all $l, n, t \in S$, and is called an *S-int weak-interior id of S* if it is both *S-int L weak-interior id of S over U* and *S-int R weak-interior id of S over U*. An ss f_K over U is called an *S-int L quasi-interior (R) id of S* if $f_K(l_n t \alpha) \supseteq f_K(n) \cap f_K(\alpha)(f_K(l_n t \alpha) \supseteq f_K(l) \cap f_K(t))$ for all $l, n, t, \alpha \in S$, and is called an *S-int quasi-interior id of S* if it is both *S-int L quasi-interior id of S over U* and *S-int R quasi-interior id of S over U* [73,74].

If $f_K(x) = U$ for all $x \in S$, then f_K is an S-int subsemigroup (id, bi-id, interior id, weak-interior id, quasi-interior id). We denote such a kind of S-int subsemigroup (id, bi-id, interior id, weak-interior id, quasi-interior id) by $\tilde{S}$ [42,43,73-75]. Moreover, $\tilde{S} = S_S$, that is, $\tilde{S}(x) = U$ for all $x \in S$ [42].

Definition 12[73-75] An ss f_K over U is called an *S-int quasi-id of S over U* if $(\tilde{S} \circ f_K) \cap (f_K \circ \tilde{S}) \subseteq f_K$. An ss f_K

over U is called an *S-int bi-interior id of S over U* if $(\tilde{S} \circ f_K \circ \tilde{S}) \cap (f_K \circ \tilde{S} \circ f_K) \subseteq f_K$. An ss f_K over U is called an *S-int L bi-quasi (R) id of S* if $(\tilde{S} \circ f_K) \cap (f_K \circ \tilde{S} \circ f_K) \subseteq f_K$, $((f_K \circ \tilde{S}) \cap (f_K \circ \tilde{S} \circ f_K)) \subseteq f_K$, and is called an *S-intbi-quasi id of S* if it is both *S-int L bi-quasi id of S over U* and *S-int R bi-quasi id of S over U*.

Theorem 4 [42] Let $f_S \in S_S(U)$. Then,

- i) $\tilde{S} \circ \tilde{S} \subseteq \tilde{S}$
- ii) $\tilde{S} \circ f_S \subseteq \tilde{S}$ and $f_S \circ \tilde{S} \subseteq \tilde{S}$
- iii) $f_S \cup \tilde{S} = \tilde{S}$ and $f_S \cap \tilde{S} = f_S$.

Theorem 5 [42,43] Let K be a nonempty subset of a semigroup S . Then, K is a subsemigroup of S iff S_K is an S-int subsemigroup.

Theorem 6 [42,43,73-75] Let $f_S \in S_S(U)$. Then,

- (1) f_S is an S-int subsemigroup iff $(f_S \circ f_S) \subseteq f_S$,
- (2) f_S is an S-int L (R) id iff $(\tilde{S} \circ f_S) \subseteq f_S$ and $(f_S \circ \tilde{S}) \subseteq f_S$,
- (3) f_S is an S-int bi-id iff $(f_S \circ f_S) \subseteq f_S$ and $(f_S \circ \tilde{S}) \subseteq f_S$,
- (4) f_S is an S-int interior id iff $(\tilde{S} \circ f_S \circ \tilde{S}) \subseteq f_S$,
- (5) f_S is an S-int L (R) weak-interior id iff $(\tilde{S} \circ f_S \circ f_S) \subseteq f_S$ $((f_S \circ f_S \circ \tilde{S}) \subseteq f_S)$,
- (6) f_S is an S-int L (R) quasi-interior id iff $(\tilde{S} \circ f_S \circ \tilde{S}) \subseteq f_S$ $((f_S \circ \tilde{S} \circ f_S \circ \tilde{S}) \subseteq f_S)$,

Theorem 7[42,43] The following assertions hold:

- (1) Every S-int L (R) two-sided id is an S-int subsemigroup (S-int bi-id/S-int quasi-id).
- (2) Every S-int id is an S-int interior id (S-int quasi-id).
- (3) Every S-int quasi id is an S-int subsemigroup (S-int bi-id).

Theorem 8[42] Let $f_S \in S_S(U)$, α be a subset of U , $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-int subsemigroup of S , then $U(f_S; \alpha)$ is a subsemigroup of S .

III. SOFT INTERSECTION BI-QUASI-INTERIOR IDEALS OF SEMIGROUPS

In this section, we present the concept of soft intersection (S-int) bi-quasi-interior ids in semigroups, provide its examples, thoroughly examine its relationships with other soft intersection ids, and analyze the concept in terms of certain SS concepts and operations.

Definition 13 A soft set f_S over U is called a soft intersection (S-int) *BQI ideal of S over U* iff $(f_S \cap f_S) \supseteq f_S(b) \cap f_S(d) \cap f_S(m) \cap f_S(n)$ for all $b, d, m, n \in S$.

S-int bi-quasi-interior of S over U is abbreviated by S-int BQI-id in what follows.

Example 1 Let the semigroup $S = \{z, u, v\}$ be defined as follows:

⋮	ℤ	ℓ	ω
ℤ	ω	ω	ω
ℓ	ℤ	ℓ	η
ω	ω	ω	η

Let $\mathfrak{k}_g$ and $\mathfrak{h}_g$ be $\mathfrak{S}_{\mathfrak{S}}$ over $U = Z_g^*$ as follows:

$$\mathfrak{t}_S = \{(z, \{\bar{1}, \bar{3}, \bar{5}\}), (u, \{\bar{1}, \bar{3}, \bar{7}\}), (w, \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\})\}, \mathfrak{a}_S = \{(z, \{\bar{1}, \bar{3}, \bar{5}\}), (u, \{\bar{1}, \bar{3}, \bar{7}\}), (w, \{\bar{1}, \bar{3}, \bar{5}\})\}$$

τ_S is an S-int $\mathfrak{BQI}$ -id of S . Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$$\begin{aligned} t_S(\mathbf{zzllluy}) &= t_S(\mathbf{uy}) \supseteq t_S(\mathbf{z}) \cap t_S(\mathbf{l}) \cap \\ t_S(\mathbf{uy}), t_S(\mathbf{llllluy}) &= t_S(\mathbf{uy}) \supseteq t_S(\mathbf{l}) \cap t_S(\mathbf{l}) \cap \\ t_S(\mathbf{uy}), t_S(\mathbf{zzzlluz}) &= t_S(\mathbf{uy}) \supseteq t_S(\mathbf{z}) \cap t_S(\mathbf{z}) \cap t_S(\mathbf{z}) \end{aligned}$$

It can be easily shown that the $\mathbf{S}_8$ set $\mathbf{t}_8$ satisfies the S-int $\mathbf{BQI}$ -id condition for all other element combinations of the set S . However, since $\mathbf{a}_S(\mathbf{u}\mathbf{l}\mathbf{u}\mathbf{l}\mathbf{u}\mathbf{l}\mathbf{u}) = \mathbf{a}_S(\mathbf{u}) \not\subseteq \mathbf{a}_S(\mathbf{u}) \cap \mathbf{a}_S(\mathbf{u}) \cap \mathbf{a}_S(\mathbf{u})\mathbf{a}_S$ is not an S-int $\mathbf{BQI}$ -id.

Theorem 9 Let $\mathfrak{f}_S \in S_S(U)$. Then, $\mathfrak{f}_S$ is an S -int BQI -id iff $\mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \cong \mathfrak{f}_S$.

Proof: Suppose that $\mathfrak{f}_S$ is an S-int $\mathbb{B}\mathbb{Q}\bar{\mathbb{I}}$ -id and $s \in S$. If $(\mathfrak{f}_S \circ \bar{S} \circ \mathfrak{f}_S \circ \bar{S} \circ \mathfrak{f}_S)(s) = \emptyset$, then $\mathfrak{f}_S \circ \bar{S} \circ \mathfrak{f}_S \circ \bar{S} \circ \mathfrak{f}_S \subseteq \mathfrak{f}_S$. Otherwise, there exist elements $\mathfrak{l}, \mathfrak{n}, \mathfrak{t}, \mathfrak{x}, \mathfrak{z}, \mathfrak{u}, \mathfrak{y} \in S$ such that $s = \mathfrak{l}\mathfrak{n}$, $\mathfrak{l} = \mathfrak{t}\mathfrak{x}$, $\mathfrak{t} = \mathfrak{z}\mathfrak{u}$ and $\mathfrak{z} = \mathfrak{u}\mathfrak{y}$, for $s \in S$. Since $\mathfrak{f}_S$ is an S-int $\mathbb{B}\mathbb{Q}\bar{\mathbb{I}}$ -id, $\mathfrak{f}_S(s) = \mathfrak{f}_S(\mathfrak{l}\mathfrak{n}) = \mathfrak{f}_S(\mathfrak{u}\mathfrak{y}\mathfrak{t}\mathfrak{x}\mathfrak{n}) \supseteq \mathfrak{f}_S(\mathfrak{u}) \cap \mathfrak{f}_S(\mathfrak{l}) \cap \mathfrak{f}_S(\mathfrak{n})$. Therefore,

