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KIRŞEHİR AHİ EVRAN ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI

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AYŞEGÜL KIZILARSLAN

YÜKSEK LİSANS TEZİ

KIRŞEHİR
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**DANIŞMAN**
**Prof. Dr. Ayşegül ÇETİNKAYA**

KIRŞEHİR
2025

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Kırşehir Ahi Evran Üniversitesi Bilimsel Araştırma ve Yayın Etiği Yönergesini okuduğumu ve anladığımı ve Kırşehir Ahi Evran Üniversitesi Fen Bilimleri Enstitüsü Tez Yazım Kurallarına uygun olarak hazırladığım bu tez çalışmasında;

- Tez içinde sunduğum verileri, bilgileri ve dokümanları akademik ve etik kurallar çerçevesinde elde ettiğimi,
- Tüm bilgi, belge, değerlendirme ve sonuçları bilimsel etik kurallarına uygun olarak sunduğumu,
- Tez çalışmasında yararlandığım eserlerin tümüne uygun atıfta bulunarak kaynak gösterdiğimi,
- Kullanılan verilerde ve ortaya çıkan sonuçlarda herhangi bir değişiklik yapmadığımı,
- Tez olarak sunduğum bu çalışmanın özgün olduğunu,

bildirir, aksi bir durumda bu konuda hakkımda yapılacak tüm yasal işlemleri ve aleyhime doğabilecek tüm hak kayıplarını kabullendiğimi beyan ederim.

01/07/2025

Ayşegül KIZILARSLAN

**İÇİNDEKİLER DİZİNİ**

**Sayfa No**

**TEŞEKKÜR**

Yüksek lisansımın her aşamasında bilgi ve tecrübesiyle bana yol gösteren, tüm içtenliği ve sabrı ile beni motive edip destekleyen, tez çalışmamın her satırında emeği olan çok kıymetli danışman hocam Prof. Dr. Ayşegül ÇETİNKAYA'ya sonsuz teşekkürlerimi sunarım.

Uzakta olmama rağmen sevgi ve desteğini içtenlikle hissettiren, bugünlere gelmemde büyük emeği olan, sabrını ve fedakarlığını asla esirgemeyen kıymetli fındık annem Hatice KIZILARSLAN'a, çok kıymetli canım babam Zekeriye KIZILARSLAN'a, varlığı bana hediye olan kız kardeşim Meryem KIZILARSLAN ve kalbimin bir parçası olan erkek kardeşim Sefa KIZILARSLAN'a; her zaman yanımda olan, kıymetli arkadaşım İsmail ŞAHİN'e sonsuz teşekkürlerimi sunarım.

Temmuz, 2025 Ayşegül KIZILARSLAN

**ÖZET**


**YÜKSEK LİSANS TEZİ**

# $k$-LAURICELLA FONKSİYONLARI VE BAZI ÖZELLİKLERİ



**Ayşegül KIZILARSLAN**

**KIRŞEHİR AHİ EVRAN ÜNİVERSİTESİ**
**FEN BİLİMLERİ ENSTİTÜSÜ**
**MATEMATİK ANABİLİM DALI**

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**Yıl: 2025 Sayfa: 73**
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Bu tezde $k$-Lauricella fonksiyonları tanımlanmış ve bazı özellikleri incelenmiştir. İlk olarak, klasik Lauricella fonksiyonlarının integral gösterimleri ve dönüşüm formülleri gibi bazı özellikleri hatırlatılmıştır. Daha sonra, Pochhammer $k$-sembolü kullanılarak $k$-Lauricella fonksiyonları tanımlanmıştır. Ayrıca, klasik Lauricella fonksiyonları ile $k$-Lauricella fonksiyonları arasındaki ilişkiler elde edilmiştir. Son olarak, $k$-Lauricella fonksiyonlarının bazı integral gösterimleri ve dönüşüm formülleri ispatlanmıştır. İspatlarda klasik yöntemlere ek olarak, bu ilişkiler ile klasik Lauricella fonksiyonlarının bilinen özelliklerinin kullanımına dayalı, daha kolay ve kısa olan ikinci bir yöntem kullanılmıştır.



**Anahtar Kelimeler:** Pochhammer sembolü, Lauricella fonksiyonları, Dönüşüm formülleri, İntegral gösterimleri.

## ABSTRACT

**MASTER'S THESIS**

# $k$-LAURICELLA FUNCTIONS AND SOME PROPERTIES

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In this thesis, $k$-Lauricella functions are defined and some of their properties are investigated. First, some properties of classical Lauricella functions such as their integral representations and transformation formulas are recalled. Then, $k$-Lauricella functions are defined using the Pochhammer $k$-symbol. Also, the relations between classical Lauricella functions and $k$-Lauricella functions are obtained. Finally, some integral representations and transformation formulas of $k$-Lauricella functions are proved. In the proofs, in addition to classical methods, a shorter and easier second method is employed, which is based on the use of these relations and known properties of classical Lauricella functions.

**Keywords:** Pochhammer symbol, Lauricella functions, Transformation formulas, Integral representations.

## SİMGELER VE KISALTMALAR DİZİNİ

| **Simgeler** | **Açıklama** |
|---|---|
| $\Gamma(x)$ | : Gamma Fonksiyonu |
| $B(x,y)$ | : Beta Fonksiyonu |
| $(\alpha)_n$ | : Pochhammer Sembolü |
| ${}_2F_1$ | : Gauss Hipergeometrik Fonksiyonu |
| ${}_1F_1$ | : Kummer Hipergeometrik Fonksiyonu |
| $F_1, F_2, F_3, F_4$ | : Appell Hipergeometrik Fonksiyonları |
| $H_A, H_B, H_C$ | : Srivastava Hipergeometrik Fonksiyonları |
| $G_1, G_2, G_3$ | : Horn Hipergeometrik Fonksiyonları |
| $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}, F_D^{(n)}$ | : Lauricella Fonksiyonları |
| $\Phi_2^{(n)}, \Psi_2^{(n)}, \Phi_D^{(n)}, \Xi_1^{(n)}, \Phi_3^{(n)}, \Psi_A^{(n)}$ | : Lauricella Fonksiyonlarının Konfluent Formları |
| $\Gamma_k(x)$ | : $k$-Gamma Fonksiyonu |
| $B_k(x,y)$ | : $k$-Beta Fonksiyonu |
| $(\alpha)_{n,k}$ | : Pochhammer $k$-Sembolü |
| ${}_2F_{1,k}$ | : $k$-Gauss Hipergeometrik Fonksiyonu |
| ${}_1F_{1,k}$ | : $k$-Kummer Hipergeometrik Fonksiyonu |
| $F_{1,k}, F_{2,k}, F_{3,k}, F_{4,k}$ | : $k$-Appell Hipergeometrik Fonksiyonları |
| $H_{A,k}, H_{B,k}, H_{C,k}$ | : $k$-Srivastava Hipergeometrik Fonksiyonları |
| $G_1^k, G_2^k, G_3^k$ | : $k$-Horn Hipergeometrik Fonksiyonları |
| $F_{A,k}^{(n)}, F_{B,k}^{(n)}, F_{C,k}^{(n)}, F_{D,k}^{(n)}$ | : $k$-Lauricella Fonksiyonları |
| $\Phi_{2,k}^{(n)}, \Psi_{2,k}^{(n)}, \Phi_{D,k}^{(n)}, \Xi_{1,k}^{(n)}, \Phi_{3,k}^{(n)}, \Psi_{A,k}^{(n)}$ | : $k$-Lauricella Fonksiyonlarının Konfluent Formları |

# 1. GİRİŞ

18. ve 19. yüzyıllarda Euler, Gauss ve Kummer gibi matematikçilerin, tek değişkenli hipergeometrik fonksiyonlar üzerine yoğun bir biçimde çalıştıkları görülmektedir. Daha sonra tek değişkenli hipergeometrik fonksiyonların, çok değişkenli versiyonlarını tanımlamak için birçok çalışma yapıldığı dikkat çekmektedir. Bu çalışmalardan biri de 1882 yılında Paul Appell tarafından iki Gauss hipergeometrik fonksiyonunun çarpımından hareketle tanımlanan iki değişkenli Appell hipergeometrik fonksiyonları ile ilgilidir [2]. Bir diğer çalışma ise 1893 yılında İtalyan matematikçi Giuseppe Lauricella tarafından Appell'in iki değişkenli fonksiyonlarını daha da genelleştirerek $n$ değişkenli Lauricella hipergeometrik fonksiyonlarının tanımlanması üzerinedir. Lauricella'nın bu konuya ilişkin çalışması "Sulle funzioni ipergeometriche a più variabili" ("Çok değişkenli hipergeometrik fonksiyonlar üzerine") başlıklı makalesinde yayınlanmıştır [15]. Lauricella fonksiyonları kuantum teorisi, elektromanyetik teori, olasılık teorisi, iletişim teorisi, optimizasyon gibi çeşitli bilimsel alanlarda önemli uygulamalara sahiptir [3, 4, 19].

Ayrıca literatür incelendiğinde hipergeometrik fonksiyonların genelleştirilmeleri ile ilgili çok sayıda çalışmaya rastlanmaktadır. Bu çalışmalardan bazıları da Pochhammer $k$-sembolü yardımıyla tanımlanan özel fonksiyonların $k$-genelleştirilmeleri ile ilgilidir. Örnek olarak $k$-Gauss hipergeometrik fonksiyonu, $k$-Kummer hipergeometrik fonksiyonu, $k$-Appell hipergeometrik fonksiyonları, $k$-Srivastava hipergeometrik fonksiyonları ve $k$-Horn hipergeometrik fonksiyonları verilebilir.

Bu tezin amacı, Pochhammer $k$-sembolü kullanılarak "$k$-Lauricella fonksiyonları" olarak adlandırılacak olan, klasik Lauricella fonksiyonlarının $k$-genelleştirmelerini vermek ve bazı özelliklerini incelemektir.

Bu tez beş ana bölümden oluşmaktadır. Birinci bölüm giriş kısmına ayrılmıştır. İkinci bölümde literatürde yer alan çeşitli fonksiyonların $k$-genelleştirmelerine yer verilmiştir. Üçüncü bölüm, dördüncü bölümde ihtiyaç duyulacak olan klasik Lauricella fonksiyonları ile ilgili Euler ve Laplace tipi integral formüllerini ve de dönüşüm formüllerini içermektedir. Dördüncü bölümde Pochhammer $k$-sembolü vasıtasıyla $k$-Lauricella fonksiyonları ve konfluent formları tanımlanmıştır. Ayrıca bu fonksiyonlar ile klasik versiyonları arasındaki ilişkiler verilmiştir. Daha sonra bu ilişkiler ve üçüncü bölümde verilen formüller kullanılarak $k$-Lauricella fonksiyonlarının Euler ve Laplace tipi integralleri ve de dönüşüm formülleri elde edilmiştir. Beşinci bölüm de sonuç ve önerilere ayrılmıştır.

## 2. ÖNCEKİ ÇALIŞMALAR

Literatür incelendiğinde son yıllarda özel fonksiyonlar ve onların genelleştirmeleri ile ilgili pek çok çalışmaya rastlanmaktadır. Bu çalışmaların bir kısmını da tek ya da çok değişkenli hipergeometrik fonksiyonların $k$-genelleştirmeleri oluşturmaktadır. Son yıllarda $k$-genelleştirmiş fonksiyonların uygulama alanlarının artmasına paralel olarak da konu ile ilgili çalışmalarda artış gözlemlenmektedir. Bu konu ile ilgili yapılan çalışmalardan bazıları şunlardır:

2007 yılında Diaz ve Pariguan [7] tarafından, $k$-Gamma fonksiyonu

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \quad Re(x) > 0 \tag{2.1}$$

$k$-Beta fonksiyonu

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt, \quad Re(x) > 0, Re(y) > 0 \tag{2.2}$$

ve Pochhammer $k$-sembolü

$$(\alpha)_{n,k} = \alpha(\alpha + k)(\alpha + 2k) \cdots (\alpha + (n-1)k), \quad n = 1,2,3,\ldots \tag{2.3}$$

$$(\alpha)_{0,k} = 1$$

şeklinde tanımlanmıştır. Burada $k \in \mathbb{R}^+$ dır. Ayrıca $k$-Gamma fonksiyonunun, $k$-Beta fonksiyonunun ve Pochhammer $k$-sembolünün aşağıda listelenen bazı önemli özellikleri verilmiştir [7, 16, 18].

$$\Gamma_k(x + k) = x\Gamma_k(x) \tag{2.4}$$

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)} \tag{2.5}$$

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha + nk)}{\Gamma_k(\alpha)}, \tag{2.6}$$

$$\frac{(\beta)_{n,k}}{(\gamma)_{n,k}} = \frac{B_k(\beta + nk, \gamma - \beta)}{B_k(\beta, \gamma - \beta)} \tag{2.7}$$

$$(\alpha)_{m+n,k} = (\alpha)_{m,k}(\alpha + mk)_{n,k} \tag{2.8}$$

$$\sum_{n=0}^{\infty} (\alpha)_{n,k} \frac{x^n}{n!} = (1 - kx)^{-\frac{\alpha}{k}}, \qquad |x| < \frac{1}{k} \tag{2.9}$$

Dahası bu $k$-genelleştirmeleri ile klasik versiyonları arasında aşağıdaki ilişkiler sunulmuştur.

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \tag{2.10}$$

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right) \tag{2.11}$$

$$(\alpha)_{n,k} = k^n\left(\frac{\alpha}{k}\right)_n \tag{2.12}$$

Aynı makalede Diaz ve Pariguan tarafından, $k$-Gauss hipergeometrik fonksiyonu

$$_2F_{1,k}(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{x^n}{n!}, \quad |x| < \frac{1}{k} \tag{2.13}$$

$k$-Kummer (konfluent) hipergeometrik fonksiyonu

$$_1F_{1,k}(\alpha;\gamma;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}}{(\gamma)_{n,k}}\frac{x^n}{n!}, \tag{2.14}$$

ve $_0F_1$ hipergeometrik fonksiyonunun $k$-genelleştirilmesi

$$_0F_{1,k}(-;\gamma;x) = \sum_{n=0}^{\infty}\frac{1}{(\gamma)_{n,k}}\frac{x^n}{n!}, \tag{2.15}$$

şeklinde tanımlanmıştır. Burada $k \in \mathbb{R}^+$ ve $\gamma \neq 0, -k, -2k, \ldots$ dır. Bu fonksiyonların söz konusu $k$-genelleştirmeleri ile klasik tanımları arasında

$$_2F_{1,k}(\alpha,\beta;\gamma;x) = {}_2F_1\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma}{k};kx\right) \tag{2.16}$$

$$_1F_{1,k}(\alpha;\gamma;x) = {}_1F_1\left(\frac{\alpha}{k};\frac{\gamma}{k};x\right) \tag{2.17}$$

$$_0F_{1,k}(-;\gamma;x) = {}_0F_1\left(-;\frac{\gamma}{k};\frac{x}{k}\right) \tag{2.18}$$

ilişkileri elde edilmiştir.

2015 yılında Mubeen ve ark. [17] tarafından, $F_1$ klasik Appell hipergeometrik fonksiyonunun $k$-genelleştirmesi

$$F_{1,k}(\alpha,\beta_1,\beta_2;\gamma;x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta_1)_{m,k}(\beta_2)_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{2.19}$$

$$(|x| < \frac{1}{k}, |y| < \frac{1}{k})$$

şeklinde tanımlanmış ve bu yeni fonksiyon için bir integral gösterimi elde edilmiştir.

2017 yılında Kıymaz ve ark. [14] tarafından, diğer $F_2$, $F_3$ ve $F_4$ klasik Appell hipergeometrik fonksiyonlarının $k$-genelleştirilmeleri

$$F_{2,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta_1)_{m,k}(\beta_2)_{n,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{2.20}$$

$$(|x|+|y|<\frac{1}{k}),$$

$$F_{3,k}(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha_1)_{m,k}(\alpha_2)_{n,k}(\beta_1)_{m,k}(\beta_2)_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{2.21}$$

$$\big(|x|<\frac{1}{k},\ |y|<\frac{1}{k}\big),$$

$$F_{4,k}(\alpha,\beta;\gamma_1,\gamma_2;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m+n,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{2.22}$$

$$(\sqrt{|x|}+\sqrt{|y|}<\frac{1}{\sqrt{k}}).$$

şeklinde tanımlanmıştır. Burada $k\in\mathbb{R}^+$ ve $i=1,2$ için $\gamma,\gamma_i\in\mathbb{C}\setminus\{k\mathbb{Z}_0^-\}$ dır. Ayrıca Appell hipergeometrik fonksiyonlarının $k$-genelleştirmeleri ile klasik tanımları arasında

$$F_{1,k}(\alpha,\beta_1,\beta_2;\gamma;x,y)=F_1\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky\right) \tag{2.23}$$

$$F_{2,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y)=F_2\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky\right) \tag{2.24}$$

$$F_{3,k}(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;x,y)=F_3\left(\frac{\alpha_1}{k},\frac{\alpha_2}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky\right) \tag{2.25}$$

$$F_{4,k}(\alpha,\beta;\gamma_1,\gamma_2;x,y)=F_4\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky\right) \tag{2.26}$$

ilişkilerinin sağlandığı gösterilmiştir. Bu ilişkiler kullanılarak $k$-Appell fonksiyonlarının birçok özelliğinin kısaca ve kolayca elde edilebileceğini dikkat çekmek adına, bu fonksiyonların bazı integral gösterimleri ve Mellin dönüşümleri verilmiştir.

2020 yılında Gürel Yılmaz ve ark. [9] tarafından, yukarıda tanımlanan $k$-Appell hipergeometrik fonksiyonları için bazı dönüşüm formülleri, indirgeme formülleri ve üretici fonksiyon bağıntıları sunulmuştur.

2022 yılında Halıcı [10] yüksek lisans tezinde, $H_{A,k}$, $H_{B,k}$ ve $H_{C,k}$ ile gösterilen $k$-Srivastava hipergeometrik fonksiyonlarını

$$H_{A,k}(\alpha,\beta_1,\beta_2;\nu_1,\nu_2;x_1,x_2,x_3)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\nu_1)_{m,k}(\nu_2)_{n+p,k}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!} \quad (2.27)$$

$$\big(r_1<1, r_2<1, r_3<(1-r_1)(1-r_2)\big),$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\nu_1,\nu_2,\nu_3;x_1,x_2,x_3)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\nu_1)_{m,k}(\nu_2)_{n,k}(\nu_3)_{p,k}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!} \quad (2.28)$$

$$\big(r_1+r_2+r_3+2\sqrt{r_1r_2r_3}<1\big),$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\nu;x_1,x_2,x_3)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\nu)_{m+n+p,k}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!} \quad (2.29)$$

$$\big(r_1<1, r_2<1, r_3<1, r_1+r_2+r_3-2\sqrt{(1-r_1)(1-r_2)(1-r_3)}<2\big)$$

biçiminde tanımlamıştır (ayrıca bkz. [11]) . Burada $k\in\mathbb{R}^+$, $i=1,2,3$ için $\nu,\nu_i\in\mathbb{C}\backslash\{k\mathbb{Z}_0^-\}$ ve $r_1=k|x_1|, r_2=k|x_2|, r_3=k|x_3|$ dir. $k$-Srivastava hipergeometrik fonksiyonları ile klasik Srivastava hipergeometrik fonksiyonları arasında

$$H_{A,k}(\alpha,\beta_1,\beta_2;\nu_1,\nu_2;x_1,x_2,x_3)=H_A(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\nu_1}{k},\frac{\nu_2}{k};kx_1,kx_2,kx_3) \quad (2.30)$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\nu_1,\nu_2,\nu_3;x_1,x_2,x_3)=H_B(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\nu_1}{k},\frac{\nu_2}{k},\frac{\nu_3}{k};kx_1,kx_2,kx_3) \quad (2.31)$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\nu;x_1,x_2,x_3)=H_C(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\nu}{k};kx_1,kx_2,kx_3) \quad (2.32)$$

ilişkileri bulunmuştur. Bu ilişkiler kullanılarak $k$-Srivastava hipergeometrik fonksiyonlarının integral gösterimleri ve yineleme formülleri uzun ispatlara gerek olmaksızın kısa ve kolay bir şekilde elde edilebileceği gösterilmiştir. Dahası bu çalışmada gerekliliğine ihtiyaç duyulan

$X_4$ klasik Exton hipergeometrik fonksiyonunun $k$-genelleştirmesi

$$X_{4,k}\left(\alpha,\beta;\nu_1,\nu_2,\nu_3;x_1,x_2,x_3\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{2m+n+p,k}(\beta)_{n+p,k}}{(\nu_1)_{m,k}(\nu_2)_{n,k}(\nu_3)_{p,k}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!} \tag{2.33}$$

şeklinde tanımlanmış ve aralarındaki ilişki

$$X_{4,k}\left(\alpha,\beta;\nu_1,\nu_2,\nu_3;x_1,x_2,x_3\right)=X_4\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\nu_1}{k},\frac{\nu_2}{k},\frac{\nu_3}{k};kx_1,kx_2,kx_3\right) \tag{2.34}$$

şeklinde verilmiştir.

2023 yılında Çatak [5] yüksek lisans tezinde, $G_1^k$, $G_2^k$ ve $G_3^k$ ile gösterilen $k$-Horn hipergeometrik fonksiyonlarını

$$G_1^k(\alpha,\beta,\gamma;z,\omega)=\sum_{n,m=0}^{\infty}(\alpha)_{m+n,k}(\beta)_{m-n,k}(\gamma)_{n-m,k}\frac{z^n}{n!}\frac{\omega^m}{m!} \tag{2.35}$$

$$G_2^k(\alpha,\lambda,\beta,\gamma;z,\omega)=\sum_{n,m=0}^{\infty}(\alpha)_{m,k}(\lambda)_{n,k}(\beta)_{m-n,k}(\gamma)_{n-m,k}\frac{z^n}{n!}\frac{\omega^m}{m!} \tag{2.36}$$

$$G_3^k(\alpha,\lambda;z,\omega)=\sum_{n,m=0}^{\infty}(\alpha)_{2n-m,k}(\lambda)_{2m-n,k}\frac{z^n}{n!}\frac{\omega^m}{m!} \tag{2.37}$$

şeklinde tanımlamıştır (ayrıca bkz. [6]). Bu fonksiyonlar için türev, integral ve yineleme formülleri elde edilmiş ve bu fonksiyonların $k$-Gauss hipergeometrik fonksiyonu ile olan ilişkileri üzerinde durulmuştur.

Belirtelim ki bu bölümde bahsi geçen tüm $k$-genelleştirmeleri $k = 1$ alınması durumunda, klasik karşılıklarına indirgenirler.

## 3. MATERYAL VE METOT

Bu bölümde, sonraki bölümde ihtiyaç duyulan klasik Lauricella fonksiyonlarının literatürde yer alan bazı özelliklerine değinilmiştir [8, 12, 13, 15, 20, 21].

### 3.1. Lauricella Fonksiyonları

"Lauricella hipergeometrik fonksiyonları" ya da kısaca "Lauricella fonksiyonları" olarak adlandırılan $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ ve $F_D^{(n)}$ fonksiyonları aşağıda verilmiştir [8, 15, 20].

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{3.1}$$

$$\big(|x_1| + \cdots + |x_n| < 1\big),$$

$$F_B^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{3.2}$$

$$\big(\max\{|x_1|, \ldots, |x_n|\} < 1\big),$$

$$F_C^{(n)}(\alpha, \beta; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{3.3}$$

$$\big(\sqrt{|x_1|} + \cdots + \sqrt{|x_n|} < 1\big),$$

$$F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \tag{3.4}$$

$$\big(\max\{|x_1|, \ldots, |x_n|\} < 1\big).$$

Burada $i = 1, \ldots, n$ için $\gamma, \gamma_i \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}$ dir. Belirtelim ki $n = 2$ özel durumu için

$$F_A^{(2)} = F_2, F_B^{(2)} = F_3, F_C^{(2)} = F_4 \text{ ve } F_D^{(2)} = F_1$$

olup, $F_1$, $F_2$, $F_3$ ve $F_4$ klasik Appell hipergeometrik fonksiyonlarıdır.

### 3.2. Lauricella Fonksiyonlarının Konfluent Formları

Lauricella fonksiyonlarının konfluent formları,

$$\Phi_2^{(n)}(\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{3.5}$$

$$\Psi_2^{(n)}(\alpha;\gamma_1,\dots,\gamma_n;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{3.6}$$

$$\Phi_D^{(n)}(\alpha,\beta_1,\dots,\beta_{n-1};\gamma;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{3.7}$$

$$\Xi_1^{(n)}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1};\gamma;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{3.8}$$

$$\Phi_3^{(n)}(\beta_1,\dots,\beta_{n-1};\gamma;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{3.9}$$

$$\Psi_A^{(n)}(\alpha,\beta_1,\dots,\beta_{n-1};\gamma_1,\dots,\gamma_n;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{3.10}$$

şeklinde tanımlanır [8, 20].

**Teorem 3.1.** Aşağıdaki eşitlikler sağlanır [8, 20].

$$\Phi_2^{(n)}(\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n) = \lim_{|\alpha|\to\infty} F_D^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;\frac{x_1}{\alpha},\dots,\frac{x_n}{\alpha}) \tag{3.11}$$

$$\Phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = \lim_{min\{|\alpha_1|,\ldots,|\alpha_n|\}\to\infty} F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha_1},\ldots,\frac{x_n}{\alpha_n}) \tag{3.12}$$

$$\Psi_2^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = \lim_{|\beta|\to\infty} F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;\frac{x_1}{\beta},\ldots,\frac{x_n}{\beta}) \tag{3.13}$$

$$\Psi_2^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = \lim_{min\{|\beta_1|,\ldots,|\beta_n|\}\to\infty} F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{\beta_1},\ldots,\frac{x_n}{\beta_n}) \tag{3.14}$$

$$\Phi_D^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n) = \lim_{|\beta_n|\to\infty} F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{3.15}$$

$$\Xi_1^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n) = \lim_{|\beta_n|\to\infty} F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{3.16}$$

$$\Phi_3^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n) = \lim_{|\beta_n|\to\infty} \Phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{3.17}$$

$$\Phi_3^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n) = \lim_{min\{|\alpha|,|\beta_n|\}\to\infty} F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha},\ldots,\frac{x_{n-1}}{\alpha},\frac{x_n}{\alpha\beta_n}) \tag{3.18}$$

$$\Phi_3^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n) = \lim_{min\{|\alpha_1|,\ldots,|\alpha_n|,|\beta_n|\}\to\infty} F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha_1},\ldots,\frac{x_{n-1}}{\alpha_{n-1}},\frac{x_n}{\alpha_n\beta_n}) \tag{3.19}$$

$$\Psi_A^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = \lim_{|\beta_n|\to\infty} F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{3.20}$$

**İspat**. (3.11)-(3.20) eşitliklerin ispatları için,

$$\lim_{|\alpha|\to\infty} \frac{(\alpha)_n}{\alpha^n} = 1 \tag{3.21}$$

özelliğinin dikkate alınması yeterlidir. Şimdi bunu görelim:

(3.11) eşitliğinin ispatı,

$$\begin{aligned}
&\lim_{|\alpha|\to\infty} F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha},\ldots,\frac{x_n}{\alpha})\\
&=\lim_{|\alpha|\to\infty}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{(\frac{x_1}{\alpha})^{m_1}}{m_1!}\cdots\frac{(\frac{x_n}{\alpha})^{m_n}}{m_n!}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\Big(\lim_{|\alpha|\to\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{\alpha^{m_1+\cdots+m_n}}\Big)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\Phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)
\end{aligned}$$

şeklindedir.

(3.12) eşitliğinin ispatı,

$$\begin{aligned}
&\lim_{min\{|\alpha_1|,\ldots,|\alpha_n|\}\to\infty} F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha_1},\ldots,\frac{x_n}{\alpha_n})\\
&=\lim_{min\{|\alpha_1|,\ldots,|\alpha_n|\}\to\infty}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{(\frac{x_1}{\alpha_1})^{m_1}}{m_1!}\cdots\frac{(\frac{x_n}{\alpha_n})^{m_n}}{m_n!}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\Big(\lim_{min\{|\alpha_1|,\ldots,|\alpha_n|\}\to\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}}{\alpha_1^{m_1}\cdots\alpha_n^{m_n}}\Big)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\Phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)
\end{aligned}$$

şeklindedir.

(3.13) eşitliğinin ispatı,

$$\begin{aligned}
&\lim_{|\beta|\to\infty} F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;\frac{x_1}{\beta},\ldots,\frac{x_n}{\beta})\\
&=\lim_{|\beta|\to\infty}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{(\frac{x_1}{\beta})^{m_1}}{m_1!}\cdots\frac{(\frac{x_n}{\beta})^{m_n}}{m_n!}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\Big(\lim_{|\beta|\to\infty}\frac{(\beta)_{m_1+\cdots+m_n}}{\beta^{m_1+\cdots+m_n}}\Big)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\Psi_2^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)
\end{aligned}$$

şeklindedir.

(3.14) eşitliğinin ispatı,

$$
\begin{aligned}
&\lim_{min\{|\beta_1|,\dots,|\beta_n|\}\to\infty} F_A^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n;\frac{x_1}{\beta_1},\dots,\frac{x_n}{\beta_n}) \\
&= \lim_{min\{|\beta_1|,\dots,|\beta_n|\}\to\infty} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{(\frac{x_1}{\beta_1})^{m_1}}{m_1!}\cdots\frac{(\frac{x_n}{\beta_n})^{m_n}}{m_n!} \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \left(\lim_{min\{|\beta_1|,\dots,|\beta_n|\}\to\infty} \frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{\beta_1^{m_1}\cdots\beta_n^{m_n}}\right) \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \Psi_2^{(n)}(\alpha;\gamma_1,\dots,\gamma_n;x_1,\dots,x_n)
\end{aligned}
$$

şeklindedir.

(3.15) eşitliğinin ispatı,

$$
\begin{aligned}
&\lim_{|\beta_n|\to\infty} F_D^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_{n-1},\frac{x_n}{\beta_n}) \\
&= \lim_{|\beta_n|\to\infty} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{(\frac{x_n}{\beta_n})^{m_n}}{m_n!} \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \left(\lim_{|\beta_n|\to\infty} \frac{(\beta_n)_{m_n}}{\beta_n^{m_n}}\right) \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \Phi_D^{(n)}(\alpha,\beta_1,\dots,\beta_{n-1};\gamma;x_1,\dots,x_n)
\end{aligned}
$$

şeklindedir.

(3.16) eşitliğinin ispatı,

$$
\begin{aligned}
&\lim_{|\beta_n|\to\infty} F_B^{(n)}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_{n-1},\frac{x_n}{\beta_n}) \\
&= \lim_{|\beta_n|\to\infty} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{(\frac{x_n}{\beta_n})^{m_n}}{m_n!} \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \left(\lim_{|\beta_n|\to\infty} \frac{(\beta_n)_{m_n}}{{\beta_n}^{m_n}}\right) \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \Xi_1^{(n)}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1};\gamma;x_1,\dots,x_n)
\end{aligned}
$$

şeklindedir.

(3.17) eşitliğinin ispatı,

$$
\begin{aligned}
&\lim_{|\beta_n|\to\infty} \Phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \\
&= \lim_{|\beta_n|\to\infty} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{(\frac{x_n}{\beta_n})^{m_n}}{m_n!} \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \Big(\lim_{|\beta_n|\to\infty}\frac{(\beta_n)_{m_n}}{\beta_n^{m_n}}\Big)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \Phi_3^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)
\end{aligned}
$$

şeklindedir.

(3.18) ve (3.19) eşitliklerinin ispatı için, (3.17) nin sağ yanındaki $\Phi_2^{(n)}$ fonksiyonu yerine sırasıyla (3.11) ve (3.12) deki eşitliklerin dikkate alınması yeterlidir.

(3.20) eşitliğinin ispatı,

$$
\begin{aligned}
&\lim_{|\beta_n|\to\infty} F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \\
&= \lim_{|\beta_n|\to\infty} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{(\frac{x_n}{\beta_n})^{m_n}}{m_n!} \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \Big(\lim_{|\beta_n|\to\infty}\frac{(\beta_n)_{m_n}}{\beta_n^{m_n}}\Big)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \\
&= \Psi_A^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)
\end{aligned}
$$

şeklindedir. ■

### 3.3. Lauricella Fonksiyonlarının Tek Katlı Laplace Tipi İntegralleri

**Teorem 3.2.** Lauricella fonksiyonlarının tek katlı Laplace tipi integralleri

$$
\begin{aligned}
F_A^{(n)}(&\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}t^{\alpha-1}{}_1F_1(\beta_1;\gamma_1;tx_1)\cdots{}_1F_1(\beta_n;\gamma_n;tx_n)\,dt \qquad (3.22)
\end{aligned}
$$

$$(Re(\alpha)>0)$$

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\frac{1}{\Gamma(\beta_n)}\int_0^\infty e^{-t}t^{\beta_n-1}\Psi_A^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_{n-1},tx_n)dt \quad (3.23)$$

$$(Re(\beta_n)>0)$$

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$=\frac{1}{\Gamma(\beta_n)}\int_0^\infty e^{-t}t^{\beta_n-1}\Xi_1^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_{n-1},tx_n)dt \quad (3.24)$$

$$(Re(\beta_n)>0)$$

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}t^{\alpha-1}\Psi_2^{(n)}(\beta;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt \quad (3.25)$$

$$(Re(\alpha)>0)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}t^{\alpha-1}\Phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma;tx_1,\ldots,tx_n)dt \quad (3.26)$$

$$(Re(\alpha)>0)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$=\frac{1}{\Gamma(\beta_n)}\int_0^\infty e^{-t}t^{\beta_n-1}\Phi_D^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_{n-1},tx_n)dt \quad (3.27)$$

$$(Re(\beta_n)>0)$$

şeklindedir [20].