$$\begin{aligned} (\mathbf{f}_S \circ \widetilde{\mathbf{S}} \circ \mathbf{f}_S \circ \widetilde{\mathbf{S}} \circ \mathbf{f}_S)(s) &= [(\mathbf{f}_S \circ \widetilde{\mathbf{S}} \circ \mathbf{f}_S \circ \widetilde{\mathbf{S}}) \circ \mathbf{f}_S](s) \\ &= \bigcup_{s=\mathbf{l} \cap \mathbf{n}_s} \left\{ (\mathbf{f}_S \circ \widetilde{\mathbf{S}} \circ \mathbf{f}_S \circ \widetilde{\mathbf{S}})(\mathbf{l}) \right. \\ &\quad \cap (\mathbf{f}_S)(\mathbf{n}_s) \Big\} \\ &= \bigcup_{s=\mathbf{l} \cap \mathbf{n}_s} \left\{ \bigcup_{\mathbf{l}=\mathbf{t} \cap \mathcal{O}} \left\{ (\mathbf{f}_S \circ \widetilde{\mathbf{S}} \circ \mathbf{f}_S)(\mathbf{t}) \right. \right. \\ &\quad \cap \widetilde{\mathbf{S}}(\mathcal{O}) \Big\} \cap \mathbf{f}_S(\mathbf{n}_s) \Big\} \\ &= \bigcup_{s=\mathbf{l} \cap \mathbf{n}_s} \left\{ \bigcup_{\mathbf{l}=\mathbf{t} \cap \mathcal{O}} \left\{ \bigcup_{\mathbf{t}=\mathbf{z} \cap \mathbf{l}} \left\{ (\mathbf{f}_S \circ \widetilde{\mathbf{S}})(\mathbf{z}) \right. \right. \right. \\ &\quad \cap (\mathbf{f}_S)(\mathbf{l}) \Big\} \cap \widetilde{\mathbf{S}}(\mathcal{O}) \cap \mathbf{f}_S(\mathbf{n}_s) \Big\} \Big\} \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{s \models n_b} \left\{ \bigcup_{t \models \partial} \left\{ \bigcup_{u \models \mathbb{L}} \left\{ \bigcup_{z \models u \models \mathbb{D}} \{ (f_S)(u) \right. \right. \right. \\
&\quad \left. \left. \cap \widetilde{S}(b) \right\} \cap (f_S)(\mathbb{L}) \cap \widetilde{S}(\partial) \cap f_S(n_b) \right\} \right\} \Big\} \Big\} \\
&= \bigcup_{s = u \models b \models \partial \models n_b} \{ f_S(u) \cap f_S(\mathbb{L}) \cap f_S(n_b) \} \\
&\subseteq \bigcup_{s = u \models b \models \partial \models n_b} \{ f_S(u \models b \models \partial \models n_b) \} \\
&= f_S(\mathbb{L} n_b) \\
&= f_S(s)
\end{aligned}$$

Thus, we have $\mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \cong \mathfrak{f}_S$. Moreover, in the case where $s = \text{ln}$ and $\text{ln} \neq \text{upl\&dn}$ for $s \in S$, since $(\mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \circ \tilde{S})(\text{ln}) = \emptyset$, $\mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \circ \tilde{S} \circ \mathfrak{f}_S \cong \mathfrak{f}_S$ is satisfied.

Conversely, assume that $\mathfrak{f}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{f}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{f}_S \subseteq \mathfrak{f}_S$. Let $s = \mathfrak{l}\mathfrak{u}\mathfrak{t}\mathfrak{a}\mathfrak{z}$ for $\mathfrak{l}, \mathfrak{u}, \mathfrak{t}, \mathfrak{a}, \mathfrak{z} \in S$. Then, we have

$$\begin{aligned}
f_S(l \circ t \circ z) &= f_S(s) \\
&\supseteq (f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(s) \\
&= [(f_S \circ \tilde{S} \circ f_S \circ \tilde{S}) \circ f_S](s) \\
&= \bigcup_{s=bz} \{ (f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(b) \cap f_S(z) \} \\
&= \bigcup_{s=bz} \left\{ \bigcup_{b=y\vartheta} \{ (f_S \circ \tilde{S})(y) \cap \tilde{S}(\vartheta) \} \cap f_S(z) \right\} \\
&= \bigcup_{s=bz} \left\{ \bigcup_{b=y\vartheta} \left\{ \bigcup_{y=n\mathfrak{t}} \{ (f_S \circ \tilde{S})(n) \cap f_S(\mathfrak{t}) \} \cap \tilde{S}(\vartheta) \right. \right. \\
&\quad \left. \left. \cap f_S(z) \right\} \right\} \\
&= \bigcup_{s=bz} \left\{ \bigcup_{b=y\vartheta} \left\{ \bigcup_{y=n\mathfrak{t}} \left\{ \bigcup_{n=\mathfrak{l}\vartheta} f_S(\mathfrak{l}) \cap \tilde{S}(\vartheta) \cap f_S(\mathfrak{t}) \right. \right. \right. \\
&\quad \left. \left. \left. \cap \tilde{S}(\vartheta) \cap f_S(z) \right\} \right\} \right\} \\
&\supseteq (f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(l \circ t \circ z) \cap f_S(z) \\
&= \bigcup_{\mathfrak{z}=\mathfrak{l}\circ\mathfrak{t}\circ z} \{ f_S(\mathfrak{l}) \cap \tilde{S}(\vartheta) \cap f_S(\mathfrak{t}) \cap \tilde{S}(\vartheta) \cap f_S(z) \} \\
&\supseteq f_S(\mathfrak{l}) \cap \tilde{S}(\vartheta) \cap f_S(\mathfrak{t}) \cap \tilde{S}(\vartheta) \cap f_S(z) \\
&= f_S(\mathfrak{l}) \cap U \cap f_S(\mathfrak{t}) \cap U \cap f_S(z) \\
&= f_S(\mathfrak{l}) \cap f_S(\mathfrak{t}) \cap f_S(z)
\end{aligned}$$

Hence, $f_S(l \cup t \cup z) \supseteq f_S(l) \cap f_S(t) \cap f_S(z)$ implying that f_S is an S-int BQI-id.

Corollary 1 $\tilde{\mathbb{S}}$ and $\emptyset_S$ are S-int $\mathbb{B}\mathbb{Q}\bar{\mathbb{I}}$ -ids.

Theorem 10 Let $\mathbb{K}$ be a subsemigroup of S . Then, $\mathbb{K}$ is a $\beta\bar{Q}\bar{I}$ id of S iff $S_{\mathbb{K}}$, the $\mathcal{S}\mathcal{C}\mathcal{I}\mathcal{I}$ of $\mathbb{K}$, is an S -int $\beta\bar{Q}\bar{I}$ -id.

Proof: Let K be a $\mathfrak{BQI}$ id of S . Then, $KS_KS_K \subseteq K$. By Theorem 3, $S_K \circ \tilde{S} \circ S_K \circ \tilde{S} \circ S_K = S_K \circ S_S \circ S_K \circ S_S \circ S_K = S_{KS_KS_K} \cong S_K$. Hence, S_K is an S-int $\mathfrak{BQI}$ -id. Conversely, let S_K be an S-int $\mathfrak{BQI}$ -id and K be a subsemigroup of S . Then, $S_K \circ$

$\tilde{S} \circ S_K \circ \tilde{S} \circ S_K \cong S_K$. Let $x \in KSKSK$. Then, $S_K(x) \supseteq (S_K \circ \tilde{S} \circ S_K \circ \tilde{S} \circ S_K)(x) = (S_K \circ S_S \circ S_K \circ S_S \circ S_K)(x) = S_{KSKSK}(x) = U$. Thus, $S_K(x) = U$ and so $x \in K$, implying that $KSKSK \subseteq K$. Hence, K is a βQI -id of S .

Example 2 Consider the semigroup in Example 1. $A = \{z, u\}$ is a βQI -id of S . By the definition of $S_{\mathcal{A}}$, $S_A = \{(z, U), (u, \emptyset)\}$. S_A is an S-int βQI -id. Conversely, by choosing the S-int βQI -id as $t_S = \{(z, U), (u, \emptyset)\}$, which is the $S_{\mathcal{A}}$ of $X = \{z, u\}$, X is a βQI -id of S .

Now, we continue with the relationships between S-int βQI -ids and other types of S-int ids of S .

Theorem 11 Every S-int bi-id is an S-int βQI -id.

Proof: Let t_S be an S-int bi-id of S . Then, $t_S \circ \tilde{S} \circ t_S \subseteq t_S$. Thus, $(t_S \circ \tilde{S} \circ t_S) \circ \tilde{S} \circ t_S \subseteq t_S \circ \tilde{S} \circ t_S \subseteq t_S$. Hence, t_S is an S-int βQI -id of S .