**İspat**.
Biliyoruz ki Pochhammer sembolünün integral gösterimi

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}t^{\alpha+n-1}dt,\qquad Re(\alpha)>0 \quad (3.28)$$

dir [13, 21]. Şimdi bu integral gösterimini birer kez kullanarak aşağıdaki ispatları verelim:

(3.22) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}(\alpha)_{m_1+\cdots+m_n}\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\alpha)}\sum_{m_1,\ldots,m_n=0}^{\infty}\int_0^{\infty}e^{-t}t^{\alpha+m_1+\cdots+m_n-1}dt\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}e^{-t}t^{\alpha-1}\Big(\sum_{m_1=0}^{\infty}\frac{(\beta_1)_{m_1}}{(\gamma_1)_{m_1}}\frac{(tx_1)^{m_1}}{m_1!}\cdots\sum_{m_n=0}^{\infty}\frac{(\beta_n)_{m_n}}{(\gamma_n)_{m_n}}\frac{(tx_n)^{m_n}}{m_n!}\Big)dt\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}e^{-t}t^{\alpha-1}{}_1F_1\left(\beta_1;\gamma_1;tx_1\right)\cdots{}_1F_1\left(\beta_n;\gamma_n;tx_n\right)dt
\end{aligned}
$$

şeklindedir.

(3.23) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}(\beta_n)_{m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\beta_n)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\int_0^{\infty}e^{-t}t^{\beta_n+m_n-1}dt\\
&\qquad\cdot\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\beta_n)}\int_0^{\infty}e^{-t}t^{\beta_n-1}\Big(\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\\
&\qquad\cdot\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{(tx_n)^{m_n}}{m_n!}\Big)dt\\
&=\frac{1}{\Gamma(\beta_n)}\int_0^{\infty}e^{-t}t^{\beta_n-1}\Psi_A^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_{n-1},tx_n)dt
\end{aligned}
$$

şeklindedir.

(3.24) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}}(\beta_n)_{m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\beta_n)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}}\int_0^{\infty}e^{-t}t^{\beta_n+m_n-1}dt\\
&\qquad\cdot\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}
\end{aligned}
$$

$$= \frac{1}{\Gamma(\beta_n)} \int_0^\infty e^{-t} t^{\beta_n - 1} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\beta_1)_{m_1} \cdots (\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!} \frac{(tx_n)^{m_n}}{m_n!} dt$$

$$= \frac{1}{\Gamma(\beta_n)} \int_0^\infty e^{-t} t^{\beta_n - 1} \Xi_1^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}; \gamma; x_1, \ldots, x_{n-1}, tx_n) dt$$

şeklindedir.

(3.25) integral gösteriminin ispatı,

$$F_C^{(n)}(\alpha, \beta; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} (\alpha)_{m_1+\cdots+m_n} \frac{(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{m_1,\ldots,m_n=0}^{\infty} \int_0^\infty e^{-t} t^{\alpha+m_1+\cdots+m_n-1} dt \frac{(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{(tx_1)^{m_1}}{m_1!} \cdots \frac{(tx_n)^{m_n}}{m_n!} dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} \Psi_2^{(n)}(\beta; \gamma_1, \ldots, \gamma_n; tx_1, \ldots, tx_n) dt$$

şeklindedir.

(3.26) integral gösteriminin ispatı aşağıdadır.

$$F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} (\alpha)_{m_1+\cdots+m_n} \frac{(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{m_1,\ldots,m_n=0}^{\infty} \int_0^\infty e^{-t} t^{\alpha+m_1+\cdots+m_n-1} dt \frac{(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{(tx_1)^{m_1}}{m_1!} \cdots \frac{(tx_n)^{m_n}}{m_n!} dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} \Phi_2^{(n)}(\beta_1, \ldots, \beta_n; \gamma; tx_1, \ldots, tx_n) dt$$

(3.27) integral gösteriminin ispatı aşağıdadır.

$$F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} (\beta_n)_{m_n} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \frac{1}{\Gamma(\beta_n)} \sum_{m_1,\ldots,m_n=0}^{\infty} \int_0^\infty e^{-t} t^{\beta_n+m_n-1} dt \frac{(\alpha)_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \frac{1}{\Gamma(\beta_n)} \int_0^\infty e^{-t} t^{\beta_n-1} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_n}} \cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!} \frac{(tx_n)^{m_n}}{m_n!} dt$$

$$= \frac{1}{\Gamma(\beta_n)} \int_0^\infty e^{-t} t^{\beta_n-1} \Phi_D^{(n)}(\alpha, \beta_1, \ldots, \beta_{n-1}; \gamma; x_1, \ldots, x_{n-1}, tx_n) dt$$

■

### 3.4. Lauricella Fonksiyonlarının Çok Katlı Laplace Tipi İntegralleri

**Teorem 3.3.** Lauricella fonksiyonlarının çok katlı Laplace tipi integralleri,

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \frac{1}{\Gamma(\beta_1) \cdots \Gamma(\beta_n)} \cdot \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\beta_1-1} \cdots t_n^{\beta_n-1} \cdot \Psi_2^{(n)}(\alpha; \gamma_1, \ldots, \gamma_n; t_1x_1, \ldots, t_nx_n) dt_1 \cdots dt_n \tag{3.29}$$

$$(Re(\beta_j) > 0, j = 1, \ldots, n)$$

$$F_B^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \frac{1}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n) \Gamma(\beta_1) \cdots \Gamma(\beta_n)} \cdot \int_0^\infty \cdots \int_0^\infty e^{-u_1-\cdots-u_n-v_1-\cdots-v_n} u_1^{\alpha_1-1} \cdots u_n^{\alpha_n-1} v_1^{\beta_1-1} \cdots v_n^{\beta_n-1} \cdot {}_0F_1(-; \gamma; u_1v_1x_1 + \cdots + u_nv_nx_n) du_1 \cdots du_n dv_1 \cdots dv_n \tag{3.30}$$

$$(\min\{R(\alpha_j), R(\beta_j)\} > 0, j = 1, \ldots, n)$$

$$F_B^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \frac{1}{\Gamma(\beta_1) \cdots \Gamma(\beta_n)} \cdot \int_0^\infty \cdots \int_0^\infty e^{-u_1-\cdots-u_n} u_1^{\beta_1-1} \cdots u_n^{\beta_n-1} \cdot \Phi_2^{(n)}(\alpha_1, \ldots, \alpha_n; \gamma; u_1x_1, \ldots, u_nx_n) du_1 \cdots du_n \tag{3.31}$$

$$(Re(\beta_j) > 0, j = 1, \ldots, n)$$

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\cdot\int_0^\infty\int_0^\infty e^{-u-v}u^{\alpha-1}v^{\beta-1}{}_0F_1(-;\gamma_1;uvx_1)\cdots{}_0F_1(-;\gamma_n;uvx_n)dudv \tag{3.32}$$

$$(\min\{R(\alpha),R(\beta)\}>0)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}$$
$$\cdot\int_0^\infty\cdots\int_0^\infty e^{-u-v_1-\cdots-v_n}u^{\alpha-1}v_1^{\beta_1-1}\cdots v_n^{\beta_n-1}$$
$$\cdot{}_0F_1(-;\gamma;(v_1x_1+\cdots+v_nx_n)u)dudv_1\cdots dv_n \tag{3.33}$$

$$(R(\alpha)>0;R(\beta_j)>0,j=1,\ldots,n)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}$$
$$\cdot\int_0^\infty\cdots\int_0^\infty e^{-t_1-\cdots-t_n}t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}$$
$$\cdot{}_1F_1(\alpha;\gamma;t_1x_1+\cdots+t_nx_n)dt_1\cdots dt_n \tag{3.34}$$

$$(Re(\beta_j)>0,j=1,\ldots,n)$$

şeklindedir [8, 13].

**İspat**. Yukarıdaki eşitliklerin ispatı için (3.28) de verilen Pochhammer sembolünün integral gösteriminin birden çok kez kullanılması yeterlidir. Şimdi bunu görelim:
(3.29) integral gösterimi, (3.28) formülünün $n$ kez kullanılmasıyla

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$\cdot\int_0^\infty\cdots\int_0^\infty e^{-t_1-\cdots-t_n}t_1^{\beta_1+m_1-1}\cdots t_n^{\beta_n+m_n-1}dt_1\cdots dt_n$$

$$= \frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\beta_1-1} \cdots t_n^{\beta_n-1}$$
$$\cdot \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{(t_1x_1)^{m_1}}{m_1!} \cdots \frac{(t_nx_n)^{m_n}}{m_n!} dt_1 \cdots dt_n$$
$$= \frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\beta_1-1} \cdots t_n^{\beta_n-1}$$
$$\cdot \Psi_2^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;t_1x_1,\ldots,t_nx_n) dt_1 \cdots dt_n$$

şeklinde elde edilir.

(3.30) integral gösterimi, (3.28) formülünün $2n$ kez kullanılmasıyla

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{1}{(\gamma)_{m_1+\cdots+m_n}} (\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$= \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}$$
$$\cdot \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{1}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \int_0^\infty \cdots \int_0^\infty e^{-u_1-\cdots-u_n-v_1-\cdots-v_n}$$
$$\cdot u_1^{\alpha_1+m_1-1}\cdots u_n^{\alpha_n+m_n-1} v_1^{\beta_1+m_1-1}\cdots v_n^{\beta_n+m_n-1} du_1\cdots du_n dv_1\cdots dv_n$$
$$= \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}$$
$$\cdot \int_0^\infty \cdots \int_0^\infty e^{-u_1-\cdots-u_n-v_1-\cdots-v_n} u_1^{\alpha_1-1}\cdots u_n^{\alpha_n-1} v_1^{\beta_1-1}\cdots v_n^{\beta_n-1}$$
$$\cdot \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{1}{(\gamma)_{m_1+\cdots+m_n}} \frac{(x_1u_1v_1)^{m_1}}{m_1!}\cdots\frac{(x_nu_nv_n)^{m_n}}{m_n!} du_1\cdots du_n dv_1\cdots dv_n$$
$$= \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}$$
$$\cdot \int_0^\infty \cdots \int_0^\infty e^{-u_1-\cdots-u_n-v_1-\cdots-v_n} u_1^{\alpha_1-1}\cdots u_n^{\alpha_n-1} v_1^{\beta_1-1}\cdots v_n^{\beta_n-1}$$
$$\cdot \sum_{m=0}^{\infty} \frac{1}{(\gamma)_m} \frac{(x_1u_1v_1+\cdots+x_nu_nv_n)^m}{m!} du_1\cdots du_n dv_1\cdots dv_n$$

şeklinde bulunur. Belirtelim ki son integraldeki seri ${}_0F_1(-;\gamma;u_1v_1x_1+\cdots+u_nv_nx_n)$ fonksiyonu ile ifade edilebilir. Ayrıca son eşitliğe geçişte

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\cdots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty} f(m)\frac{(x_1+\cdots+x_n)^m}{m!} \tag{3.35}$$

formülünün [21] kullanıldığını vurgulayalım.

(3.31) integral gösterimi, (3.28) formülünün $n$ kez kullanılmasıyla

$$\begin{aligned}
&F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\cdot\int_0^\infty\cdots\int_0^\infty e^{-u_1-\cdots-u_n}u_1^{\beta_1+m_1-1}\cdots u_n^{\beta_n+m_n-1}du_1\cdots du_n\\
&=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\int_0^\infty\cdots\int_0^\infty e^{-u_1-\cdots-u_n}u_1^{\beta_1-1}\cdots u_n^{\beta_n-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{(u_1x_1)^{m_1}}{m_1!}\cdots\frac{(u_nx_n)^{m_n}}{m_n!}du_1\cdots du_n\\
&=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\int_0^\infty\cdots\int_0^\infty e^{-u_1-\cdots-u_n}u_1^{\beta_1-1}\cdots u_n^{\beta_n-1}\\
&\qquad\cdot\Phi_2^{(n)}(\alpha_1,\ldots,\alpha_n;\gamma;u_1x_1,\ldots,u_nx_n)du_1\cdots du_n
\end{aligned}$$

şeklinde elde edilir.

(3.32) integral gösterimi, (3.28) formülünün iki kez kullanılmasıyla

$$\begin{aligned}
&F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}(\alpha)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\\
&\qquad\cdot\int_0^\infty\int_0^\infty e^{-u-v}u^{\alpha+m_1+\cdots+m_n-1}v^{\beta+m_1+\cdots+m_n-1}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}dudv\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty e^{-u-v}u^{\alpha-1}v^{\beta-1}\\
&\qquad\cdot\sum_{m_1=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}}\frac{(uvx_1)^{m_1}}{m_1!}\cdots\sum_{m_n=0}^{\infty}\frac{1}{(\gamma_n)_{m_n}}\frac{(uvx_n)^{m_n}}{m_n!}dudv\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty e^{-u-v}u^{\alpha-1}v^{\beta-1}{}_0F_1(-;\gamma_1;uvx_1)\cdots{}_0F_1(-;\gamma_n;uvx_n)dudv
\end{aligned}$$

şeklinde bulunur.

(3.33) integral gösterimi, (3.28) formülünün $n+1$ kez kullanılmasıyla ve (3.35) formülünün dikkate alınmasıyla

$$
\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{1}{(\gamma)_{m_1+\cdots+m_n}}(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{1}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\cdot\int_0^\infty\cdots\int_0^\infty e^{-u-v_1-\cdots-v_n}u^{\alpha+m_1+\cdots+m_n-1}v_1^{\beta_1+m_1-1}\cdots v_n^{\beta_n+m_n-1}du dv_1\cdots dv_n\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\int_0^\infty\cdots\int_0^\infty e^{-u-v_1-\cdots-v_n}u^{\alpha-1}v_1^{\beta_1-1}\cdots v_n^{\beta_n-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{1}{(\gamma)_{m_1+\cdots+m_n}}\frac{(uv_1x_1)^{m_1}}{m_1!}\cdots\frac{(uv_nx_n)^{m_n}}{m_n!}du dv_1\cdots dv_n\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\int_0^\infty\cdots\int_0^\infty e^{-u-v_1-\cdots-v_n}u^{\alpha-1}v_1^{\beta_1-1}\cdots v_n^{\beta_n-1}\\
&\qquad\cdot\sum_{m=0}^{\infty}\frac{1}{(\gamma)_m}\frac{(uv_1x_1+\cdots+uv_nx_n)^m}{m!}du dv_1\cdots dv_n\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\int_0^\infty\cdots\int_0^\infty e^{-u-v_1-\cdots-v_n}u^{\alpha-1}v_1^{\beta_1-1}\cdots v_n^{\beta_n-1}\\
&\qquad\cdot{}_0F_1(-;\gamma;(v_1x_1+\cdots+v_nx_n)u)du dv_1\cdots dv_n
\end{aligned}
$$

şeklinde elde edilir.

(3.34) integral gösterimi, (3.28) formülünün $n$ kez kullanılmasıyla ve (3.35) formülünün dikkate alınmasıyla

$$
\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma)_{m_1+\cdots+m_n}}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\cdot\int_0^\infty\cdots\int_0^\infty e^{-t_1-\cdots-t_n}t_1^{\beta_1+m_1-1}\cdots t_n^{\beta_n+m_n-1}dt_1\cdots dt_n\\
&=\frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}\int_0^\infty\cdots\int_0^\infty e^{-t_1-\cdots-t_n}t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{(t_1x_1)^{m_1}}{m_1!}\cdots\frac{(t_nx_n)^{m_n}}{m_n!}dt_1\cdots dt_n
\end{aligned}
$$

$$= \frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}$$
$$\cdot \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m} \frac{(t_1x_1+\cdots+t_nx_n)^m}{m!} dt_1\cdots dt_n$$
$$= \frac{1}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}$$
$$\cdot {}_1F_1(\alpha;\gamma;t_1x_1+\cdots+t_nx_n)dt_1\cdots dt_n$$

şeklinde bulunur. ■

## 3.5. $F_A^{(n)}$ ve $F_D^{(n)}$ Fonksiyonlarının Euler Tipi İntegralleri

**Teorem 3.4.** $F_A^{(n)}$ ve $F_D^{(n)}$ Lauricella fonksiyonlarının Euler tipi integralleri şunlardır [8,12]:

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$= \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)} \int_0^1 \cdots \int_0^1 t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}$$
$$\cdot (1-t_1)^{\gamma_1-\beta_1-1}\cdots(1-t_n)^{\gamma_n-\beta_n-1}(1-x_1t_1-\cdots-x_nt_n)^{-\alpha}dt_1\cdots dt_n \tag{3.36}$$

$$(Re(\gamma_j) > Re(\beta_j) > 0, j=1,\ldots,n)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-x_1t)^{-\beta_1}\cdots(1-x_nt)^{-\beta_n}dt \tag{3.37}$$

$$(Re(\gamma) > Re(\alpha) > 0)$$

**İspat**. İspata başlamadan önce

$$\frac{(\beta)_n}{(\gamma)_n} = \frac{B(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)} = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt \tag{3.38}$$

$$(Re(\gamma) > Re(\beta) > 0)$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.39}$$

$$\sum_{n=0}^{\infty}(\alpha)_n \frac{x^n}{n!} = (1-x)^{-\alpha}, \qquad |x|<1 \tag{3.40}$$

özelliklerinin sağlandığını hatırlatalım [1, 21]. Bu özelliklerin yardımıyla teoremin ispatı aşağıdaki şekilde yapılır:

(3.36) integral gösterimi, $F_A^{(n)}$ nin tanımında sırasıyla (3.38), (3.35), (3.39) ve (3.40) özelliklerinin kullanılmasıyla

$$
\begin{aligned}
&F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) \\
&= \sum_{m_1, \ldots, m_n=0}^{\infty} (\alpha)_{m_1+\cdots+m_n} \frac{(\beta_1)_{m_1}}{(\gamma_1)_{m_1}} \cdots \frac{(\beta_n)_{m_n}}{(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1, \ldots, m_n=0}^{\infty} (\alpha)_{m_1+\cdots+m_n} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \frac{1}{B(\beta_1, \gamma_1 - \beta_1) \cdots B(\beta_n, \gamma_n - \beta_n)} \\
&\cdot \int_0^1 \cdots \int_0^1 t_1^{\beta_1+m_1-1} \cdots t_n^{\beta_n+m_n-1} (1-t_1)^{\gamma_1-\beta_1-1} \cdots (1-t_n)^{\gamma_n-\beta_n-1} dt_1 \cdots dt_n \\
&= \frac{1}{B(\beta_1, \gamma_1 - \beta_1) \cdots B(\beta_n, \gamma_n - \beta_n)} \int_0^1 \cdots \int_0^1 (1-t_1)^{\gamma_1-\beta_1-1} \cdots (1-t_n)^{\gamma_n-\beta_n-1} \\
&\qquad \cdot t_1^{\beta_1-1} \cdots t_n^{\beta_n-1} \sum_{m_1, \ldots, m_n=0}^{\infty} (\alpha)_{m_1+\cdots+m_n} \frac{(x_1 t_1)^{m_1}}{m_1!} \cdots \frac{(x_n t_n)^{m_n}}{m_n!} dt_1 \cdots dt_n \\
&= \frac{1}{B(\beta_1, \gamma_1 - \beta_1) \cdots B(\beta_n, \gamma_n - \beta_n)} \int_0^1 \cdots \int_0^1 t_1^{\beta_1-1} \cdots t_n^{\beta_n-1} \\
&\qquad \cdot (1-t_1)^{\gamma_1-\beta_1-1} \cdots (1-t_n)^{\gamma_n-\beta_n-1} \sum_{m=0}^{\infty} (\alpha)_m \frac{(x_1 t_1 + \cdots + x_n t_n)^m}{m!} dt_1 \cdots dt_n \\
&= \frac{\Gamma(\gamma_1) \cdots \Gamma(\gamma_n)}{\Gamma(\beta_1)\Gamma(\gamma_1 - \beta_1) \cdots \Gamma(\beta_n)\Gamma(\gamma_n - \beta_n)} \int_0^1 \cdots \int_0^1 t_1^{\beta_1-1} \cdots t_n^{\beta_n-1} \\
&\qquad \cdot (1-t_1)^{\gamma_1-\beta_1-1} \cdots (1-t_n)^{\gamma_n-\beta_n-1} (1 - x_1 t_1 - \cdots - x_n t_n)^{-\alpha} dt_1 \cdots dt_n
\end{aligned}
$$

şeklinde elde edilir.

Benzer şekilde (3.37) integral gösterimi, $F_D^{(n)}$ nin tanımında sırasıyla (3.38), (3.39) ve (3.40) özelliklerinin kullanılmasıyla

$$
\begin{aligned}
&F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}}{(\gamma)_{m_1+\cdots+m_n}} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \\
&= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 t^{\alpha+m_1+\cdots+m_n-1} (1-t)^{\gamma-\alpha-1} dt \\
&\qquad\qquad\qquad \cdot (\beta_1)_{m_1} \cdots (\beta_n)_{m_n} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} \sum_{m_1=0}^{\infty} (\beta_1)_{m_1} \frac{(x_1 t)^{m_1}}{m_1!} \cdots \sum_{m_n=0}^{\infty} (\beta_n)_{m_n} \frac{(x_n t)^{m_n}}{m_n!} dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1 - x_1 t)^{-\beta_1} \cdots (1 - x_n t)^{-\beta_n} dt
\end{aligned}
$$

şeklinde bulunur. ■

### 3.6. Lauricella Fonksiyonlarının Ekstra Parametre İçeren İntegralleri

**Teorem 3.5.** $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ Lauricella fonksiyonlarının ekstra $\lambda$ parametresi içeren integral gösterimlerinin 1.grubu sırasıyla,

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{B(\beta_1,\lambda-\beta_1)}$$

$$\cdot\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}F_A^{(n)}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt \tag{3.41}$$

$$(Re(\lambda)>Re(\beta_1)>0)$$

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{B(\beta_1,\lambda-\beta_1)}$$

$$\cdot\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt \tag{3.42}$$

$$(Re(\lambda)>Re(\beta_1)>0)$$

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{B(\lambda,\gamma_1-\lambda)}$$

$$\cdot\int_0^1 t^{\lambda-1}(1-t)^{\gamma_1-\lambda-1}F_C^{(n)}(\alpha,\beta;\lambda,\gamma_2,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt \tag{3.43}$$

$$(Re(\gamma_1)>Re(\lambda)>0)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{B(\beta_1,\lambda-\beta_1)}$$

$$\cdot\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}F_D^{(n)}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt \tag{3.44}$$

$$(Re(\lambda)>Re(\beta_1)>0)$$

şeklindedir.

**İspat**. Bu integral gösterimleri, $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ ve $F_D^{(n)}$ Lauricella fonksiyonlarının sırasıyla (3.1), (3.2), (3.3) ve (3.4) deki seri gösterimlerinin pay ve paydalarının $(\lambda)_{m_1}$ ile çarpılmasıyla ve (3.38) özelliğinin kullanılmasıyla aşağıdaki şekilde ispatlanır.

(3.41) integral gösteriminin ispatı

$$\begin{aligned}
&F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1}}{(\lambda)_{m_1}}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{B(\beta_1+m_1,\lambda-\beta_1)}{B(\beta_1,\lambda-\beta_1)}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\cdot\frac{1}{B(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\beta_1+m_1-1}(1-t)^{\lambda-\beta_1-1}dt\\
&=\frac{1}{B(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{(tx_1)^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots\frac{x_n^{m_n}}{m_n!}dt\\
&=\frac{1}{B(\beta_1,\lambda-\beta_1)}\\
&\qquad\cdot\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}F_A^{(n)}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,x_2\ldots,x_n)dt
\end{aligned}$$

şeklindedir.

(3.42) integral gösteriminin ispatı,

$$\begin{aligned}
&F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1}}{(\lambda)_{m_1}}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{B(\beta_1+m_1,\lambda-\beta_1)}{B(\beta_1,\lambda-\beta_1)}\\
&=\frac{1}{B(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{(tx_1)^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots\frac{x_n^{m_n}}{m_n!}dt\\
&=\frac{1}{B(\beta_1,\lambda-\beta_1)}\\
&\qquad\cdot\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt
\end{aligned}$$

şeklindedir.

(3.43) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1}}{(\lambda)_{m_1}}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\lambda)_{m_1}(\gamma_2)_{m_2}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{B(\lambda+m_1,\gamma_1-\lambda)}{B(\lambda,\gamma_1-\lambda)}\\
&=\frac{1}{B(\lambda,\gamma_1-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\gamma_1-\lambda-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\lambda)_{m_1}(\gamma_2)_{m_2}\cdots(\gamma_n)_{m_n}}\frac{(tx_1)^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots\frac{x_n^{m_n}}{m_n!}dt\\
&=\frac{1}{B(\lambda,\gamma_1-\lambda)}\\
&\qquad\cdot\int_0^1 t^{\lambda-1}(1-t)^{\gamma_1-\lambda-1}F_C^{(n)}(\alpha,\beta;\lambda,\gamma_2,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt
\end{aligned}
$$

şeklindedir.

(3.44) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1}}{(\lambda)_{m_1}}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{B(\beta_1+m_1,\lambda-\beta_1)}{B(\beta_1,\lambda-\beta_1)}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\cdot\frac{1}{B(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\beta_1+m_1-1}(1-t)^{\lambda-\beta_1-1}dt\\
&=\frac{1}{B(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}\\
&\qquad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\lambda)_{m_1}(\beta_2)_{m_2}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{(tx_1)^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\cdots\frac{x_n^{m_n}}{m_n!}dt\\
&=\frac{1}{B(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\beta_1-1}(1-t)^{\lambda-\beta_1-1}F_D^{(n)}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt
\end{aligned}
$$

şeklindedir. ■

**Teorem 3.6.** $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ Lauricella fonksiyonlarının ekstra $\lambda$ parametresi içeren integral gösterimlerinin 2.grubu sırasıyla,

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{B(\alpha,\lambda-\alpha)}$$
$$\cdot\int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1}F_A^{(n)}(\lambda,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt \quad (3.45)$$

$$(Re(\lambda)>Re(\alpha)>0)$$

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{B(\lambda,\gamma-\lambda)}$$
$$\cdot\int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt \quad (3.46)$$

$$(Re(\gamma)>Re(\lambda)>0)$$

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{B(\alpha,\lambda-\alpha)}$$
$$\cdot\int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1}F_C^{(n)}(\lambda,\beta;\gamma_1,...,\gamma_n;tx_1,...,tx_n)dt \quad (3.47)$$

$$(Re(\lambda)>Re(\alpha)>0)$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{B(\lambda,\gamma-\lambda)}$$
$$\cdot\int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt \quad (3.48)$$

$$(Re(\gamma)>Re(\lambda)>0)$$

şeklindedir [8].

**İspat**. Bu integral gösterimleri, $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ ve $F_D^{(n)}$ Lauricella fonksiyonlarının sırasıyla (3.1), (3.2), (3.3) ve (3.4) deki seri gösterimlerinin pay ve paydalarının $(\lambda)_{m_1+\cdots+m_n}$ ile çarpılmasıyla ve (3.38) özelliğinin kullanılmasıyla aşağıdaki şekilde ispatlanır.
(3.45) integral gösteriminin ispatı,

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1+\cdots+m_n}}{(\lambda)_{m_1+\cdots+m_n}}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$\cdot \frac{B(\alpha+m_1+\cdots+m_n,\lambda-\alpha)}{B(\alpha,\lambda-\alpha)}$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$\cdot \frac{1}{B(\alpha,\lambda-\alpha)} \int_0^1 t^{\alpha+m_1+\cdots+m_n-1}(1-t)^{\lambda-\alpha-1}dt$$
$$= \frac{1}{B(\alpha,\lambda-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1}$$
$$\cdot \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{(tx_1)^{m_1}}{m_1!}\cdots\frac{(tx_n)^{m_n}}{m_n!}dt$$
$$= \frac{1}{B(\alpha,\lambda-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1} F_A^{(n)}(\lambda,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt$$

şeklindedir.

(3.46) integral gösteriminin ispatı,

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1+\cdots+m_n}}{(\lambda)_{m_1+\cdots+m_n}}$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\lambda)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$\cdot \frac{B(\lambda+m_1+\cdots+m_n,\gamma-\lambda)}{B(\lambda,\gamma-\lambda)}$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\lambda)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$\cdot \frac{1}{B(\lambda,\gamma-\lambda)} \int_0^1 t^{\lambda+m_1+\cdots+m_n-1}(1-t)^{\gamma-\lambda-1}dt$$
$$= \frac{1}{B(\lambda,\gamma-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}$$
$$\cdot \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\lambda)_{m_1+\cdots+m_n}} \frac{(tx_1)^{m_1}}{m_1!}\cdots\frac{(tx_n)^{m_n}}{m_n!}dt$$
$$= \frac{1}{B(\lambda,\gamma-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1} F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt$$

şeklindedir.

(3.47) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1+\cdots+m_n}}{(\lambda)_{m_1+\cdots+m_n}}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\lambda)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\qquad\cdot\frac{B(\alpha+m_1+\cdots+m_n,\lambda-\alpha)}{B(\alpha,\lambda-\alpha)}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\lambda)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\qquad\cdot\frac{1}{B(\alpha,\lambda-\alpha)}\int_0^1 t^{\alpha+m_1+\cdots+m_n-1}(1-t)^{\lambda-\alpha-1}dt\\
&=\frac{1}{B(\alpha,\lambda-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1}\\
&\quad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\lambda)_{m_1+\cdots+m_n}(\beta)_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\frac{(tx_1)^{m_1}}{m_1!}\cdots\frac{(tx_n)^{m_n}}{m_n!}dt\\
&=\frac{1}{B(\alpha,\lambda-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1}F_C^{(n)}(\lambda,\beta;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt
\end{aligned}
$$

şeklindedir.