We show with a counterexample that the converse of Theorem 11 is not true:

Example 3 Let the semigroup $S = \{\mathfrak{u}, \wp, \mathfrak{v}, \bar{\mathfrak{w}}\}$ be defined as follows:

$\otimes$	$\mathfrak{u}$	$\wp$	$\mathfrak{v}$	$\bar{\mathfrak{w}}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\wp$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{v}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\wp$
$\bar{\mathfrak{w}}$	$\mathfrak{u}$	$\mathfrak{u}$	$\wp$	$\mathfrak{v}$

Let t_S be S_S over $U = \mathbb{Z}$ as follows: $t_S = \{(\mathfrak{u}, \{1, 2, 3, 4, 5\}), (\wp, \{2, 3\}), (\mathfrak{v}, \{4, 5\}), (\bar{\mathfrak{w}}, \{1\})\}$. Here, t_S is an S-int βQI -id. In fact,

$$\begin{aligned} (t_S \circ \tilde{S} \circ t_S \circ \tilde{S} \circ t_S)(\mathfrak{u}) &= t_S(\mathfrak{u}) \cup t_S(\wp) \cup t_S(\mathfrak{v}) \cup t_S(\bar{\mathfrak{w}}) \subseteq t_S(\mathfrak{u}), \\ (t_S \circ \tilde{S} \circ t_S \circ \tilde{S} \circ t_S)(\wp) &= \emptyset \subseteq t_S(\wp), \\ (t_S \circ \tilde{S} \circ t_S \circ \tilde{S} \circ t_S)(\mathfrak{v}) &= \emptyset \subseteq t_S(\mathfrak{v}), \\ (t_S \circ \tilde{S} \circ t_S \circ \tilde{S} \circ t_S)(\bar{\mathfrak{w}}) &= \emptyset \subseteq t_S(\bar{\mathfrak{w}}) \end{aligned}$$

Thus, t_S is an S-int βQI -id of S . However, since $(t_S \circ \tilde{S} \circ t_S)(\wp) = t_S(\bar{\mathfrak{w}}) \not\subseteq t_S(\wp)$, t_S is not an S-int bi-id.

Theorem 12 Let $t_S \in S_S(U)$ and S be an $\mathcal{R}$ semigroup. Then, the following conditions are equivalent:

- t_S is an S-int bi-id.
- t_S is an S-int βQI -id.

Proof: (1) implies (2) is by Theorem 11. Let t_S is an S-int βQI -id and $\mathfrak{l}, \mathfrak{n}, \mathfrak{t} \in S$. By assumption, there exists $\mathfrak{m}, \mathfrak{n} \in S$ such that $\mathfrak{l} = \mathfrak{l}\mathfrak{m}\mathfrak{l}$ and $\mathfrak{n} = \mathfrak{n}\mathfrak{m}\mathfrak{n}$. Thus,

$$\begin{aligned} t_S(\mathfrak{l}\mathfrak{n}\mathfrak{t}) &= t_S((\mathfrak{l}\mathfrak{m}\mathfrak{l})\mathfrak{n}\mathfrak{t}) = t_S(\mathfrak{l}\mathfrak{m}\mathfrak{l}\mathfrak{n}\mathfrak{t}) \\ &\supseteq t_S(\mathfrak{l}) \cap t_S(\mathfrak{l}) \cap t_S(\mathfrak{t}) = t_S(\mathfrak{l}) \cap t_S(\mathfrak{t}) \\ t_S(\mathfrak{l}\mathfrak{n}) &= t_S(\mathfrak{l}\mathfrak{m}\mathfrak{l}(\mathfrak{n}\mathfrak{y})\mathfrak{n}) \supseteq t_S(\mathfrak{l}) \cap t_S(\mathfrak{l}) \cap t_S(\mathfrak{n}) \\ &= t_S(\mathfrak{l}) \cap t_S(\mathfrak{n}) \end{aligned}$$

Thus, t_S is an S-int bi-id.

Proposition 1 Every S-int L id is an S-int βQI -id.

Proof: Let t_S be an S-int L id of S . Then, by Theorem 7, t_S is an S-int bi-id. The rest of the proof is obvious by Theorem 11. Hence, t_S is an S-int βQI -id of S .

We show with a counterexample that the converse of Proposition 1 is not true:

Example 4 Let the semigroup $\mathfrak{K} = \{\mathfrak{d}, \mathfrak{k}\}$ be defined as follows:

$\star$	$\mathfrak{d}$	$\mathfrak{k}$
$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$
$\mathfrak{k}$	$\mathfrak{k}$	$\mathfrak{k}$

Let $t_{\mathfrak{K}}$ be S_S over $U = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{Z} \right\}$ as follows: $t_{\mathfrak{K}} = \left\{ \left(\mathfrak{d}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\} \right), \left(\mathfrak{k}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} \right) \right\}$. Here, $t_{\mathfrak{K}}$ is an S-int βQI -id. In fact,

$$\begin{aligned} (t_{\mathfrak{K}} \circ \tilde{S} \circ t_{\mathfrak{K}} \circ \tilde{S} \circ t_{\mathfrak{K}})(\mathfrak{d}) &= t_{\mathfrak{K}}(\mathfrak{d}) \subseteq t_{\mathfrak{K}}(\mathfrak{d}), \\ (t_{\mathfrak{K}} \circ \tilde{S} \circ t_{\mathfrak{K}} \circ \tilde{S} \circ t_{\mathfrak{K}})(\mathfrak{k}) &= t_{\mathfrak{K}}(\mathfrak{k}) \subseteq t_{\mathfrak{K}}(\mathfrak{k}) \end{aligned}$$

Thus, $t_{\mathfrak{K}}$ is an S-int βQI -id of $\mathfrak{K}$. However, since $t_{\mathfrak{K}}(\mathfrak{k}\mathfrak{d}) = t_{\mathfrak{K}}(\mathfrak{k}) \not\subseteq t_{\mathfrak{K}}(\mathfrak{d})$, $t_{\mathfrak{K}}$ is not an S-int L id.

Proposition 2 Let $t_S \in S_S(U)$ and S be an $\mathcal{R}$ semigroup and $\mathcal{R}$ simple semigroup. Then, the following conditions are equivalent:

- t_S is an S-int L id.
- t_S is an S-int βQI -id.

Proof: (1) implies (2) is by Proposition 1. Let t_S is an S-int βQI -id and $\mathfrak{l}, \mathfrak{n}, \mathfrak{t} \in S$. By assumption, there exists $\mathfrak{m}, \mathfrak{n} \in S$ such that $\mathfrak{l} = \mathfrak{l}\mathfrak{m}\mathfrak{l}$ and $\mathfrak{n} = \mathfrak{n}\mathfrak{m}\mathfrak{n}$. Thus, $t_S(\mathfrak{l}\mathfrak{n}) = t_S((\mathfrak{l}\mathfrak{m}\mathfrak{l})(\mathfrak{n}\mathfrak{m}\mathfrak{n})) = t_S(\mathfrak{l}\mathfrak{m}\mathfrak{l}\mathfrak{n}\mathfrak{m}\mathfrak{n}) \supseteq t_S(\mathfrak{l}) \cap t_S(\mathfrak{n}) \cap t_S(\mathfrak{n}) = t_S(\mathfrak{l}) \cap t_S(\mathfrak{n})$. Thus, t_S is an S-int L id.

Proposition 3 Every S-int $\mathcal{R}$ id is an S-int βQI -id.

Proof: Let t_S be an S-int $\mathcal{R}$ id of S . Then, by Theorem 7, t_S is an S-int bi-id. The rest of the proof is obvious by Theorem 11. Hence, t_S is an S-int βQI -id of S .

We show with a counterexample that the converse of Proposition 3 is not true:

Example 5 Consider the S_S in Example 1. It was shown in Example 1 that t_S is an S-int βQI -id. Since, $t_S(\mathfrak{l}z) = t_S(z) \not\subseteq t_S(\mathfrak{l})$, t_S is not an S-int $\mathcal{R}$ id.

Proposition 4 Let $t_S \in S_S(U)$ and S be an $\mathcal{R}$ semigroup and L simple semigroup. Then, the following conditions are equivalent:

- t_S is an S-int $\mathcal{R}$ id.
- t_S is an S-int βQI -id.

Proof: (1) implies (2) is by Proposition 3. Let t_S is an S-int βQI -id and $\mathfrak{l}, \mathfrak{n} \in S$. By assumption, there exists $\mathfrak{m}, \mathfrak{n} \in S$ such that $\mathfrak{n} = \mathfrak{m}\mathfrak{l}$ and $\mathfrak{l} = \mathfrak{l}\mathfrak{m}\mathfrak{l}$. Thus, $t_S(\mathfrak{l}\mathfrak{n}) =$

$\mathfrak{k}_S((\mathfrak{l}\mathfrak{n}\mathfrak{l})(\mathfrak{m}\mathfrak{l})) = \mathfrak{k}_S(\mathfrak{l}\mathfrak{n}\mathfrak{l}\mathfrak{m}\mathfrak{l}) \supseteq \mathfrak{k}_S(\mathfrak{l}) \cap \mathfrak{k}_S(\mathfrak{l}) \cap \mathfrak{k}_S(\mathfrak{l}) = \mathfrak{k}_S(\mathfrak{l})$ Thus, $\mathfrak{k}_S$ is an S-int $\mathbb{R}$ id.

Theorem 13 Every S-int id is an S-int $\mathbb{BQI}$ -id.

Proof: It is followed by Proposition 1 and Proposition 3. Here note that the converse of Theorem 13 is not true follows from Example 4 and Example 5.

Theorem 14 shows that the converse of Theorem 13 holds for $\mathcal{R}$ groups.

Theorem 14 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Proposition 1 and Proposition 3. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id of a group S . Then, by Theorem 1, S is both an $\mathbb{L}$ simple and an $\mathbb{R}$ simple semigroup. The rest of the proof follows from Proposition 2 and Proposition 4.

Theorem 15 Every S-int interior id is an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int interior id of S . Then, $\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \subseteq \mathfrak{k}_S$. Thus, $\mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S \circ \mathfrak{k}_S \circ \mathfrak{k}_S \subseteq \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \subseteq \mathfrak{k}_S$. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id of S .