(3.48) integral gösteriminin ispatı,

$$
\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\frac{(\lambda)_{m_1+\cdots+m_n}}{(\lambda)_{m_1+\cdots+m_n}}\\
&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\lambda)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\qquad\cdot\frac{B(\lambda+m_1+\cdots+m_n,\gamma-\lambda)}{B(\lambda,\gamma-\lambda)}\\
&=\frac{1}{B(\lambda,\gamma-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}dt\\
&\quad\cdot\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\lambda)_{m_1+\cdots+m_n}}\frac{(tx_1)^{m_1}}{m_1!}\cdots\frac{(tx_n)^{m_n}}{m_n!}\\
&=\frac{1}{B(\lambda,\gamma-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt
\end{aligned}
$$

şeklindedir. ■

## 3.7. $F_A^{(n)}$ Fonksiyonu için Dönüşüm Formülleri

**Teorem 3.7.** $F_A^{(n)}$ için $\binom{n}{1} = n$ tane dönüşüm formülü aşağıdadır [8].

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = (1 - x_1)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha, \gamma_1 - \beta_1, \beta_2, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{x_1 - 1}, \frac{x_2}{1 - x_1}, \ldots, \frac{x_n}{1 - x_1})$$

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = (1 - x_2)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha, \beta_1, \gamma_2 - \beta_2, \beta_3, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{1 - x_2}, \frac{x_2}{x_2 - 1}, \frac{x_3}{1 - x_2}, \ldots, \frac{x_n}{1 - x_2})$$
$$\vdots$$

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = (1 - x_n)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_{n-1}, \gamma_n - \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{1 - x_n}, \ldots, \frac{x_{n-1}}{1 - x_n}, \frac{x_n}{x_n - 1})$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için sırasıyla (3.36) integralinde $\binom{n}{1}$ olası dönüşümün

$$t_1 = 1 - u_1, t_2 = u_2, \ldots, t_n = u_n$$
$$t_1 = u_1, t_2 = 1 - u_2, t_3 = u_3, \ldots, t_n = u_n$$
$$\vdots$$
$$t_1 = u_1, \ldots, t_{n-1} = u_{n-1}, t_n = 1 - u_n$$

şeklinde yapılması yeterlidir. Gerçekten de (3.36) integraline ilk dönüşüm uygulanırsa,

$$F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \frac{\Gamma(\gamma_1) \cdots \Gamma(\gamma_n)}{\Gamma(\beta_1) \cdots \Gamma(\beta_n) \Gamma(\gamma_1 - \beta_1) \cdots \Gamma(\gamma_n - \beta_n)}$$
$$\cdot \int_0^1 \cdots \int_0^1 t_1^{\beta_1 - 1} \cdots t_n^{\beta_n - 1} (1 - t_1)^{\gamma_1 - \beta_1 - 1} \cdots (1 - t_n)^{\gamma_n - \beta_n - 1}$$
$$\cdot (1 - x_1 t_1 - \cdots - x_n t_n)^{-\alpha} dt_1 \cdots dt_n$$
$$= \frac{\Gamma(\gamma_1) \cdots \Gamma(\gamma_n)}{\Gamma(\beta_1) \cdots \Gamma(\beta_n) \Gamma(\gamma_1 - \beta_1) \cdots \Gamma(\gamma_n - \beta_n)}$$
$$\cdot \int_0^1 \cdots \int_0^1 (1 - u_1)^{\beta_1 - 1} u_2^{\beta_2 - 1} \cdots u_n^{\beta_n - 1} u_1^{\gamma_1 - \beta_1 - 1} (1 - u_2)^{\gamma_2 - \beta_2 - 1} \cdots (1 - u_n)^{\gamma_n - \beta_n - 1}$$
$$\cdot (1 - x_1(1 - u_1) - x_2 u_2 - \cdots - x_n u_n)^{-\alpha} du_1 \cdots du_n$$

$$= (1-x_1)^{-\alpha}\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)}$$
$$\cdot\int_0^1\cdots\int_0^1 u_1^{\gamma_1-\beta_1-1}u_2^{\beta_2-1}\cdots u_n^{\beta_n-1}(1-u_1)^{\beta_1-1}(1-u_2)^{\gamma_2-\beta_2-1}\cdots(1-u_n)^{\gamma_n-\beta_n-1}$$
$$\cdot(1-\frac{x_1}{x_1-1}u_1-\frac{x_2}{1-x_1}u_2-\cdots-\frac{x_n}{1-x_1}u_n)^{-\alpha}du_1\cdots du_n$$
$$= (1-x_1)^{-\alpha}F_A^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1-1},\frac{x_2}{1-x_1},\ldots,\frac{x_n}{1-x_1})$$

ilk dönüşüm formülü elde edilir. Diğerleri de benzer şekilde yapılır. ■

**Teorem 3.8.** $F_A^{(n)}$ in bir diğer $\binom{n}{2}$ tane dönüşüm formülü,

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = (1-x_1-x_2)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\beta_3,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1+x_2-1},$$
$$\frac{x_2}{x_1+x_2-1},\frac{x_3}{1-x_1-x_2},\ldots,\frac{x_n}{1-x_1-x_2})$$

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = (1-x_1-x_3)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1+x_3-1},$$
$$\frac{x_2}{1-x_1-x_3},\frac{x_3}{x_1+x_3-1},\frac{x_4}{1-x_1-x_3},\ldots,\frac{x_n}{1-x_1-x_3})$$
$$\vdots$$
$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = (1-x_1-x_n)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\ldots,\beta_{n-1},\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1+x_n-1},$$
$$\frac{x_2}{1-x_1-x_n},\ldots,\frac{x_{n-1}}{1-x_1-x_n},\frac{x_n}{x_1+x_n-1})$$

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = (1-x_2-x_3)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-x_2-x_3},$$
$$\frac{x_2}{x_2+x_3-1},\frac{x_3}{x_2+x_3-1},\frac{x_4}{1-x_2-x_3},\ldots,\frac{x_n}{1-x_2-x_3})$$
$$\vdots$$

dır [8].

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için (3.36 ) integralinde $\binom{n}{2}$ olası dönüşümün

$$
\begin{aligned}
& t_1 = 1-u_1, t_2 = 1-u_2, t_3 = u_3, \dots, t_n = u_n \\
& t_1 = 1-u_1, t_2 = u_2, t_3 = 1-u_3, t_4 = u_4, \dots, t_n = u_n \\
& \vdots \\
& t_1 = 1-u_1, t_2 = u_2, \dots, t_{n-1} = u_{n-1}, t_n = 1-u_n \\
& t_1 = u_1, t_2 = 1-u_2, t_3 = 1-u_3, t_4 = u_4, \dots, t_n = u_n \\
& \vdots
\end{aligned}
$$

şeklinde uygulanması yeterlidir. Gerçekten de (3.36) integraline ilk dönüşüm uygulanırsa,

$$
\begin{aligned}
& F_A^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) = \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)} \\
& \cdot \int_0^1 \cdots \int_0^1 t_1^{\beta_1-1} t_2^{\beta_2-1} t_3^{\beta_3-1} \cdots t_n^{\beta_n-1} \\
& \cdot (1-t_1)^{\gamma_1-\beta_1-1}(1-t_2)^{\gamma_2-\beta_2-1}(1-t_3)^{\gamma_3-\beta_3-1}\cdots(1-t_n)^{\gamma_n-\beta_n-1} \\
& \cdot (1-x_1t_1-\cdots-x_nt_n)^{-\alpha} dt_1\cdots dt_n \\
& = \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)} \\
& \cdot \int_0^1 \cdots \int_0^1 (1-u_1)^{\beta_1-1}(1-u_2)^{\beta_2-1} u_3^{\beta_3-1}\cdots u_n^{\beta_n-1} \\
& \cdot u_1^{\gamma_1-\beta_1-1} u_2^{\gamma_2-\beta_2-1}(1-u_3)^{\gamma_3-\beta_3-1}\cdots(1-u_n)^{\gamma_n-\beta_n-1} \\
& \cdot (1-x_1(1-u_1)-x_2(1-u_2)-x_3u_3-\cdots-x_nu_n)^{-\alpha} du_1\cdots du_n \\
& = (1-x_1-x_2)^{-\alpha}\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)} \\
& \cdot \int_0^1 \cdots \int_0^1 u_1^{\gamma_1-\beta_1-1} u_2^{\gamma_2-\beta_2-1} u_3^{\beta_3-1}\cdots u_n^{\beta_n-1} \\
& \cdot (1-u_1)^{\beta_1-1}(1-u_2)^{\beta_2-1}(1-u_3)^{\gamma_3-\beta_3-1}\cdots(1-u_n)^{\gamma_n-\beta_n-1} \\
& \cdot (1-\frac{x_1}{x_1+x_2-1}u_1-\frac{x_2}{x_1+x_2-1}u_2 \\
& \qquad -\frac{x_3}{1-x_1-x_2}u_3-\cdots-\frac{x_n}{1-x_1-x_2}u_n)^{-\alpha} du_1\cdots du_n \\
& = (1-x_1-x_2)^{-\alpha} F_A^{(n)}(\alpha, \gamma_1-\beta_1, \gamma_2-\beta_2, \beta_3, \dots, \beta_n; \gamma_1, \dots, \gamma_n; \\
& \frac{x_1}{x_1+x_2-1}, \frac{x_2}{x_1+x_2-1}, \frac{x_3}{1-x_1-x_2}, \dots, \frac{x_n}{1-x_1-x_2})
\end{aligned}
$$

ilk dönüşüm formülü elde edilir. Diğerleri de benzer şekilde yapılır. ■

**Teorem 3.9.** $F_A^{(n)}$ in bir diğer $\binom{n}{3}$ tane dönüşüm formülü aşağıda listelenmiştir [8].

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_1-x_2-x_3)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;$$
$$\frac{x_1}{x_1+x_2+x_3-1},\frac{x_2}{x_1+x_2+x_3-1},\frac{x_3}{x_1+x_2+x_3-1},$$
$$\frac{x_4}{1-x_1-x_2-x_3},\ldots,\frac{x_n}{1-x_1-x_2-x_3})$$

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_1-x_2-x_4)^{-\alpha}$$
$$.F_A^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\beta_3,\gamma_4-\beta_4,\beta_5,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;$$
$$\frac{x_1}{x_1+x_2+x_4-1},\frac{x_2}{x_1+x_2+x_4-1},\frac{x_3}{1-x_1-x_2-x_4},\frac{x_4}{x_1+x_2+x_4-1},$$
$$\frac{x_5}{1-x_1-x_2-x_4},\ldots,\frac{x_n}{1-x_1-x_2-x_4})$$
$$\vdots$$
$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_1-x_2-x_n)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\beta_3,\ldots,\beta_{n-1},\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1+x_2+x_n-1},$$
$$\frac{x_2}{x_1+x_2+x_n-1},\frac{x_3}{1-x_1-x_2-x_n},\ldots,\frac{x_{n-1}}{1-x_1-x_2-x_n},\frac{x_n}{x_1+x_2+x_n-1})$$

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_2-x_3-x_4)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\gamma_4-\beta_4,\beta_5,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-x_2-x_3-x_4},$$
$$\frac{x_2}{x_2+x_3+x_4-1},\frac{x_3}{x_2+x_3+x_4-1},\frac{x_4}{x_2+x_3+x_4-1},\frac{x_5}{1-x_2-x_3-x_4},\ldots,$$
$$\frac{x_n}{1-x_2-x_3-x_4})$$
$$\vdots$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için (3.36) integralinde $\binom{n}{3}$ olası dönüşümün

$$t_1=1-u_1,t_2=1-u_2,t_3=1-u_3,t_4=u_4,\ldots,t_n=u_n$$
$$t_1=1-u_1,t_2=1-u_2,t_3=u_3,t_4=1-u_4,t_5=u_5,\ldots,t_n=u_n$$
$$\vdots$$
$$t_1=1-u_1,t_2=1-u_2,t_3=u_3,\ldots,t_{n-1}=u_{n-1},t_n=1-u_n$$
$$t_1=u_1,t_2=1-u_2,t_3=1-u_3,t_4=1-u_4,t_5=u_5,\ldots,t_n=u_n$$
$$\vdots$$

şeklinde yapılması yeterlidir. Gerçekten de (3.36) integraline ilk dönüşüm uygulanırsa,

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)}$$
$$\cdot\int_0^1\cdots\int_0^1(1-u_1)^{\beta_1-1}(1-u_2)^{\beta_2-1}(1-u_3)^{\beta_3-1}u_4^{\beta_4-1}\cdots u_n^{\beta_n-1}$$
$$\cdot u_1^{\gamma_1-\beta_1-1}u_2^{\gamma_2-\beta_2-1}u_3^{\gamma_3-\beta_3-1}(1-u_4)^{\gamma_4-\beta_4-1}\cdots(1-u_n)^{\gamma_n-\beta_n-1}$$
$$\cdot(1-x_1(1-u_1)-x_2(1-u_2)-x_3(1-u_3)-x_4u_4-\cdots-x_nu_n)^{-\alpha}du_1\cdots du_n$$
$$=(1-x_1-x_2-x_3)^{-\alpha}\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)}$$
$$\cdot\int_0^1\cdots\int_0^1 u_1^{\gamma_1-\beta_1-1}u_2^{\gamma_2-\beta_2-1}u_3^{\gamma_3-\beta_3-1}u_4^{\beta_4-1}\cdots u_n^{\beta_n-1}$$
$$\cdot(1-u_1)^{\beta_1-1}(1-u_2)^{\beta_2-1}(1-u_3)^{\beta_3-1}(1-u_4)^{\gamma_4-\beta_4-1}\cdots(1-u_n)^{\gamma_n-\beta_n-1}$$
$$\cdot(1-\frac{x_1}{x_1+x_2+x_3-1}u_1-\frac{x_2}{x_1+x_2+x_3-1}u_2-\frac{x_3}{x_1+x_2+x_3-1}u_3$$
$$-\frac{x_4}{1-x_1-x_2-x_3}u_4-\cdots-\frac{x_n}{1-x_1-x_2-x_3}u_n)^{-\alpha}du_1\cdots du_n$$
$$=(1-x_1-x_2-x_3)^{-\alpha}F_A^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;$$
$$\frac{x_1}{x_1+x_2+x_3-1},\frac{x_2}{x_1+x_2+x_3-1},\frac{x_3}{x_1+x_2+x_3-1},$$
$$\frac{x_4}{1-x_1-x_2-x_3},\ldots,\frac{x_n}{1-x_1-x_2-x_3})$$

ilk dönüşüm formülü elde edilir. Diğerleri de benzer şekilde yapılır. ■

**Teorem 3.10.** $F_A^{(n)}$ in bir diğer $\binom{n}{n-1}=n$ tane dönüşüm formülleri aşağıdadır [8].

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_2-\cdots-x_n)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\ldots,\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-x_2-\cdots-x_n},$$
$$\frac{x_2}{x_2+\cdots+x_n-1},\ldots,\frac{x_n}{x_2+\cdots+x_n-1})$$
$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_1-x_3-\cdots-x_n)^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\gamma_3-\beta_3,\ldots,\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1+x_3+\cdots+x_n-1},$$
$$\frac{x_2}{1-x_1-x_3-\cdots-x_n},\frac{x_3}{x_1+x_3+\cdots+x_n-1},\ldots,\frac{x_n}{x_1+x_3+\cdots+x_n-1})$$
$$\vdots$$
$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-x_1-\cdots-x_{n-1})^{-\alpha}$$
$$\cdot F_A^{(n)}(\alpha,\gamma_1-\beta_1,\ldots,\gamma_{n-1}-\beta_{n-1},\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{x_1+\cdots+x_{n-1}-1},$$
$$\frac{x_{n-1}}{x_1+\cdots+x_{n-1}-1},\frac{x_n}{1-x_1-\cdots-x_{n-1}})$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için (3.36) integralinde $\binom{n}{n-1}$ olası dönüşüm

$$t_1 = u_1, t_2 = 1 - u_2, t_3 = 1 - u_3, \ldots, t_n = 1 - u_n$$
$$t_1 = 1 - u_1, t_2 = u_2, t_3 = 1 - u_3, \ldots, t_n = 1 - u_n$$
$$\vdots$$
$$t_1 = 1 - u_1, t_2 = 1 - u_2, \ldots, t_{n-1} = 1 - u_{n-1}, t_n = u_n$$

dır. Gerçekten de dönüşümlerden ilki uygulanırsa,

$$\begin{aligned}
F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) &= \frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)} \\
&\cdot \int_0^1 \cdots \int_0^1 u_1^{\beta_1-1}(1-u_2)^{\beta_2-1}\cdots(1-u_n)^{\beta_n-1} \\
&\cdot (1-u_1)^{\gamma_1-\beta_1-1}u_2^{\gamma_2-\beta_2-1}\cdots u_n^{\gamma_n-\beta_n-1} \\
&\cdot (1 - x_1u_1 - x_2(1-u_2) - \cdots - x_n(1-u_n))^{-\alpha}du_1\cdots du_n \\
&= (1 - x_2 - \cdots - x_n)^{-\alpha}\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)} \\
&\cdot \int_0^1 \cdots \int_0^1 u_1^{\beta_1-1}u_2^{\gamma_2-\beta_2-1}\cdots u_n^{\gamma_n-\beta_n-1} \\
&\cdot (1-u_1)^{\gamma_1-\beta_1-1}(1-u_2)^{\beta_2-1}\cdots(1-u_n)^{\beta_n-1}(1 - \frac{x_1}{1-x_2-\cdots-x_n}u_1 \\
&- \frac{x_2}{x_2+\cdots+x_n-1}u_2 - \cdots - \frac{x_n}{x_2+\cdots+x_n-1}u_n)^{-\alpha}du_1\cdots du_n \\
&= (1 - x_2 - \cdots - x_n)^{-\alpha}F_A^{(n)}(\alpha, \beta_1, \gamma_2-\beta_2, \ldots, \gamma_n-\beta_n; \gamma_1, \ldots, \gamma_n; \\
&\frac{x_1}{1-x_2-\cdots-x_n}, \frac{x_2}{x_2+\cdots+x_n-1}, \ldots, \frac{x_n}{x_2+\cdots+x_n-1})
\end{aligned}$$

ilk dönüşüm formülü elde edilir. Diğerleri de benzer şekilde yapılır. ■

**Teorem 3.11.** $F_A^{(n)}$ in diğer $\binom{n}{n} = 1$ tane dönüşüm formülü,

$$\begin{aligned}
F_A^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) &= (1 - x_1 - \cdots - x_n)^{-\alpha} \\
&\cdot F_A^{(n)}(\alpha, \gamma_1-\beta_1, \ldots, \gamma_n-\beta_n; \gamma_1, \ldots, \gamma_n; \\
&\frac{x_1}{x_1+\cdots+x_n-1}, \ldots, \frac{x_n}{x_1+\cdots+x_n-1})
\end{aligned}$$

dır [8].

**İspat**. Yukarıdaki dönüşüm formülünün ispatı için $\binom{n}{n}$ olası dönüşüm

$$t_1 = 1 - u_1, \ldots, t_n = 1 - u_n$$

olup, (3.36) integraline bu dönüşüm uygulanırsa

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)}$$
$$\cdot\int_0^1\cdots\int_0^1(1-u_1)^{\beta_1-1}\cdots(1-u_n)^{\beta_n-1}u_1^{\gamma_1-\beta_1-1}\cdots u_n^{\gamma_n-\beta_n-1}$$
$$\cdot(1-x_1(1-u_1)-\cdots-x_n(1-u_n))^{-\alpha}du_1\cdots du_n$$
$$=(1-x_1-\cdots-x_n)^{-\alpha}\frac{\Gamma(\gamma_1)\cdots\Gamma(\gamma_n)}{\Gamma(\beta_1)\cdots\Gamma(\beta_n)\Gamma(\gamma_1-\beta_1)\cdots\Gamma(\gamma_n-\beta_n)}$$
$$\cdot\int_0^1\cdots\int_0^1 u_1^{\gamma_1-\beta_1-1}\cdots u_n^{\gamma_n-\beta_n-1}(1-u_1)^{\beta_1-1}\cdots(1-u_n)^{\beta_n-1}$$
$$\cdot(1-\frac{x_1}{x_1+\cdots+x_n-1}u_1-\cdots-\frac{x_n}{x_1+\cdots+x_n-1}u_n)^{-\alpha}du_1\cdots du_n$$
$$=(1-x_1-\cdots-x_n)^{-\alpha}F_A^{(n)}(\alpha,\gamma_1-\beta_1,\ldots,\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;$$
$$\frac{x_1}{x_1+\cdots+x_n-1},\ldots,\frac{x_n}{x_1+\cdots+x_n-1})$$

istenilen dönüşüm formülü elde edilir. ■

**Sonuç 3.12.** Bu kısımda $F_A^{(n)}$ için verilen toplam dönüşüm sayısı

$$\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n-1$$

dir [8].

## 3.8. $F_D^{(n)}$ Fonksiyonu için Dönüşüm Formülleri

**Teorem 3.13.** $F_D^{(n)}$ için bir dönüşüm formülü

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=(1-x_1)^{-\beta_1}\cdots(1-x_n)^{-\beta_n}$$
$$\cdot F_D^{(n)}(\gamma-\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1})$$

şeklindedir [8].

**İspat**. Yukarıdaki dönüşüm formülünün ispatı için uygun dönüşüm

$$t=1-u\Longrightarrow dt=-du$$

dir. (3.37) integralinde bu dönüşüm uygulanır ve gerekli düzenlemeler yapılırsa,

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-x_1t)^{-\beta_1}\cdots(1-x_nt)^{-\beta_n}dt$$

$$
\begin{aligned}
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{\gamma-\alpha-1} (1-x_1+x_1u)^{-\beta_1} \cdots (1-x_n+x_nu)^{-\beta_n} du \\
&= (1-x_1)^{-\beta_1} \cdots (1-x_n)^{-\beta_n} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\gamma-\alpha-1} (1-u)^{\alpha-1} \\
&\qquad \cdot (1-\frac{x_1}{x_1-1}u)^{-\beta_1} \cdots (1-\frac{x_n}{x_n-1}u)^{-\beta_n} du \\
&= (1-x_1)^{-\beta_1} \cdots (1-x_n)^{-\beta_n} F_D^{(n)}(\gamma-\alpha, \beta_1, \ldots, \beta_n; \gamma; \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1})
\end{aligned}
$$

istenilen dönüşüm formülü elde edilir. ■

**Teorem 3.14.** $F_D^{(n)}$ için $n$ tane dönüşüm formülü aşağıdadır [8].

$$
\begin{aligned}
&F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = (1-x_1)^{-\alpha} \\
&\cdot F_D^{(n)}(\alpha, \gamma-\beta_1-\cdots-\beta_n, \beta_2, \ldots, \beta_n; \gamma; \frac{x_1}{x_1-1}, \frac{x_1-x_2}{x_1-1}, \ldots, \frac{x_1-x_n}{x_1-1}) \\
&F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = (1-x_2)^{-\alpha} \\
&\cdot F_D^{(n)}(\alpha, \beta_1, \gamma-\beta_1-\cdots-\beta_n, \beta_3, \ldots, \beta_n; \gamma; \frac{x_2-x_1}{x_2-1}, \frac{x_2}{x_2-1}, \frac{x_2-x_3}{x_2-1}, \ldots, \frac{x_2-x_n}{x_2-1}) \\
&\vdots \\
&F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = (1-x_n)^{-\alpha} \\
&\cdot F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_{n-1}, \gamma-\beta_1-\cdots-\beta_n; \gamma; \frac{x_n-x_1}{x_n-1}, \ldots, \frac{x_n-x_{n-1}}{x_n-1}, \frac{x_n}{x_n-1})
\end{aligned}
$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için (3.37) integralinde sırasıyla

$$
t = \frac{u}{1-x_1+x_1u}, \ldots, t = \frac{u}{1-x_n+x_nu}
$$

dönüşümleri yapılmalıdır. Gerçekten de dönüşümlerden ilki olan

$$
t = \frac{u}{1-x_1+x_1u} \Longrightarrow dt = \frac{1-x_1}{(1-x_1+x_1u)^2} du
$$

dönüşümü (3.37) integraline uygulanır ve

$$
\begin{aligned}
1-t &= 1-\frac{u}{1-x_1+x_1u} = \frac{(1-u)(1-x_1)}{1-x_1+x_1u} \\
1-x_1t &= 1-\frac{x_1u}{1-x_1+x_1u} = \frac{1-x_1}{1-x_1+x_1u} \\
1-x_2t &= 1-\frac{x_2u}{1-x_1+x_1u} = \frac{1-x_1+x_1u-x_2u}{1-x_1+x_1u} = \frac{1-x_1}{1-x_1+x_1u}\Big(1-\frac{x_1-x_2}{x_1-1}u\Big) \\
&\vdots \\
1-x_nt &= 1-\frac{x_nu}{1-x_1+x_1u} = \frac{1-x_1+x_1u-x_nu}{1-x_1+x_1u} = \frac{1-x_1}{1-x_1+x_1u}\Big(1-\frac{x_1-x_n}{x_1-1}u\Big)
\end{aligned}
$$

oldukları da dikkate alınırsa,

$$
\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-x_1t)^{-\beta_1}(1-x_2t)^{-\beta_2}\cdots(1-x_nt)^{-\beta_n}dt\\
&=(1-x_1)^{-\alpha}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}\\
&\quad\cdot(1-\frac{x_1}{x_1-1}u)^{-\gamma+\beta_1+\cdots+\beta_n}(1-\frac{x_1-x_2}{x_1-1}u)^{-\beta_2}\cdots(1-\frac{x_1-x_n}{x_1-1}u)^{-\beta_n}du\\
&=(1-x_1)^{-\alpha}F_D^{(n)}(\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\ldots,\beta_n;\gamma;\frac{x_1}{x_1-1},\frac{x_1-x_2}{x_1-1},\ldots,\frac{x_1-x_n}{x_1-1})
\end{aligned}
$$

elde edilir ki bu da ilk dönüşüm formülünün ispatını tamamlar. Diğer dönüşüm formülleri de benzer şekilde yapılır. ■

**Teorem 3.15.** $F_D^{(n)}$ için $n$ tane dönüşüm formülü aşağıdadır [8].

$$
\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=(1-x_1)^{\gamma-\alpha-\beta_1}(1-x_2)^{-\beta_2}\cdots(1-x_n)^{-\beta_n}\\
&\quad\cdot F_D^{(n)}(\gamma-\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\ldots,\beta_n;\gamma;x_1,\frac{x_2-x_1}{x_2-1},\ldots,\frac{x_n-x_1}{x_n-1})\\
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=(1-x_1)^{-\beta_1}(1-x_2)^{\gamma-\alpha-\beta_2}(1-x_3)^{-\beta_3}\cdots(1-x_n)^{-\beta_n}\\
&\quad\cdot F_D^{(n)}(\gamma-\alpha,\beta_1,\gamma-\beta_1-\cdots-\beta_n,\beta_3,\ldots,\beta_n;\gamma;\frac{x_1-x_2}{x_1-1},x_2,\frac{x_3-x_2}{x_3-1},\ldots,\frac{x_n-x_2}{x_n-1})\\
&\vdots\\
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=(1-x_1)^{-\beta_1}\cdots(1-x_{n-1})^{-\beta_{n-1}}(1-x_n)^{\gamma-\alpha-\beta_n}\\
&\quad\cdot F_D^{(n)}(\gamma-\alpha,\beta_1,\ldots,\beta_{n-1},\gamma-\beta_1-\cdots-\beta_n;\gamma;\frac{x_1-x_n}{x_1-1},\ldots,\frac{x_{n-1}-x_n}{x_{n-1}-1},x_n)
\end{aligned}
$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için (3.37) integralinde sırasıyla

$$t=\frac{1-u}{1-x_1u},\ldots,t=\frac{1-u}{1-x_nu}$$

dönüşümleri yapılmalıdır. Gerçekten de dönüşümlerden ilki olan

$$t=\frac{1-u}{1-x_1u}\Longrightarrow dt=\frac{x_1-1}{(1-x_1u)^2}du$$

dönüşümü uygulandığında,

$$
\begin{aligned}
1-t&=1-\frac{1-u}{1-x_1u}=\frac{u(1-x_1)}{1-x_1u}\\
1-x_1t&=1-x_1\Big(\frac{1-u}{1-x_1u}\Big)=\frac{1-x_1}{1-x_1u}
\end{aligned}
$$

$$1-x_2t=1-x_2\Big(\frac{1-u}{1-x_1u}\Big)=\frac{1-x_2}{1-x_1u}\Big(1-\frac{x_2-x_1}{x_2-1}u\Big)$$

$$\vdots$$

$$1-x_nt=1-x_n\Big(\frac{1-u}{1-x_1u}\Big)=\frac{1-x_n}{1-x_1u}\Big(1-\frac{x_n-x_1}{x_n-1}u\Big)$$

oldukları (3.37) integralinde dikkate alınır ve gerekli düzenlemeler yapılırsa

$$\begin{aligned}
&F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-x_1t)^{-\beta_1}(1-x_2t)^{-\beta_2}\cdots(1-x_nt)^{-\beta_n}dt\\
&=(1-x_1)^{\gamma-\alpha-\beta_1}(1-x_2)^{-\beta_2}\cdots(1-x_n)^{-\beta_n}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\gamma-\alpha-1}(1-u)^{\alpha-1}\\
&\quad\cdot(1-x_1u)^{\beta_1+\cdots+\beta_n-\gamma}\Big(1-\frac{x_2-x_1}{x_2-1}u\Big)^{-\beta_2}\cdots\Big(1-\frac{x_n-x_1}{x_n-1}u\Big)^{-\beta_n}du\\
&=(1-x_1)^{\gamma-\alpha-\beta_1}(1-x_2)^{-\beta_2}\cdots(1-x_n)^{-\beta_n}\\
&\quad\cdot F_D^{(n)}(\gamma-\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\ldots,\beta_n;\gamma;x_1,\frac{x_2-x_1}{x_2-1},\ldots,\frac{x_n-x_1}{x_n-1})
\end{aligned}$$

bulunur, bu da ilk dönüşüm formülünün kendisidir. Diğerleri de benzer şekilde ispatlanır. ■

**Sonuç 3.16.** Bu kısımda $F_D^{(n)}$ için verilen toplam dönüşüm sayısı $2n+1$ dir [8].

# 4. BULGULAR VE TARTIŞMA

Bu bölümde, Pochhammer $k$-sembolü kullanılarak bir önceki bölümde bahsi geçen fonksiyonların ve özelliklerinin $k$-analogları verilmiştir. Bu bölüm boyunca $k \in \mathbb{R}^+$ dır.

## 4.1. $k$-Lauricella Fonksiyonları

"$k$-Lauricella hipergeometrik fonksiyonları" ya da kısaca "$k$-Lauricella fonksiyonları" olarak adlandırılan $F_{A,k}^{(n)}$, $F_{B,k}^{(n)}$, $F_{C,k}^{(n)}$ ve $F_{D,k}^{(n)}$ fonksiyonları sırasıyla,

$$\begin{aligned} &F_{A,k}^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) \\ &= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma_1)_{m_1,k}\cdots(\gamma_n)_{m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \end{aligned} \tag{4.1}$$

$$\left(|x_1| + \dots + |x_n| < \frac{1}{k}\right),$$

$$\begin{aligned} &F_{B,k}^{(n)}(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n) \\ &= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1,k}\cdots(\alpha_n)_{m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\dots+m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \end{aligned} \tag{4.2}$$

$$\left(\max\{|x_1|, \dots, |x_n|\} < \frac{1}{k}\right),$$

$$\begin{aligned} &F_{C,k}^{(n)}(\alpha, \beta; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) \\ &= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n,k}(\beta)_{m_1+\dots+m_n,k}}{(\gamma_1)_{m_1,k}\cdots(\gamma_n)_{m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \end{aligned} \tag{4.3}$$

$$\left(\sqrt{|x_1|} + \dots + \sqrt{|x_n|} < \frac{1}{\sqrt{k}}\right),$$

$$\begin{aligned} &F_{D,k}^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n) \\ &= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\dots+m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \end{aligned} \tag{4.4}$$

$$\left(\max\{|x_1|, \dots, |x_n|\} < \frac{1}{k}\right).$$

şeklinde tanımlanır. Burada $i = 1, \dots, n$ için $\gamma, \gamma_i \in \mathbb{C} \setminus \{k\mathbb{Z}_0^-\}$ dır. Belirtelim ki $n = 2$ durumunda bu fonksiyonlar, $k$-Appell hipergeometrik fonksiyonlarına indirgenir.

**Teorem 4.1.** $F^{(n)}_{A,k}$, $F^{(n)}_{B,k}$, $F^{(n)}_{C,k}$, $F^{(n)}_{D,k}$ $k$-Lauricella fonksiyonları ile $F^{(n)}_{A}$, $F^{(n)}_{B}$, $F^{(n)}_{C}$, $F^{(n)}_{D}$ klasik Lauricella fonksiyonu arasındaki ilişkiler sırasıyla,

$$F^{(n)}_{A,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = F^{(n)}_{A}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n) \tag{4.5}$$

$$F^{(n)}_{B,k}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = F^{(n)}_{B}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n) \tag{4.6}$$

$$F^{(n)}_{C,k}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = F^{(n)}_{C}(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n) \tag{4.7}$$

$$F^{(n)}_{D,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = F^{(n)}_{D}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n) \tag{4.8}$$

şeklindedir.

**İspat**. Bu eşitliklerin ispatı için (2.12) de verilen $k$-Pochhammer sembolü ile klasik Pochhammer sembolü arasındaki

$$(\alpha)_{n,k} = k^n\left(\frac{\alpha}{k}\right)_n$$

ilişkinin dikkate alınması yeterlidir. Gerçekten de söz konusu ilişkinin kullanılmasıyla

$$\begin{aligned}
&F^{(n)}_{A,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma_1)_{m_1,k}\cdots(\gamma_n)_{m_n,k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{k^{m_1+\cdots+m_n}(\frac{\alpha}{k})_{m_1+\cdots+m_n}k^{m_1}(\frac{\beta_1}{k})_{m_1}\cdots k^{m_n}(\frac{\beta_n}{k})_{m_n}}{k^{m_1}(\frac{\gamma_1}{k})_{m_1}\cdots k^{m_n}(\frac{\gamma_n}{k})_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}
\end{aligned}$$

olup gerekli düzenlemelerden sonra

$$\begin{aligned}
&F^{(n)}_{A,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\frac{\alpha}{k})_{m_1+\cdots+m_n}(\frac{\beta_1}{k})_{m_1}\cdots(\frac{\beta_n}{k})_{m_n}}{(\frac{\gamma_1}{k})_{m_1}\cdots(\frac{\gamma_n}{k})_{m_n}}\frac{(kx_1)^{m_1}}{m_1!}\cdots\frac{(kx_n)^{m_n}}{m_n!}\\
&= F^{(n)}_{A}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)
\end{aligned}$$

elde edilir ki, bu da (4.5) eşitliğinin ispatını tamamlar. Benzer şekilde (4.6), (4.7) ve (4.8) eşitlikleri de ispatlanır. ■

Teorem 4.1. de verilen eşitliklerin her iki yanında parametrelerin $k$ katı, değişkenlerin $\frac{1}{k}$ katı alınarak aşağıdaki sonuca ulaşılır.