We show with a counterexample that the converse of Theorem 15 is not true:

Example 6 Consider the $\mathbb{S}\mathbb{S}$ $\mathfrak{k}_S$ in Example 1. It was shown in Example 1 that $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ id. Since $\mathfrak{k}_S(\mathfrak{l}\mathfrak{u}\mathfrak{z}) = \mathfrak{k}_S(\mathfrak{z}) \not\subseteq \mathfrak{k}_S(\mathfrak{l})$, $\mathfrak{k}_S$ is not an S-int interior id.

Theorem 16 shows that the converse of Theorem 15 holds for the groups.

Theorem 16 Let $\mathfrak{k}_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int interior id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Theorem 15. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id and $\mathfrak{l}, \mathfrak{n}, \mathfrak{t} \in S$. By assumption, there exists $\mathfrak{m} \in S$ such that $\mathfrak{l} = \mathfrak{n}\mathfrak{m}$ and $\mathfrak{t} = \mathfrak{m}\mathfrak{n}$. Thus, $\mathfrak{k}_S(\mathfrak{l}\mathfrak{n}\mathfrak{t}) = \mathfrak{k}_S((\mathfrak{n}\mathfrak{m})\mathfrak{n}(\mathfrak{m}\mathfrak{n})) = \mathfrak{k}_S(\mathfrak{n}\mathfrak{m}\mathfrak{n}\mathfrak{m}\mathfrak{n}) \supseteq \mathfrak{k}_S(\mathfrak{n}) \cap \mathfrak{k}_S(\mathfrak{n}) \cap \mathfrak{k}_S(\mathfrak{n}) = \mathfrak{k}_S(\mathfrak{n})$. Thus, $\mathfrak{k}_S$ is an S-int interior id.

Theorem 17 Every S-int quasi-id is an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int quasi-id of S . Then, by Theorem 7, $\mathfrak{k}_S$ is an S-int bi-id. The rest of the proof is obvious by Theorem 11. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id of S .

We show with a counterexample that the converse of Theorem 17 is not true:

Example 7 Consider the $\mathbb{S}\mathbb{S}$ $\mathfrak{k}_S$ in Example 3. It was shown in Example 3 that $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) = \mathfrak{k}_S(\bar{\omega}) \not\subseteq \mathfrak{k}_S(\emptyset)$, $\mathfrak{k}_S$ is not an S-int quasi-id.

$(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) = \mathfrak{k}_S(\bar{\omega}) \cup \mathfrak{k}_S(\mathfrak{n}) \not\subseteq \mathfrak{k}_S(\emptyset)$. $\mathfrak{k}_S$ is not an S-int quasi-id.

Theorem 18 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int quasi-id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Theorem 17. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since S is an $\mathcal{R}$ group, then, by Theorem 14, $\mathfrak{k}_S$ is an S-int id. The rest of the proof is obvious by Theorem 7 $\mathfrak{k}_S$ is an S-int quasi-id of S .

Theorem 19 Every S-int bi-interior id is an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int bi-interior id of S . Then, $(\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$. Since, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}$ and $\mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S$ it is obtained $\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \subseteq (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id of S .

Conversely, an S-int $\mathbb{BQI}$ ideal is not necessarily an S-int bi-interior id.

Example 8 Consider the $\mathbb{S}\mathbb{S}$ $\mathfrak{k}_S$ in Example 3. It was shown in Example 3 that $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since, $(\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) = \mathfrak{k}_S(\bar{\omega}) \not\subseteq \mathfrak{k}_S(\emptyset)$, $\mathfrak{k}_S$ is not an S-int bi-interior id.

The following theorem offers another characterization of S-int $\mathbb{BQI}$ ideals.

Theorem 20 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int bi-interior id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Theorem 19. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since S is an $\mathcal{R}$ group, then, by Theorem 12, $\mathfrak{k}_S$ is an S-int bi-id. The rest of the proof is obvious by Theorem 7 $\mathfrak{k}_S$ is an S-int bi-interior id of S .

This characterization will be useful for proving subsequent results.

Proposition 5 Every S-int $\mathbb{L}$ bi-quasi id is an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int $\mathbb{BQI}$ -id of S . Then, $(\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \subseteq \mathfrak{k}_S$. Since, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \subseteq \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}$ and $\mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S$ it is obtained $\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \subseteq (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{L}$ bi-quasi id of S .

We show with a counterexample that the converse of Proposition 5 is not true:

Example 9 Consider the $\mathbb{S}\mathbb{S}$ $\mathfrak{k}_S$ in Example 3. It was shown in Example 3 that $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since, $(\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) = \mathfrak{k}_S(\bar{\omega}) \not\subseteq \mathfrak{k}_S(\emptyset)$, $\mathfrak{k}_S$ is not an S-int $\mathbb{L}$ bi-quasi id.

$\mathfrak{k}_S)(\wp) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S)(\wp) = \mathfrak{k}_S(\bar{w}) \not\subseteq \mathfrak{k}_S(\wp)$. $\mathfrak{k}_S$ is not an S-int $\mathbb{L}$ bi-quasi id.

Proposition 6 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathbb{R}$ simple $\mathcal{R}$ semigroup. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int $\mathbb{L}$ bi-quasi id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Proposition 5. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since S is $\mathbb{R}$ simple $\mathcal{R}$ semigroup, then, by Theorem 12, $\mathfrak{k}_S$ is an S-int bi-id and by Proposition 2, $\mathfrak{k}_S$ is an S-int $\mathbb{L}$ id. Since, $(\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$ and $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$ it is obtained $(\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$.

Proposition 7 Every S-int $\mathbb{R}$ bi-quasi id is also an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int $\mathbb{BQI}$ -id of S . Then, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$. Since, $\mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S \circ \tilde{\mathfrak{S}}$ and $\mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S$ it is obtained $\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \subseteq (\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{R}$ bi-quasi id of S .

We show with a counterexample that the converse of Proposition 7 is not true:

Example 10 Consider the $S_S \mathfrak{k}_S$ in Example 3. It was shown in Example 3 that $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}})(\wp) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S)(\wp) = \mathfrak{k}_S(\bar{w}) \not\subseteq \mathfrak{k}_S(\wp)$ $\mathfrak{k}_S$ is not an S-int $\mathbb{R}$ bi-quasi id.

Proposition 8 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathbb{L}$ simple $\mathcal{R}$ semigroup. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int $\mathbb{R}$ bi-quasi id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Proposition 7. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id. Since S is an $\mathbb{L}$ simple $\mathcal{R}$ semigroup, then, by Theorem 12, $\mathfrak{k}_S$ is an S-int bi-id and by Proposition 4, $\mathfrak{k}_S$ is an S-int $\mathbb{R}$ id. Since, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \subseteq \mathfrak{k}_S$ and $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$ it is obtained $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \cap (\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S$.

Theorem 21 Every S-int bi-quasi id is also an S-int $\mathbb{BQI}$ -id.

Proof: It follows by Proposition 5 and Proposition 7.

Theorem 22 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathcal{R}$ group. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int bi-quasi id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Note that the converse of Theorem 21 is not true, following from Example 9 and Example 10. Theorem 22 shows that the converse of Theorem 21 holds for $\mathcal{R}$ groups.

Proposition 9 Every S-int $\mathbb{L}$ quasi-interior id is also an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int $\mathbb{L}$ quasi-interior id of S . Then, $\tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S$. Thus, $(\mathfrak{k}_S \circ \tilde{\mathfrak{S}}) \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \subseteq \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S$. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id of S .

Proposition 10 Let $\mathfrak{k}_S \in S_S(U)$ and S be an $\mathbb{R}$ simple semigroup. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int $\mathbb{L}$ quasi-interior id.
2. $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Proposition 9. Let $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id and $\mathfrak{l}, \mathfrak{n}, \mathfrak{t}, \mathfrak{d} \in S$. By assumption, there exists $\mathfrak{m} \in S$ such that $\mathfrak{l} = \mathfrak{n}\mathfrak{m}$. Thus, $\mathfrak{k}_S(\mathfrak{l}\mathfrak{n}\mathfrak{t}\mathfrak{d}) = \mathfrak{k}_S((\mathfrak{n}\mathfrak{m})\mathfrak{n}\mathfrak{t}\mathfrak{d}) = \mathfrak{k}_S(\mathfrak{n}\mathfrak{m}\mathfrak{n}\mathfrak{t}\mathfrak{d}) \supseteq \mathfrak{k}_S(\mathfrak{n}) \cap \mathfrak{k}_S(\mathfrak{n}) \cap \mathfrak{k}_S(\mathfrak{d}) = \mathfrak{k}_S(\mathfrak{n}) \cap \mathfrak{k}_S(\mathfrak{d})$. Thus, $\mathfrak{k}_S$ is an S-int $\mathbb{L}$ quasi-interior id.

Proposition 11 Every S-int $\mathbb{R}$ quasi-interior id is also an S-int $\mathbb{BQI}$ -id.

Proof: Let $\mathfrak{k}_S$ be an S-int $\mathbb{R}$ quasi-interior id of S . Then, $\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \subseteq \mathfrak{k}_S$. Thus, $\mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ (\tilde{\mathfrak{S}} \circ \mathfrak{k}_S) \subseteq \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{k}_S \circ \tilde{\mathfrak{S}} \subseteq \mathfrak{k}_S$. Hence, $\mathfrak{k}_S$ is an S-int $\mathbb{BQI}$ -id of S .