**Sonuç 4.2.** Aşağıdaki ilişkiler geçerlidir.

$$F_A^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$= F_{A,k}^{(n)}(k\alpha,k\beta_1,\ldots,k\beta_n;k\gamma_1,\ldots,k\gamma_n;\frac{x_1}{k},\ldots,\frac{x_n}{k}) \tag{4.9}$$

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$= F_{B,k}^{(n)}(k\alpha_1,\ldots,k\alpha_n,k\beta_1,\ldots,k\beta_n;k\gamma;\frac{x_1}{k},\ldots,\frac{x_n}{k}) \tag{4.10}$$

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$= F_{C,k}^{(n)}(k\alpha,k\beta;k\gamma_1,\ldots,k\gamma_n;\frac{x_1}{k},\ldots,\frac{x_n}{k}) \tag{4.11}$$

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$= F_{D,k}^{(n)}(k\alpha,k\beta_1,\ldots,k\beta_n;k\gamma;\frac{x_1}{k},\ldots,\frac{x_n}{k}) \tag{4.12}$$

## 4.2. $k$-Lauricella Fonksiyonlarının Konfluent Formları

$k$-Lauricella fonksiyonlarının konfluent formları,

$$\Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\cdots+m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.13}$$

$$\Psi_{2,k}^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n,k}}{(\gamma_1)_{m_1,k}\cdots(\gamma_n)_{m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.14}$$

$$\Phi_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_{n-1})_{m_{n-1},k}}{(\gamma)_{m_1+\cdots+m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.15}$$

$$\Xi_{1,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1,k}\cdots(\alpha_n)_{m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_{n-1})_{m_{n-1},k}}{(\gamma)_{m_1+\cdots+m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.16}$$

$$\Phi_{3,k}^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1,k}\cdots(\beta_{n-1})_{m_{n-1},k}}{(\gamma)_{m_1+\cdots+m_n,k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.17}$$

$$\Psi_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_{n-1})_{m_{n-1},k}}{(\gamma_1)_{m_1,k}\cdots(\gamma_n)_{m_n,k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.18}$$

şeklinde tanımlanır.

**Teorem 4.3.** $k$-Lauricella ile klasik Lauricella fonksiyonlarının konfluent formları arasındaki ilişkiler aşağıda listelenmiştir.

$$\Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$
$$=\Phi_2^{(n)}(\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};x_1,\ldots,x_n) \tag{4.19}$$

$$\Psi_{2,k}^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\Psi_2^{(n)}(\frac{\alpha}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};x_1,\ldots,x_n) \tag{4.20}$$

$$\Phi_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$=\Phi_D^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma}{k};kx_1,\ldots,kx_{n-1},x_n) \tag{4.21}$$

$$\Xi_{1,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$=\Xi_1^{(n)}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma}{k};kx_1,\ldots,kx_{n-1},x_n) \tag{4.22}$$

$$\Phi_{3,k}^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$=\Phi_3^{(n)}(\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma}{k};x_1,\ldots,x_{n-1},\frac{x_n}{k}) \tag{4.23}$$

$$\Psi_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$=\Psi_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_{n-1},x_n). \tag{4.24}$$

**İspat**. (2.12) de verilen Pochhammer $k$-sembolü ile klasik Pochhammer sembolü arasındaki ilişkinin kullanılmasıyla

$$\begin{aligned}
&\Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&\quad=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\cdots+m_n,k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\quad=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{k^{m_1}(\frac{\beta_1}{k})_{m_1}\cdots k^{m_n}(\frac{\beta_n}{k})_{m_n}}{k^{m_1+\cdots+m_n}(\frac{\gamma}{k})_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\quad=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\frac{\beta_1}{k})_{m_1}\cdots(\frac{\beta_n}{k})_{m_n}}{(\frac{\gamma}{k})_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\quad=\Phi_2^{(n)}(\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};x_1,\ldots,x_n)
\end{aligned}$$

bulunur ki bu da (4.19) un ispatını tamamlar. (4.20)-(4.24) eşitlikleri de benzer şekilde ispatlanabilir. ■

**Teorem 4.4.** Aşağıdaki eşitlikler sağlanır.

$$\Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\lim_{|\alpha|\to\infty}F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha},\ldots,\frac{x_n}{\alpha}) \tag{4.25}$$

$$\Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\lim_{min\{|\alpha_1|,\ldots,|\alpha_n|\}\to\infty}F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha_1},\ldots,\frac{x_n}{\alpha_n}) \tag{4.26}$$

$$\Psi_{2,k}^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\lim_{|\beta|\to\infty}F_{C,k}^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;\frac{x_1}{\beta},\ldots,\frac{x_n}{\beta}) \tag{4.27}$$

$$\Psi_{2,k}^{(n)}(\alpha;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\lim_{min\{|\beta_1|,\ldots,|\beta_n|\}\to\infty}F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{\beta_1},\ldots,\frac{x_n}{\beta_n}) \tag{4.28}$$

$$\Phi_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)=\lim_{|\beta_n|\to\infty}F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{4.29}$$

$$\Xi_{1,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$= \lim_{|\beta_n|\to\infty} F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{4.30}$$

$$\Phi_{3,k}^{(n)}(\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_n)$$
$$= \lim_{|\beta_n|\to\infty} \Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n})$$
$$= \lim_{min\{|\alpha|,|\beta_n|\}\to\infty} F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha},\ldots,\frac{x_{n-1}}{\alpha},\frac{x_n}{\alpha\beta_n})$$
$$= \lim_{min\{|\alpha_1|,\ldots,|\alpha_n|,|\beta_n|\}\to\infty} F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha_1},\ldots,\frac{x_{n-1}}{\alpha_{n-1}},\frac{x_n}{\alpha_n\beta_n}) \tag{4.31}$$

$$\Psi_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)$$
$$= \lim_{|\beta_n|\to\infty} F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_{n-1},\frac{x_n}{\beta_n}) \tag{4.32}$$

**İspat**. Teorem 3.1. in ispatına benzer şekilde ispat yapılır. Bunun için de

$$\lim_{|\alpha|\to\infty} \frac{(\alpha)_{n,k}}{\alpha^n} = 1 \tag{4.33}$$

özelliğinin kullanılması yeterlidir. Gerçekten de sırasıyla (4.4), (4.33) ve (4.13) kullanılmasıyla

$$\lim_{|\alpha|\to\infty} F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{x_1}{\alpha},\ldots,\frac{x_n}{\alpha})$$
$$= \lim_{|\alpha|\to\infty} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n,k}(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\cdots+m_n,k}} \frac{(\frac{x_1}{\alpha})^{m_1}}{m_1!}\cdots\frac{(\frac{x_n}{\alpha})^{m_n}}{m_n!}$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\cdots+m_n,k}} \left(\lim_{|\alpha|\to\infty} \frac{(\alpha)_{m_1+\cdots+m_n,k}}{\alpha^{m_1+\cdots+m_n}}\right) \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\beta_1)_{m_1,k}\cdots(\beta_n)_{m_n,k}}{(\gamma)_{m_1+\cdots+m_n,k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$
$$= \Phi_{2,k}^{(n)}(\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$

elde edilir ki bu da (4.25) eşitliğinin ispatını tamamlar. Diğer (4.26)-(4.32) eşitliklerinin ispatlarıda benzer şekilde elde edilir. ■

## 4.3. $k$-Lauricella Fonksiyonlarının Tek Katlı Laplace Tipi İntegralleri

**Teorem 4.5.** $k$-Lauricella fonksiyonlarının tek katlı Laplace tipi integralleri aşağıda listelenmiştir.

$$F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = \frac{k^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$$
$$\int_0^\infty e^{-t}t^{\frac{\alpha}{k}-1}{}_1F_{1,k}\left(\beta_1;\gamma_1;ktx_1\right)\cdots{}_1F_{1,k}\left(\beta_n;\gamma_n;ktx_n\right)dt \tag{4.34}$$

$$(Re(\alpha) > 0)$$

$$F_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \frac{k^{\frac{\beta_n}{k}}}{k\Gamma_k(\beta_n)}$$
$$\cdot \int_0^\infty e^{-t} t^{\frac{\beta_n}{k}-1} \Psi_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_{n-1}; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_{n-1}, ktx_n)dt \tag{4.35}$$

$$(Re(\beta_n) > 0)$$

$$F_{B,k}^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \frac{k^{\frac{\beta_n}{k}}}{k\Gamma_k(\beta_n)}$$
$$\cdot \int_0^\infty e^{-t} t^{\frac{\beta_n}{k}-1} \Xi_{1,k}^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}; \gamma; x_1, \ldots, x_{n-1}, ktx_n)dt \tag{4.36}$$

$$(Re(\beta_n) > 0)$$

$$F_{C,k}^{(n)}(\alpha, \beta; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n)$$
$$= \frac{k^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^\infty e^{-t} t^{\frac{\alpha}{k}-1} \Psi_{2,k}^{(n)}(\beta; \gamma_1, \ldots, \gamma_n; ktx_1, \ldots, ktx_n)dt \tag{4.37}$$

$$(Re(\alpha) > 0)$$

$$F_{D,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$$
$$= \frac{k^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^\infty e^{-t} t^{\frac{\alpha}{k}-1} \Phi_{2,k}^{(n)}(\beta_1, \ldots, \beta_n; \gamma; ktx_1, \ldots, ktx_n)dt \tag{4.38}$$

$$(Re(\alpha) > 0)$$

$$F_{D,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$$
$$= \frac{k^{\frac{\beta_n}{k}}}{k\Gamma_k(\beta_n)} \int_0^\infty e^{-t} t^{\frac{\beta_n}{k}-1} \Phi_{D,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_{n-1}; \gamma; x_1, \ldots, x_{n-1}, ktx_n)dt \tag{4.39}$$

$$(Re(\beta_n) > 0)$$

**İspat**. Pochhammer $k$-sembolünün integral gösterimi (2.1) ve (2.6) dan

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha + nk)}{\Gamma_k(\alpha)} = \frac{1}{\Gamma_k(\alpha)} \int_0^\infty e^{-\frac{t^k}{k}} t^{\alpha+nk-1}dt \tag{4.40}$$

dir. Teorem 3.2. nin ispatına benzer şekilde, yukarıdaki integral gösterimlerinin ispatı için birinci yol olarak (4.40) integral gösterimini birer kez kullanmak yeterlidir.

Ayrıca bu ispatlar için ikinci bir yol olarak da (4.5), (4.6), (4.7) ve (4.8) ilişkilerine ve Teorem 3.2. deki integral formüllerinin kullanılmasına dayanan daha kolay ve kısa bir metod kullanılabilir. Şimdi de bu metodun integral formüllerinde nasıl kullanıldığını görelim: (4.34) integral gösterimi, sırasıyla (4.5), (3.22), (2.10) ve (2.17) kullanılarak

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{\Gamma(\frac{\alpha}{k})}\int_0^\infty e^{-t}t^{\frac{\alpha}{k}-1}{}_1F_1\left(\frac{\beta_1}{k};\frac{\gamma_1}{k};ktx_1\right)\cdots{}_1F_1\left(\frac{\beta_n}{k};\frac{\gamma_n}{k};ktx_n\right)dt\\
&=\frac{1}{k^{1-\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_0^\infty e^{-t}t^{\frac{\alpha}{k}-1}{}_1F_{1,k}\left(\beta_1;\gamma_1;ktx_1\right)\cdots{}_1F_{1,k}\left(\beta_n;\gamma_n;ktx_n\right)dt
\end{aligned}
$$

şeklinde elde edilir.

(4.35) integral gösterimi, sırasıyla (4.5), (3.23), (2.10) ve (4.24) kullanılarak

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{\Gamma(\frac{\beta_n}{k})}\int_0^\infty e^{-t}t^{\frac{\beta_n}{k}-1}\Psi_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_{n-1},ktx_n)dt\\
&=\frac{1}{k^{1-\frac{\beta_n}{k}}\Gamma_k(\beta_n)}\int_0^\infty e^{-t}t^{\frac{\beta_n}{k}-1}\Psi_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_{n-1},ktx_n)dt
\end{aligned}
$$

şeklinde bulunur.

(4.36) integral gösterimi, sırasıyla (4.6), (3.24), (2.10) ve (4.22) kullanılarak

$$
\begin{aligned}
&F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F_B^{(n)}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{\Gamma(\frac{\beta_n}{k})}\int_0^\infty e^{-t}t^{\frac{\beta_n}{k}-1}\Xi_1^{(n)}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma}{k};kx_1,\ldots,kx_{n-1},ktx_n)dt\\
&=\frac{1}{k^{1-\frac{\beta_n}{k}}\Gamma_k(\beta_n)}\int_0^\infty e^{-t}t^{\frac{\beta_n}{k}-1}\Xi_{1,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_{n-1},ktx_n)dt
\end{aligned}
$$

şeklinde elde edilir.

Şimdi de bir sonraki (4.37) integral gösterimini bulalım. (4.7) ve (3.25) kullanılırsa

$$
\begin{aligned}
&F_{C,k}^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F_C^{(n)}(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{\Gamma(\frac{\alpha}{k})}\int_0^\infty e^{-t}t^{\frac{\alpha}{k}-1}\Psi_2^{(n)}(\frac{\beta}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};ktx_1,\ldots,ktx_n)dt
\end{aligned}
$$

olup, (2.10) ve (4.20) nin dikkate alınmasıyla da istenilen integral gösterimine ulaşılır.
(4.38) integral gösterimi, sırasıyla (4.8), (3.26), (2.10) ve (4.19) kullanılarak

$$
\begin{aligned}
&F^{(n)}_{D,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F^{(n)}_D(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{\Gamma(\frac{\alpha}{k})}\int_0^\infty e^{-t}t^{\frac{\alpha}{k}-1}\Phi^{(n)}_2(\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};ktx_1,\ldots,ktx_n)dt\\
&=\frac{1}{k^{1-\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_0^\infty e^{-t}t^{\frac{\alpha}{k}-1}\Phi^{(n)}_{2,k}(\beta_1,\ldots,\beta_n;\gamma;ktx_1,\ldots,ktx_n)dt
\end{aligned}
$$

şeklinde elde edilir.
(4.39) integral gösterimi, sırasıyla (4.8), (3.27), (2.10) ve (4.21) kullanılarak

$$
\begin{aligned}
&F^{(n)}_{D,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F^{(n)}_D(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{\Gamma(\frac{\beta_n}{k})}\int_0^\infty e^{-t}t^{\frac{\beta_n}{k}-1}\Phi^{(n)}_D(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_{n-1}}{k};\frac{\gamma}{k};kx_1,\ldots,kx_{n-1},ktx_n)dt\\
&=\frac{1}{k^{1-\frac{\beta_n}{k}}\Gamma_k(\beta_n)}\int_0^\infty e^{-t}t^{\frac{\beta_n}{k}-1}\Phi^{(n)}_{D,k}(\alpha,\beta_1,\ldots,\beta_{n-1};\gamma;x_1,\ldots,x_{n-1},ktx_n)dt
\end{aligned}
$$