Example 11 Let $S = \{\mathfrak{N}, \mathfrak{a}, \mathfrak{r}, 3, \mathfrak{u}\}$ be defined as follows:

**	$\mathfrak{N}$	$\mathfrak{a}$	$\mathfrak{r}$	3	$\mathfrak{u}$
$\mathfrak{N}$	$\mathfrak{N}$	3	$\mathfrak{N}$	3	3
$\mathfrak{a}$	$\mathfrak{N}$	$\mathfrak{a}$	$\mathfrak{N}$	3	3
$\mathfrak{r}$	$\mathfrak{N}$	3	$\mathfrak{r}$	3	$\mathfrak{u}$
3	$\mathfrak{N}$	3	$\mathfrak{N}$	3	3
$\mathfrak{u}$	$\mathfrak{N}$	3	$\mathfrak{r}$	3	$\mathfrak{u}$

Let f_S be S_S over U as follows: $f_S = \{(\mathfrak{N}, U), (\mathfrak{a}, \emptyset), (\mathfrak{r}, U), (3, \emptyset), (\mathfrak{u}, \emptyset)\}$. Here, f_S is an S-int $\mathbb{BQI}$ -id. However, since $f_S(\mathfrak{r}\mathfrak{a}\mathfrak{r}\mathfrak{a}) = f_S(3) \not\subseteq f_S(\mathfrak{r})$, f_S is not an S-int $\mathbb{R}$ quasi-interior id.

Proposition 12 Let $f_S \in S_S(U)$ and S be an $\mathbb{L}$ simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int $\mathbb{R}$ quasi-interior id.
2. f_S is an S-int $\mathbb{BQI}$ -id.

Proof: (1) implies (2) is by Proposition 11. Let f_S is an S-int $\mathbb{BQI}$ -id and $\mathfrak{l}, \mathfrak{n}, \mathfrak{t}, \mathfrak{d} \in S$. By assumption, there exists $\mathfrak{m} \in S$ such that $\mathfrak{d} = \mathfrak{m}\mathfrak{t}$. Thus, $f_S(\mathfrak{l}\mathfrak{n}\mathfrak{t}\mathfrak{d}) = f_S(\mathfrak{l}\mathfrak{n}\mathfrak{t}(\mathfrak{m}\mathfrak{t})) = f_S(\mathfrak{l}\mathfrak{n}\mathfrak{t}\mathfrak{m}\mathfrak{t}) \supseteq f_S(\mathfrak{l}) \cap f_S(\mathfrak{t}) \cap f_S(\mathfrak{t}) = f_S(\mathfrak{l}) \cap f_S(\mathfrak{t})$. Thus, f_S is an S-int $\mathbb{R}$ quasi-interior id.

Theorem 23 Every S-int quasi-interior id is also an S-int $\mathbb{BQI}$ -id.

Proof: It follows from Proposition 9 and Proposition 11.

Theorem 24 Let $\mathfrak{k}_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. $\mathfrak{k}_S$ is an S-int quasi-interior id.

2. $\mathfrak{k}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proof: (1) implies (2) is by Proposition 9 and Proposition 11. Let S be a group. The rest of the proof is obvious by Proposition 10 and Proposition 12. Note here that the converse of Theorem 23 is not true, following from Example 11. Theorem 24 shows that the converse of Theorem 23 holds for groups.

The following theorem analyzes the behavior of S-int $\mathfrak{BQI}$ ideals under soft set operations. The following theorem analyzes the behavior of S-int $\mathfrak{BQI}$ ideals under soft set operations. We prove that finite soft AND-products, Cartesian products, and intersections of S-int $\mathfrak{BQI}$ ideals remain S-int $\mathfrak{BQI}$ ideals. In contrast, finite soft OR-products and unions of S-int $\mathfrak{BQI}$ ideals are not S-int $\mathfrak{BQI}$ ideals.

Theorem 25 Let $\mathfrak{g}_S$ and $\mathfrak{g}_T$ be S-int $\mathfrak{BQI}$ -ids of S and T , respectively. Then, $\mathfrak{g}_S \wedge \mathfrak{g}_T$ is an S-int $\mathfrak{BQI}$ -ids of $S \times T$.

Proof: Let $(\mathfrak{u}_1, \mathfrak{b}_1), (\mathfrak{u}_2, \mathfrak{b}_2), (\mathfrak{u}_3, \mathfrak{b}_3), (\mathfrak{u}_4, \mathfrak{b}_4), (\mathfrak{u}_5, \mathfrak{b}_5) \in S \times T$. Then,

$$\begin{aligned} & \mathfrak{g}_{S \wedge T}((\mathfrak{u}_1, \mathfrak{b}_1)(\mathfrak{u}_2, \mathfrak{b}_2)(\mathfrak{u}_3, \mathfrak{b}_3)(\mathfrak{u}_4, \mathfrak{b}_4)(\mathfrak{u}_5, \mathfrak{b}_5)) \\ &= \mathfrak{g}_{S \wedge T}(\mathfrak{u}_1 \mathfrak{u}_2 \mathfrak{u}_3 \mathfrak{u}_4 \mathfrak{u}_5, \mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3 \mathfrak{b}_4 \mathfrak{b}_5) \\ &= \mathfrak{g}_S(\mathfrak{u}_1 \mathfrak{u}_2 \mathfrak{u}_3 \mathfrak{u}_4 \mathfrak{u}_5) \cap \mathfrak{g}_T(\mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3 \mathfrak{b}_4 \mathfrak{b}_5) \\ &= (\mathfrak{g}_S(\mathfrak{u}_1) \cap \mathfrak{g}_S(\mathfrak{u}_2) \cap \mathfrak{g}_S(\mathfrak{u}_3) \cap \mathfrak{g}_S(\mathfrak{u}_4) \cap \mathfrak{g}_S(\mathfrak{u}_5)) \\ & \quad \cap (\mathfrak{g}_T(\mathfrak{b}_1) \cap \mathfrak{g}_T(\mathfrak{b}_2) \cap \mathfrak{g}_T(\mathfrak{b}_3) \cap \mathfrak{g}_T(\mathfrak{b}_4) \cap \mathfrak{g}_T(\mathfrak{b}_5)) \\ &= (\mathfrak{g}_S(\mathfrak{u}_1) \cap \mathfrak{g}_T(\mathfrak{b}_1)) \cap (\mathfrak{g}_S(\mathfrak{u}_2) \cap \mathfrak{g}_T(\mathfrak{b}_2)) \cap (\mathfrak{g}_S(\mathfrak{u}_3) \cap \mathfrak{g}_T(\mathfrak{b}_3)) \\ & \quad \cap (\mathfrak{g}_S(\mathfrak{u}_4) \cap \mathfrak{g}_T(\mathfrak{b}_4)) \cap (\mathfrak{g}_S(\mathfrak{u}_5) \cap \mathfrak{g}_T(\mathfrak{b}_5)) \\ &= \mathfrak{g}_{S \wedge T}(\mathfrak{u}_1, \mathfrak{b}_1) \cap \mathfrak{g}_{S \wedge T}(\mathfrak{u}_2, \mathfrak{b}_2) \cap \mathfrak{g}_{S \wedge T}(\mathfrak{u}_3, \mathfrak{b}_3) \cap \mathfrak{g}_{S \wedge T}(\mathfrak{u}_4, \mathfrak{b}_4) \cap \mathfrak{g}_{S \wedge T}(\mathfrak{u}_5, \mathfrak{b}_5) \end{aligned}$$

Thus, $\mathfrak{g}_S \wedge \mathfrak{g}_T$ is an S-int $\mathfrak{BQI}$ -id of $S \times T$.

Note her that $\mathfrak{g}_S \vee \mathfrak{g}_T$ is not always an S-int $\mathfrak{BQI}$ -id with Example 12.

Example 12 Let's consider the subsemigroups in Example 1 and Example 4. Let $\mathfrak{k}_S$ and $\mathfrak{k}_R$ be Ss over $U = Z_6 = \{0, 1, 2, 3, 4, 5\}$ as follows:

$$\mathfrak{k}_S = \{(\mathfrak{z}, \{1, 3\}), (\mathfrak{l}, \{2, 4, 5\}), (\mathfrak{u}, \{1, 2, 3, 4, 5\})\}, \mathfrak{k}_R = \{(\mathfrak{d}, \{1, 3, 4, 5\}), (\mathfrak{k}, \{2\})\}$$

$\mathfrak{k}_S$ and $\mathfrak{k}_R$ are S-int $\mathfrak{BQI}$ -ids. Here, since

$$\begin{aligned} & \mathfrak{k}_{S \vee R}((\mathfrak{l}, \mathfrak{d})(\mathfrak{l}, \mathfrak{d})(\mathfrak{l}, \mathfrak{d})(\mathfrak{l}, \mathfrak{d})(\mathfrak{z}, \mathfrak{k})) \\ & \neq \mathfrak{k}_{S \vee R}(\mathfrak{l}, \mathfrak{d}) \cap \mathfrak{k}_{S \vee R}(\mathfrak{l}, \mathfrak{d}) \cap \mathfrak{k}_{S \vee R}(\mathfrak{z}, \mathfrak{k}) \end{aligned}$$

$\mathfrak{k}_S \vee \mathfrak{k}_R$ is not an S-int $\mathfrak{BQI}$ -id.

Theorem 26 Let $\mathfrak{k}_S$ and $\mathfrak{k}_T$ be S-int $\mathfrak{BQI}$ -ids of S and T over U , respectively. Then, $\mathfrak{k}_S \times \mathfrak{k}_T$ is an S-int $\mathfrak{BQI}$ -id of $S \times T$ over $U \times U$.

Proof: Let $\mathfrak{k}_S \times \mathfrak{k}_T = \mathfrak{k}_{S \times T}$, where $\mathfrak{k}_{S \times T}(\mathfrak{u}, \mathfrak{b}) = \mathfrak{k}_S(\mathfrak{u}) \times \mathfrak{k}_T(\mathfrak{b})$ for all $(\mathfrak{u}, \mathfrak{b}) \in S \times T$. Then, for all $(\mathfrak{u}_1, \mathfrak{b}_1), (\mathfrak{u}_2, \mathfrak{b}_2), (\mathfrak{u}_3, \mathfrak{b}_3), (\mathfrak{u}_4, \mathfrak{b}_4), (\mathfrak{u}_5, \mathfrak{b}_5) \in S \times T$,

$$\begin{aligned} & \mathfrak{k}_{S \times T}((\mathfrak{u}_1, \mathfrak{b}_1)(\mathfrak{u}_2, \mathfrak{b}_2)(\mathfrak{u}_3, \mathfrak{b}_3)(\mathfrak{u}_4, \mathfrak{b}_4)(\mathfrak{u}_5, \mathfrak{b}_5)) \\ &= \mathfrak{k}_{S \times T}(\mathfrak{u}_1 \mathfrak{u}_2 \mathfrak{u}_3 \mathfrak{u}_4 \mathfrak{u}_5, \mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3 \mathfrak{b}_4 \mathfrak{b}_5) \\ &= \mathfrak{k}_S(\mathfrak{u}_1 \mathfrak{u}_2 \mathfrak{u}_3 \mathfrak{u}_4 \mathfrak{u}_5) \times \mathfrak{k}_T(\mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3 \mathfrak{b}_4 \mathfrak{b}_5) \end{aligned}$$

$$\begin{aligned} & \supseteq (\mathfrak{k}_S(\mathfrak{u}_1) \cap \mathfrak{k}_S(\mathfrak{u}_2) \cap \mathfrak{k}_S(\mathfrak{u}_3) \cap \mathfrak{k}_S(\mathfrak{u}_4) \cap \mathfrak{k}_S(\mathfrak{u}_5)) \\ & \quad \times (\mathfrak{k}_T(\mathfrak{b}_1) \cap \mathfrak{k}_T(\mathfrak{b}_2) \cap \mathfrak{k}_T(\mathfrak{b}_3) \cap \mathfrak{k}_T(\mathfrak{b}_4) \cap \mathfrak{k}_T(\mathfrak{b}_5)) \\ &= (\mathfrak{k}_S(\mathfrak{u}_1) \cap \mathfrak{k}_T(\mathfrak{b}_1)) \cap (\mathfrak{k}_S(\mathfrak{u}_2) \cap \mathfrak{k}_T(\mathfrak{b}_2)) \cap (\mathfrak{k}_S(\mathfrak{u}_3) \cap \mathfrak{k}_T(\mathfrak{b}_3)) \\ & \quad \cap (\mathfrak{k}_S(\mathfrak{u}_4) \cap \mathfrak{k}_T(\mathfrak{b}_4)) \cap (\mathfrak{k}_S(\mathfrak{u}_5) \cap \mathfrak{k}_T(\mathfrak{b}_5)) \\ &= \mathfrak{k}_{S \times T}(\mathfrak{u}_1, \mathfrak{b}_1) \cap \mathfrak{k}_{S \times T}(\mathfrak{u}_2, \mathfrak{b}_2) \cap \mathfrak{k}_{S \times T}(\mathfrak{u}_3, \mathfrak{b}_3) \cap \mathfrak{k}_{S \times T}(\mathfrak{u}_4, \mathfrak{b}_4) \cap \mathfrak{k}_{S \times T}(\mathfrak{u}_5, \mathfrak{b}_5) \end{aligned}$$

Hence, $\mathfrak{k}_S \times \mathfrak{k}_T = \mathfrak{k}_{S \times T}$ is an S-int $\mathfrak{BQI}$ -id of $S \times T$ over $U \times U$.

Theorem 27 Let ω_S and $\mathfrak{h}_S$ be S-int $\mathfrak{BQI}$ -ids. Then, $\omega_S \tilde{\cap} \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proof: Let ω_S and $\mathfrak{h}_S$ be S-int $\mathfrak{BQI}$ -ids of S . Then, $\omega_S \circ \tilde{\mathfrak{S}} \circ \omega_S \subseteq \omega_S$ and $\mathfrak{h}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{h}_S \subseteq \mathfrak{h}_S$. Thus, $(\omega_S \tilde{\cap} \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \tilde{\cap} \mathfrak{h}_S) \subseteq (\omega_S \tilde{\cap} \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \tilde{\cap} \mathfrak{h}_S) \subseteq \omega_S \circ \tilde{\mathfrak{S}} \circ \omega_S \subseteq \omega_S$ and $(\omega_S \tilde{\cap} \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \tilde{\cap} \mathfrak{h}_S) \subseteq (\omega_S \tilde{\cap} \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \tilde{\cap} \mathfrak{h}_S) \subseteq \mathfrak{h}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{h}_S \subseteq \mathfrak{h}_S$. Hence, $(\omega_S \tilde{\cap} \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \tilde{\cap} \mathfrak{h}_S) \subseteq \omega_S \tilde{\cap} \mathfrak{h}_S$. Thus, $\omega_S \tilde{\cap} \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Corollary 2 Let ω_S be an S-int $\mathfrak{R}$ id ($\mathfrak{L}$ id/id/bi-id/interior-id/quasi-id/bi-interior/left bi-quasi/right bi-quasi/bi-quasi/left quasi-interior/right quasi-interior/quasi-interior) and $\mathfrak{h}_S$ be an S-int $\mathfrak{R}$ id ($\mathfrak{L}$ id/id/bi-id/interior-id/quasi-id/bi-interior/left bi-quasi/right bi-quasi/bi-quasi/left quasi-interior/right quasi-interior/quasi-interior). Then, $\omega_S \tilde{\cap} \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proposition 13 Let ω_S be an S-int $\mathfrak{L}$ id and $\mathfrak{h}_S$ be an Ss. Then, $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proof: Let ω_S be S-int $\mathfrak{L}$ id of S . Then, $\tilde{\mathfrak{S}} \circ \omega_S \subseteq \omega_S$ and $\mathfrak{k}_S \circ \mathfrak{k}_S \subseteq \mathfrak{k}_S$. Thus, $(\omega_S \circ \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \circ \mathfrak{h}_S) \subseteq (\omega_S \circ \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \circ \mathfrak{h}_S) \subseteq \omega_S \circ \mathfrak{h}_S \circ (\tilde{\mathfrak{S}} \circ \omega_S) \subseteq \omega_S \circ \mathfrak{h}_S \circ (\tilde{\mathfrak{S}} \circ \omega_S) \subseteq \omega_S \circ \mathfrak{h}_S \circ \omega_S \subseteq \omega_S \circ \mathfrak{h}_S \subseteq \omega_S \circ \mathfrak{h}_S$. Thus, $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proposition 14 Let ω_S be an S-int $\mathfrak{R}$ id and $\mathfrak{h}_S$ be an Ss. Then, $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proof: Let ω_S be S-int $\mathfrak{R}$ id of S . Then, $\omega_S \circ \tilde{\mathfrak{S}} \subseteq \omega_S$ and $\omega_S \circ \omega_S \subseteq \omega_S$. Thus, $(\omega_S \circ \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \circ \mathfrak{h}_S) \subseteq (\omega_S \circ \mathfrak{h}_S) \circ \tilde{\mathfrak{S}} \circ (\omega_S \circ \mathfrak{h}_S) \subseteq \omega_S \circ \mathfrak{h}_S \circ (\tilde{\mathfrak{S}} \circ \omega_S) \subseteq \omega_S \circ \mathfrak{h}_S \circ (\tilde{\mathfrak{S}} \circ \omega_S) \subseteq \omega_S \circ \mathfrak{h}_S \circ \omega_S \subseteq \omega_S \circ \mathfrak{h}_S \subseteq \omega_S \circ \mathfrak{h}_S$. Thus, $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Theorem 28 Let ω_S be an S-int id and $\mathfrak{h}_S$ be an Ss. Then, $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proposition 15 Let $\mathfrak{h}_S$ be an S-int $\mathfrak{L}$ id and ω_S be an Ss. Then, $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathfrak{BQI}$ -id.

Proof: Let $\mathfrak{h}_S$ be S-int $\mathfrak{L}$ id of S . Then, $\tilde{\mathfrak{S}} \circ \mathfrak{h}_S \subseteq \mathfrak{h}_S$ and $\mathfrak{h}_S \circ \mathfrak{h}_S \subseteq \mathfrak{h}_S$.

Then, $(\omega_S \circ h_S) \circ \tilde{S} \circ (\omega_S \circ h_S) \circ \tilde{S} \circ (\omega_S \circ h_S) \cong \omega_S \circ h_S \circ (\tilde{S} \circ \tilde{S}) \circ h_S \circ (\tilde{S} \circ \tilde{S}) \circ h_S \cong \omega_S \circ h_S \circ (\tilde{S} \circ h_S) \circ (\tilde{S} \circ h_S) \cong \omega_S \circ (h_S \circ h_S) \circ h_S \cong \omega_S \circ (h_S \circ h_S) \cong \omega_S \circ h_S$. Thus, $\omega_S \circ h_S$ is an S-int $\beta Q\bar{I}$ -id.

Proposition 16 Let h_S be an S-int $\mathcal{R}$ id and ω_S be an $\mathcal{S}\mathcal{S}$. Then, $\omega_S \circ h_S$ is an S-int $\beta Q\bar{I}$ -id.

Proof: Let h_S be S-int $\mathcal{R}$ id of S . Then, $h_S \circ \tilde{S} \cong h_S$ and $h_S \circ h_S \cong h_S$. Thus, $(\omega_S \circ h_S) \circ \tilde{S} \circ (\omega_S \circ h_S) \circ \tilde{S} \circ (\omega_S \circ h_S) \cong \omega_S \circ (h_S \circ \tilde{S}) \circ \omega_S \circ (h_S \circ \tilde{S}) \circ \omega_S \circ h_S \cong \omega_S \circ h_S \circ \omega_S \circ h_S \cong \omega_S \circ (h_S \circ \tilde{S}) \circ (h_S \circ \tilde{S}) \circ h_S \cong \omega_S \circ (h_S \circ h_S) \circ h_S \cong \omega_S \circ (h_S \circ h_S) \cong \omega_S \circ h_S$. Thus, $\omega_S \circ h_S$ is an S-int $\beta Q\bar{I}$ -id.

Theorem 29 Let h_S be an S-int id and ω_S be an $\mathcal{S}\mathcal{S}$. Then, $\omega_S \circ h_S$ is an S-int $\beta Q\bar{I}$ -id.

Proposition 17 Let ω_S and $\mathfrak{h}_S$ be $\mathcal{S}\mathcal{S}$ s over U . If $\omega_S \circ \mathfrak{h}_S$ is an S-int $\mathcal{L}$ id, then it is an S-int $\beta Q\bar{I}$ -id.

Proof: Let $\mathfrak{h}_S \circ \mathfrak{h}_S$ is an S-int $\mathcal{L}$ id. Then, $(\omega_S \circ \mathfrak{h}_S) \circ [\tilde{S} \circ (\omega_S \circ \mathfrak{h}_S)] \circ [\tilde{S} \circ (\omega_S \circ \mathfrak{h}_S)] \cong [(\omega_S \circ \mathfrak{h}_S) \circ (\omega_S \circ \mathfrak{h}_S)] \circ (\omega_S \circ \mathfrak{h}_S) \cong (\omega_S \circ \mathfrak{h}_S) \circ (\omega_S \circ \mathfrak{h}_S) \cong (\omega_S \circ \mathfrak{h}_S)$, implying that $\mathfrak{h}_S \circ \mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id.

Theorem 30 Let $\mathfrak{h}_S$ be an S-int subsemigroup over U , α be a subset of U , $I_{\mathfrak{h}}(\mathfrak{h}_S)$ be the image of $\mathfrak{h}_S$ such that $\alpha \in Im(\mathfrak{h}_S)$. If $\mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id of S , then $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is a $\beta Q\bar{I}$ id of S .

Proof: Since, $\mathfrak{h}_S(x) = \alpha$ for some $x \in S$, $\emptyset \neq \mathcal{U}(\mathfrak{h}_S; \alpha) \subseteq S$. Let $k \in \mathcal{U}(\mathfrak{h}_S; \alpha) \cdot S \cdot \mathcal{U}(\mathfrak{h}_S; \alpha) \cdot S \cdot \mathcal{U}(\mathfrak{h}_S; \alpha)$. Then, there exist $x, y, z \in \mathcal{U}(\mathfrak{h}_S; \alpha)$ and $b \in S$ such that $k = xbybz$. Thus, $\mathfrak{h}_S(x) \supseteq \alpha$, $\mathfrak{h}_S(y) \supseteq \alpha$ and $\mathfrak{h}_S(z) \supseteq \alpha$. Since $\mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id, $\mathfrak{h}_S(k) = \mathfrak{h}_S(xbybz) \supseteq \mathfrak{h}_S(x) \cap \mathfrak{h}_S(y) \cap \mathfrak{h}_S(z) \supseteq \alpha \cap \alpha \cap \alpha \supseteq \alpha$. Hence, $\mathfrak{h}_S(k) \supseteq \alpha$, implying that $k \in \mathcal{U}(\mathfrak{h}_S; \alpha)$. Therefore, $\mathcal{U}(\mathfrak{h}_S; \alpha) \cdot S \cdot \mathcal{U}(\mathfrak{h}_S; \alpha) \cdot S \cdot \mathcal{U}(\mathfrak{h}_S; \alpha) \subseteq \mathcal{U}(\mathfrak{h}_S; \alpha)$. Moreover, since $\mathfrak{h}_S$ is an S-int subsemigroup over U , by Theorem 8, $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is a subsemigroup of S . Thus, $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is a $\beta Q\bar{I}$ id.

We illustrate Theorem 30 with Example 13.

Example 13 Consider Example 1. It was shown in Example 1 that $\mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id.

By considering the image set of $\mathfrak{h}_S$, that is, $Im(\mathfrak{h}_S) = \{\{\bar{1}, \bar{3}, \bar{5}\}, \{\bar{1}, \bar{3}, \bar{7}\}, \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}\}$, we obtain the following:

$$\mathcal{U}(\mathfrak{h}_S; \alpha) = \begin{cases} \{z, u\}, & \alpha = \{\bar{1}, \bar{3}, \bar{5}\} \\ \{l, u\}, & \alpha = \{\bar{1}, \bar{3}, \bar{7}\} \\ \{u\}, & \alpha = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\} \end{cases}$$

Here, $\{z, u\}$, $\{l, u\}$ and $\{u\}$ are all $\beta Q\bar{I}$ ids of S . In fact, since $\{z, u\} \cdot \{z, u\} \subseteq \{z, u\}$, $\{l, u\} \cdot \{l, u\} \subseteq \{l, u\}$, $\{u\} \cdot \{u\} \subseteq \{u\}$ each $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is a subsemigroup of S . Similarly, since $\{z, u\} \cdot S \cdot \{z, u\} \cdot S \cdot \{z, u\} = \{u\} \subseteq \{z, u\}$, $\{l, u\} \cdot S \cdot \{l, u\} \cdot S \cdot \{l, u\} = \{u\} \subseteq \{l, u\}$, $\{u\} \cdot S \cdot \{u\} = \{u\} \subseteq \{u\}$ each $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is a $\beta Q\bar{I}$ id of S . Now,

consider the $\mathcal{S}\mathcal{S}$ $\mathfrak{h}_S$ in Example 1. By taking into account $Im(\mathfrak{h}_S) = \{\{\bar{1}, \bar{3}, \bar{5}\}, \{\bar{1}, \bar{3}, \bar{7}\}\}$ we obtain the following:

$$\mathcal{U}(\mathfrak{h}_S; \alpha) = \begin{cases} \{z, u\} & \alpha = \{\bar{1}, \bar{3}, \bar{5}\} \\ \{l, u\} & \alpha = \{\bar{1}, \bar{3}, \bar{7}\} \end{cases}$$

Here, $\{l, u\}$ is not a $\beta Q\bar{I}$ id of S . In fact, since $\{l, u\} \cdot S \cdot \{l, u\} \cdot S \cdot \{l, u\} = \{z, l, u\} \not\subseteq \{l, u\}$ one of the $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is not a $\beta Q\bar{I}$ id of S , hence it is not a $\beta Q\bar{I}$ id of S . It is seen that each of $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is not a $\beta Q\bar{I}$ id of S . On the other hand, in Example 1 it was shown that $\mathfrak{h}_S$ is not an S-int $\beta Q\bar{I}$ -id of S .

Definition 14 Let $\mathfrak{h}_S$ be an S-int subsemigroup and S-int $\beta Q\bar{I}$ -id of S . Then, the $\beta Q\bar{I}$ ids $\mathcal{U}(\mathfrak{h}_S; \alpha)$ are called upper α - $\beta Q\bar{I}$ ids of $\mathfrak{h}_S$.

Proposition 18 Let $\mathfrak{h}_S$ be an $\mathcal{S}\mathcal{S}$ over U , $\mathcal{U}(\mathfrak{h}_S; \alpha)$ be upper α - $\beta Q\bar{I}$ of $\mathfrak{h}_S$ for each $\alpha \subseteq U$ and $Im(\mathfrak{h}_S)$ be an ordered set by inclusion. Then, $\mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id.

Proof: Let $x, y, z, b, c \in S$ and $\mathfrak{h}_S(x) = \alpha_1$, $\mathfrak{h}_S(y) = \alpha_2$ and $\mathfrak{h}_S(z) = \alpha_3$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathcal{U}(\mathfrak{h}_S; \alpha_1)$ and $y \in \mathcal{U}(\mathfrak{h}_S; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2 \subseteq \alpha_3$, $x, y, z \in \mathcal{U}(\mathfrak{h}_S; \alpha_1)$ and since $\mathcal{U}(\mathfrak{h}_S; \alpha)$ is a $\beta Q\bar{I}$ of S for all $\alpha \subseteq U$, it follows that $xbycz \in \mathcal{U}(\mathfrak{h}_S; \alpha_1)$. Hence, $\mathfrak{h}_S(xbycz) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 \cap \alpha_3$. Thus, $\mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id.

Proposition 19 Let $\mathfrak{h}_S$ and $\mathfrak{h}_T$ be $\mathcal{S}\mathcal{S}$ over U , and Ψ be a semigroup isomorphism from S to T . If $\mathfrak{h}_S$ is an S-int $\beta Q\bar{I}$ -id of S , then $\Psi(\mathfrak{h}_S)$ is an S-int $\beta Q\bar{I}$ -id of T .

Proof: Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \in T$. Since Ψ is surjective, there exist $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \in S$ such that $\Psi(\mathfrak{h}_1) = \mathfrak{h}_1$, $\Psi(\mathfrak{h}_2) = \mathfrak{h}_2$, $\Psi(\mathfrak{h}_3) = \mathfrak{h}_3$, $\Psi(\mathfrak{h}_4) = \mathfrak{h}_4$ and $\Psi(\mathfrak{h}_5) = \mathfrak{h}_5$. Then,

$$\begin{aligned} (\Psi(\mathfrak{h}_S))(\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5) &= \bigcup \{\mathfrak{h}_S(\mathfrak{h}): \mathfrak{h} \in S, \Psi(\mathfrak{h}_S) \\ &= \mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5\} \\ &= \bigcup \{\mathfrak{h}_S(\mathfrak{h}): \mathfrak{h} \in S, \mathfrak{h} \\ &= \Psi^{-1}(\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5)\} \\ &= \bigcup \{\mathfrak{h}_S(\mathfrak{h}): \mathfrak{h} \in S, \mathfrak{h} \\ &= \Psi^{-1}(\Psi(\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5)) \\ &= \mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5\} \\ &= \bigcup \{\mathfrak{h}_S(\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5): \mathfrak{h}_i \in S, \Psi(\mathfrak{h}_i) = \mathfrak{h}_i, i \\ &= 1, 2, 3, 4, 5\} \\ &\supseteq \bigcup \{\mathfrak{h}_S(\mathfrak{h}_1) \cap \mathfrak{h}_S(\mathfrak{h}_3) \cap \mathfrak{h}_S(\mathfrak{h}_5): \mathfrak{h}_1, \mathfrak{h}_3, \mathfrak{h}_5 \in S, \Psi(\mathfrak{h}_1) \\ &= \mathfrak{h}_1, \Psi(\mathfrak{h}_3) = \mathfrak{h}_3 \text{ and } \Psi(\mathfrak{h}_5) = \mathfrak{h}_5\} \\ &= (\Psi(\mathfrak{h}_S))(\mathfrak{h}_1) \cap (\Psi(\mathfrak{h}_S))(\mathfrak{h}_3) \\ &\cap (\Psi(\mathfrak{h}_S))(\mathfrak{h}_5) \end{aligned}$$

Hence, $\Psi(\mathfrak{h}_S)$ is an S-int $\beta Q\bar{I}$ -id of T .

Proposition 20 Let $\mathfrak{h}_S$ and $\mathfrak{h}_T$ be $\mathcal{S}\mathcal{S}$ over U , and Ψ be a semigroup isomorphism from S to T . If $\mathfrak{h}_T$ is an S-int $\beta Q\bar{I}$ -id of S , then $\Psi^{-1}(\mathfrak{h}_T)$ is an S-int $\beta Q\bar{I}$ -id of T .

Proof: Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \in S$. Then,

$$(\Psi^{-1}(\mathfrak{h}_T))(\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5) = \mathfrak{h}_T(\Psi(\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_4 \mathfrak{h}_5))$$

$$\begin{aligned}
 &= \mathfrak{t}_T(\Psi(\mathfrak{t}_1)\Psi(\mathfrak{t}_2)\Psi(\mathfrak{t}_3)\Psi(\mathfrak{t}_4)\Psi(\mathfrak{t}_5)) \\
 &\quad \supseteq \mathfrak{t}_T(\Psi(\mathfrak{t}_1)) \cap \mathfrak{t}_T(\Psi(\mathfrak{t}_3)) \\
 &\quad \cap \mathfrak{t}_T(\Psi(\mathfrak{t}_5))) \\
 &\quad = (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{t}_1) \\
 &\quad \cap (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{t}_3) \cap (\Psi^{-1}(\mathfrak{t}_T))(\mathfrak{t}_5)
 \end{aligned}$$

Thus, $\Psi^{-1}(\mathfrak{t}_T)$ is an S-int $\mathfrak{BQI}$ -id of T .

Theorem 31 For a semigroup S , the following conditions are equivalent:

1. S is a $\mathcal{R}$ semigroup.
2. $\mathfrak{t}_S = \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S$ for every S-int $\mathfrak{BQI}$ -id of S .

Proof: First assume that (1) holds. Let S be an $\mathcal{R}$ semigroup, $\mathfrak{t}_S$ be an S-int $\mathfrak{BQI}$ -id of S and $\mathfrak{m} \in S$. Then, $\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \subseteq \mathfrak{t}_S$ and there exists an element $\mathfrak{n} \in S$ such that $\mathfrak{m} = \mathfrak{m}\mathfrak{n}\mathfrak{m}$. Thus,

$$\begin{aligned}
 &(\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S)(\mathfrak{m}) \\
 &= \bigcup_{\mathfrak{m}=\mathfrak{l}\mathfrak{n}} \{(\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}})(\mathfrak{l}) \cap \mathfrak{t}_S(\mathfrak{n})\} \\
 &\quad \supseteq (\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}})(\mathfrak{m}\mathfrak{n}) \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &= \bigcup_{\mathfrak{m}\mathfrak{n}=\mathfrak{p}q} \{(\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S)(\mathfrak{p}) \\
 &\quad \cap \tilde{\mathfrak{S}}(q)\} \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &\quad \supseteq (\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S)(\mathfrak{m}) \cap \tilde{\mathfrak{S}}(\mathfrak{m}\mathfrak{n}\mathfrak{m}) \\
 &\quad \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &= \bigcup_{\mathfrak{m}=\mathfrak{l}\mathfrak{n}} \{(\mathfrak{t}_S \circ \tilde{\mathfrak{S}})(\mathfrak{l}) \cap \mathfrak{t}_S(\mathfrak{n})\} \\
 &\quad \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &\quad \supseteq (\mathfrak{t}_S \circ \tilde{\mathfrak{S}})(\mathfrak{m}\mathfrak{n}) \cap \mathfrak{t}_S(\mathfrak{m}) \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &= \bigcup_{\mathfrak{m}\mathfrak{n}=\mathfrak{p}q} \{\mathfrak{t}_S(\mathfrak{p}) \cap \tilde{\mathfrak{S}}(q)\} \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &\quad \supseteq \mathfrak{t}_S(\mathfrak{m}) \cap \tilde{\mathfrak{S}}(\mathfrak{m}\mathfrak{n}\mathfrak{m}) \cap \mathfrak{t}_S(\mathfrak{m}) \\
 &= \mathfrak{t}_S(\mathfrak{m})
 \end{aligned}$$

Therefore, $\mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \supseteq \mathfrak{t}_S$ implying that $\mathfrak{t}_S = \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S$.

Conversely, let $\mathfrak{t}_S = \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S \circ \tilde{\mathfrak{S}} \circ \mathfrak{t}_S$ where $\mathfrak{t}_S$ is an S-int $\mathfrak{BQI}$ -id of S . To show that S is an $\mathcal{R}$ semigroup, we need to show that $A = ASASA$ for every $\mathfrak{BQI}$ id A of S . It is obvious that $ASASA \subseteq A$. Thus, it is enough to show that $A \subseteq ASASA$. Let $\mathfrak{l} \in A$, and A be any $\mathfrak{BQI}$ id of S . Then, S_A is an S-int $\mathfrak{BQI}$ of S . Hence, $S_A(\mathfrak{l}) = (S_A \circ \tilde{\mathfrak{S}} \circ S_A \circ \tilde{\mathfrak{S}} \circ S_A)(\mathfrak{l}) = (S_A \circ S_S \circ S_A \circ S_S \circ S_A)(\mathfrak{l}) = S_{ASASA}(\mathfrak{l}) = U$ implying that $\mathfrak{l} \in ASASA$. Hence, $A = ASASA$, so S is an $\mathcal{R}$ semigroup.

IV. CONCLUSION

In this study, the concept of the soft intersection (S-int) bi-quasi-interior ($\mathfrak{BQI}$) ideal in semigroups is proposed and the relations of several types of S-int ideals with S-int $\mathfrak{BQI}$ ideals are provided. We established equivalent definitions and investigated the fundamental properties of this ideal. It is obtained that every S-int bi-ideal, S-int ideal, S-int interior ideal, S-int quasi-ideal, S-int bi-interior ideal, S-int bi-quasi ideal, and S-int quasi-interior ideal of a semigroup is an S-int $\mathfrak{BQI}$ ideal. We also showed, by means of counterexamples, that the converses do not hold in general.

The conditions for the converses to hold are also explored. In addition, we proved that the class of S-int $\mathfrak{BQI}$ ideals is closed under finite soft AND-products, Cartesian products, and intersections, but not under finite soft OR-products or unions. Moreover, it is shown that if a subsemigroup of a semigroup is a $\mathfrak{BQI}$ ideal, then its soft characteristic function is also an S-int $\mathfrak{BQI}$ ideal, and the converse is also true. The results presented here build a bridge between classical semigroup theory and soft set theory, offering new directions for future research. Future work could explore more characterizations of an S-int $\mathfrak{BQI}$ ideal with certain types of semigroups like intra-regular, weakly-regular, quasi-regular, semisimple and duo semigroups.

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