şeklinde bulunur. ■

## 4.4. $k$-Lauricella Fonksiyonlarının Çok Katlı Laplace Tipi İntegralleri

**Teorem 4.6.** $k$-Lauricella fonksiyonlarının çok katlı Laplace tipi integralleri aşağıda listelenmiştir.

$$
\begin{aligned}
&F^{(n)}_{A,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{k^{\frac{\beta_1+\cdots+\beta_n}{k}}}{k^n\Gamma_k(\beta_1)\cdots\Gamma_k(\beta_n)}\\
&\quad\cdot\int_0^\infty\cdots\int_0^\infty e^{-t_1-\cdots-t_n}t_1^{\frac{\beta_1}{k}-1}\cdots t_n^{\frac{\beta_n}{k}-1}\\
&\quad\cdot\Psi^{(n)}_{2,k}(\alpha;\gamma_1,\ldots,\gamma_n;kt_1x_1,\ldots,kt_nx_n)dt_1\cdots dt_n
\end{aligned}
\tag{4.41}
$$

$$(Re(\beta_j)>0, j=1,\ldots,n)$$

$$
\begin{aligned}
&F^{(n)}_{B,k}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{k^{\frac{\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n}{k}}}{k^{2n}\Gamma_k(\alpha_1)\cdots\Gamma_k(\alpha_n)\Gamma_k(\beta_1)\cdots\Gamma_k(\beta_n)}\\
&\quad\cdot\int_0^\infty\cdots\int_0^\infty e^{-u_1-\cdots-u_n-v_1-\cdots-v_n}u_1^{\frac{\alpha_1}{k}-1}\cdots u_n^{\frac{\alpha_n}{k}-1}v_1^{\frac{\beta_1}{k}-1}\cdots v_n^{\frac{\beta_n}{k}-1}\\
&\quad\cdot{}_0F_{1,k}(-;\gamma;k^2u_1v_1x_1+\cdots+k^2u_nv_nx_n)du_1\cdots du_ndv_1\cdots dv_n
\end{aligned}
\tag{4.42}
$$

$$(\min\{R(\alpha_j), R(\beta_j)\} > 0, j = 1, \ldots, n)$$

$$F_{B,k}^{(n)}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \frac{k^{\frac{\beta_1+\cdots+\beta_n}{k}}}{k^n \Gamma_k(\beta_1)\cdots\Gamma_k(\beta_n)}$$
$$\cdot \int_0^\infty \cdots \int_0^\infty e^{-u_1-\cdots-u_n} u_1^{\frac{\beta_1}{k}-1} \cdots u_n^{\frac{\beta_n}{k}-1}$$
$$\cdot \Phi_{2,k}^{(n)}(\alpha_1, \ldots, \alpha_n; \gamma; ku_1x_1, \ldots, ku_nx_n) du_1 \cdots du_n \tag{4.43}$$

$$(Re(\beta_j) > 0, j = 1, \ldots, n)$$

$$F_{C,k}^{(n)}(\alpha, \beta; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) = \frac{k^{\frac{\alpha+\beta}{k}}}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_0^\infty \int_0^\infty e^{-u-v} u^{\frac{\alpha}{k}-1} v^{\frac{\beta}{k}-1}$$
$$\cdot {}_0F_{1,k}(-; \gamma_1; k^2uvx_1) \cdots {}_0F_{1,k}(-; \gamma_n; k^2uvx_n) dudv \tag{4.44}$$

$$(\min\big\{R(\alpha), R(\beta)\big\} > 0)$$

$$F_{D,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \frac{k^{\frac{\alpha+\beta_1+\cdots+\beta_n}{k}}}{k^{n+1} \Gamma_k(\alpha)\Gamma_k(\beta_1)\cdots\Gamma_k(\beta_n)}$$
$$\cdot \int_0^\infty \cdots \int_0^\infty e^{-u-v_1-\cdots-v_n} u^{\frac{\alpha}{k}-1} v_1^{\frac{\beta_1}{k}-1} \cdots v_n^{\frac{\beta_n}{k}-1}$$
$$\cdot {}_0F_{1,k}(-; \gamma; (v_1x_1 + \cdots + v_nx_n)k^2u) dudv_1 \cdots dv_n \tag{4.45}$$

$$(R(\alpha) > 0; R(\beta_j) > 0, j = 1, \ldots, n)$$

$$F_{D,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) = \frac{k^{\frac{\beta_1+\cdots+\beta_n}{k}}}{k^n \Gamma_k(\beta_1)\cdots\Gamma_k(\beta_n)}$$
$$\cdot \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\frac{\beta_1}{k}-1} \cdots t_n^{\frac{\beta_n}{k}-1}$$
$$\cdot {}_1F_{1,k}(\alpha; \gamma; kt_1x_1 + \cdots + kt_nx_n) dt_1 \cdots dt_n \tag{4.46}$$

$$(Re(\beta_j) > 0, j = 1, \ldots, n)$$

**İspat**. Teorem 3.3. ün ispatına benzer şekilde, yukarıdaki integral gösterimlerinin ispatı için birinci yol olarak (4.40) integral gösteriminin birden çok kez kullanmak yeterlidir.

Ayrıca bu ispatlar için ikinci bir yol olarak da (4.5), (4.6), (4.7) ve (4.8) ilişkilerine ve Teorem 3.3. deki integral formüllerinin kullanılmasına dayanan daha kolay ve kısa bir metod kullanılabilir. Şimdi de bu metodun ilk integral formülünde nasıl kullanıldığını görelim:

(4.41) integral gösterimi, sırasıyla (4.5), (3.29), (2.10) ve (4.20) kullanılarak

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) \\
&\quad = F_A^{(n)}(\frac{\alpha}{k}, \frac{\beta_1}{k}, \ldots, \frac{\beta_n}{k}; \frac{\gamma_1}{k}, \ldots, \frac{\gamma_n}{k}; kx_1, \ldots, kx_n) \\
&\quad = \frac{1}{\Gamma(\frac{\beta_1}{k}) \cdots \Gamma(\frac{\beta_n}{k})} \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\frac{\beta_1}{k}-1} \cdots t_n^{\frac{\beta_n}{k}-1} \\
&\qquad \cdot \Psi_2^{(n)}(\frac{\alpha}{k}; \frac{\gamma_1}{k}, \ldots, \frac{\gamma_n}{k}; kt_1x_1, \ldots, kt_nx_n) dt_1 \cdots dt_n \\
&\quad = \frac{k^{\frac{\beta_1+\cdots+\beta_n}{k}}}{k^n \Gamma_k(\beta_1) \cdots \Gamma_k(\beta_n)} \int_0^\infty \cdots \int_0^\infty e^{-t_1-\cdots-t_n} t_1^{\frac{\beta_1}{k}-1} \cdots t_n^{\frac{\beta_n}{k}-1} \\
&\qquad \cdot \Psi_{2,k}^{(n)}(\alpha; \gamma_1, \ldots, \gamma_n; kt_1x_1, \ldots, kt_nx_n) dt_1 \cdots dt_n
\end{aligned}$$

şeklinde elde edilir. Diğerleri de benzer şekilde ispatlanır. ■

## 4.5. $F_{A,k}^{(n)}$ ve $F_{D,k}^{(n)}$ Fonksiyonlarının Euler Tipi İntegralleri

**Teorem 4.7.** $F_{A,k}^{(n)}$ ve $F_{D,k}^{(n)}$ $k$-Lauricella fonksiyonlarının Euler tipi integralleri

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) \\
&= \frac{\Gamma_k(\gamma_1) \cdots \Gamma_k(\gamma_n)}{k^n \Gamma_k(\beta_1) \cdots \Gamma_k(\beta_n) \Gamma_k(\gamma_1 - \beta_1) \cdots \Gamma_k(\gamma_n - \beta_n)} \int_0^1 \cdots \int_0^1 t_1^{\frac{\beta_1}{k}-1} \cdots t_n^{\frac{\beta_n}{k}-1} \\
&\cdot (1-t_1)^{\frac{\gamma_1-\beta_1}{k}-1} \cdots (1-t_n)^{\frac{\gamma_n-\beta_n}{k}-1} (1 - kx_1t_1 - \cdots - kx_nt_n)^{\frac{-\alpha}{k}} dt_1 \cdots dt_n
\end{aligned} \quad (4.47)$$

$$(Re(\gamma_j) > Re(\beta_j) > 0, j = 1, \ldots, n)$$

ve

$$\begin{aligned}
&F_{D,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) \\
&= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)} \int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\gamma-\alpha}{k}-1}(1-kx_1t)^{-\frac{\beta_1}{k}} \cdots (1-kx_nt)^{-\frac{\beta_n}{k}} dt
\end{aligned} \quad (4.48)$$

$$(Re(\gamma) > Re(\alpha) > 0)$$

dir.

**İspat**. (2.2) ve (2.7) özelliklerinin kullanılmasıyla, $Re(\gamma) > Re(\beta) > 0$ için

$$\frac{(\beta)_{n,k}}{(\gamma)_{n,k}} = \frac{B_k(\beta + nk, \gamma - \beta)}{B_k(\beta, \gamma - \beta)} = \frac{1}{kB_k(\beta, \gamma - \beta)} \int_0^1 t^{\frac{\beta+nk}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1} dt \quad (4.49)$$

olduğu görülür. Yukarıdaki integral gösterimlerinin ispatı için birinci yol olarak (4.49) ile birlikte (2.5) ve (2.9) özelliklerini Teorem 3.4. ün ispatındaki gibi kullanmak yeterlidir.

Ayrıca bu ispatlar için ikinci bir yol olarak da (4.5), (4.8) ve (2.10) ilişkilerini ve Teorem 3.4. deki integral gösterimlerinin kullanılmasına dayanan daha kolay ve kısa bir yöntem seçilebilir. Şimdi bu yöntemin nasıl uygulandığını görelim:
(4.47) integral gösterimi, sırasıyla (4.5), (3.36) ve (2.10) kullanılarak

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{\Gamma(\frac{\gamma_1}{k})\cdots\Gamma(\frac{\gamma_n}{k})}{\Gamma(\frac{\beta_1}{k})\cdots\Gamma(\frac{\beta_n}{k})\Gamma(\frac{\gamma_1-\beta_1}{k})\cdots\Gamma(\frac{\gamma_n-\beta_n}{k})}\int_0^1\cdots\int_0^1 t_1^{\frac{\beta_1}{k}-1}\cdots t_n^{\frac{\beta_n}{k}-1}\\
&\cdot(1-t_1)^{\frac{\gamma_1-\beta_1}{k}-1}\cdots(1-t_n)^{\frac{\gamma_n-\beta_n}{k}-1}(1-kx_1t_1-\cdots-kx_nt_n)^{-\frac{\alpha}{k}}dt_1\cdots dt_n\\
&=\frac{\Gamma_k(\gamma_1)\cdots\Gamma_k(\gamma_n)}{k^n\Gamma_k(\beta_1)\cdots\Gamma_k(\beta_n)\Gamma_k(\gamma_1-\beta_1)\cdots\Gamma_k(\gamma_n-\beta_n)}\int_0^1\cdots\int_0^1 t_1^{\frac{\beta_1}{k}-1}\cdots t_n^{\frac{\beta_n}{k}-1}\\
&\cdot(1-t_1)^{\frac{\gamma_1-\beta_1}{k}-1}\cdots(1-t_n)^{\frac{\gamma_n-\beta_n}{k}-1}(1-kx_1t_1-\cdots-kx_nt_n)^{\frac{-\alpha}{k}}dt_1\cdots dt_n
\end{aligned}$$

şeklinde elde edilir.
(4.48) integral gösterimi, sırasıyla (4.8), (3.37) ve (2.10) kullanılarak

$$\begin{aligned}
&F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F_D^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{\Gamma(\frac{\gamma}{k})}{\Gamma(\frac{\alpha}{k})\Gamma(\frac{\gamma-\alpha}{k})}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\gamma-\alpha}{k}-1}(1-kx_1t)^{-\frac{\beta_1}{k}}\cdots(1-kx_nt)^{-\frac{\beta_n}{k}}dt\\
&=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\gamma-\alpha}{k}-1}(1-kx_1t)^{-\frac{\beta_1}{k}}\cdots(1-kx_nt)^{-\frac{\beta_n}{k}}dt
\end{aligned}$$

şeklinde bulunur. ■

## 4.6. $k$-Lauricella Fonksiyonlarının Ekstra Parametre İçeren İntegralleri

**Teorem 4.8.** $F_{A,k}^{(n)}$, $F_{B,k}^{(n)}$, $F_{C,k}^{(n)}$, $F_{D,k}^{(n)}$ Lauricella fonksiyonlarının ekstra $\lambda$ parametresi içeren integral gösterimlerinin 1.grubu sırasıyla aşağıda verilmiştir.

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{kB_k(\beta_1,\lambda-\beta_1)}\\
&\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F_{A,k}^{(n)}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt \qquad (4.50)
\end{aligned}$$

$$(Re(\lambda)>Re(\beta_1)>0)$$

$$F^{(n)}_{B,k}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{kB_k(\beta_1,\lambda-\beta_1)}$$
$$\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F^{(n)}_{B,k}(\alpha_1,\ldots,\alpha_n,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt \quad (4.51)$$

$$(Re(\lambda)>Re(\beta_1)>0)$$

$$F^{(n)}_{C,k}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{kB_k(\lambda,\gamma_1-\lambda)}$$
$$\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma_1-\lambda}{k}-1}F^{(n)}_{C,k}(\alpha,\beta;\lambda,\gamma_2,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt \quad (4.52)$$

$$(Re(\gamma_1)>Re(\lambda)>0)$$

$$F^{(n)}_{D,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{kB_k(\beta_1,\lambda-\beta_1)}$$
$$\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F^{(n)}_{D,k}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt \quad (4.53)$$

$$(Re(\lambda)>Re(\beta_1)>0)$$

**İspat**. Yukarıdaki integral gösterimlerinin ispatı için birinci yol olarak (4.1), (4.2), (4.3) ve (4.4) deki $k$-Lauricella fonksiyonlarının seri tanımlarının pay ve paydalarını $(\lambda)_{m_1,k}$ ile çarptıktan sonra (4.49) özelliğinin Teorem 3.5. in ispatındaki gibi kullanılması yeterlidir.

Ayrıca bu ispatlar için ikinci bir yol olarak da (4.5), (4.6), (4.7),(4.8) ilişkilerinin ve Teorem 3.5. deki integral gösterimlerinin ve de (2.11) ilişkisinin kullanılmasına dayanan daha kolay ve kısa bir yöntem seçilebilir. Şimdi bu yöntemin nasıl uygulandığını görelim: (4.50) integral gösteriminin ispatı için (4.5), (3.41), (2.11) ve (4.9) eşitliklerinin kullanılmasıyla

$$\begin{aligned}
&F^{(n)}_{A,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F^{(n)}_{A}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\beta_1}{k},\frac{\lambda-\beta_1}{k})}\\
&\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F^{(n)}_{A}(\frac{\alpha}{k},\frac{\lambda}{k},\frac{\beta_2}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};ktx_1,kx_2,\ldots,kx_n)dt\\
&=\frac{1}{kB_k(\beta_1,\lambda-\beta_1)}\\
&\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F^{(n)}_{A,k}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt
\end{aligned}$$

şeklinde elde edilir.

(4.51) integral gösteriminin ispatı için (4.6), (3.42), (2.11) ve (4.10) eşitliklerinin dikkate alınmasıyla

$$\begin{aligned}
&F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F_B^{(n)}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\beta_1}{k},\frac{\lambda-\beta_1}{k})}\\
&\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F_B^{(n)}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\lambda}{k},\frac{\beta_2}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};ktx_1,kx_2,\ldots,kx_n)dt\\
&=\frac{1}{kB_k(\beta_1,\lambda-\beta_1)}\\
&\cdot\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt
\end{aligned}$$

şeklinde bulunur.

(4.52) integral gösteriminin ispatı için (4.7), (3.43), (2.11) ve (4.11) eşitliklerinin kullanılmasıyla

$$\begin{aligned}
&F_{C,k}^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F_C^{(n)}(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\lambda}{k},\frac{\gamma_1-\lambda}{k})}\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma_1-\lambda}{k}-1}F_C^{(n)}(\frac{\alpha}{k},\frac{\beta}{k};\frac{\lambda}{k},\frac{\gamma_2}{k},\ldots,\frac{\gamma_n}{k};ktx_1,kx_2,\ldots,kx_n)dt\\
&=\frac{1}{kB_k(\lambda,\gamma_1-\lambda)}\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma_1-\lambda}{k}-1}F_{C,k}^{(n)}(\alpha,\beta;\lambda,\gamma_2,\ldots,\gamma_n;tx_1,x_2,\ldots,x_n)dt
\end{aligned}$$

şeklinde elde edilir.

(4.53) integral gösteriminin ispatı için (4.8), (3.44), (2.11) ve (4.12) eşitliklerinin kullanılmasıyla

$$\begin{aligned}
&F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F_D^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\beta_1}{k},\frac{\lambda-\beta_1}{k})}\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F_D^{(n)}(\frac{\alpha}{k},\frac{\lambda}{k},\frac{\beta_2}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};ktx_1,kx_2,\ldots,kx_n)dt\\
&=\frac{1}{kB_k(\beta_1,\lambda-\beta_1)}\int_0^1 t^{\frac{\beta_1}{k}-1}(1-t)^{\frac{\lambda-\beta_1}{k}-1}F_{D,k}^{(n)}(\alpha,\lambda,\beta_2,\ldots,\beta_n;\gamma;tx_1,x_2,\ldots,x_n)dt
\end{aligned}$$

şeklinde bulunur. ■

**Teorem 4.9.** $F_{A,k}^{(n)}$, $F_{B,k}^{(n)}$, $F_{C,k}^{(n)}$, $F_{D,k}^{(n)}$ Lauricella fonksiyonlarının ekstra $\lambda$ parametresi içeren integral gösterimlerinin 2.grubu sırasıyla,

$$F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{kB_k(\alpha,\lambda-\alpha)}$$
$$\cdot\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\lambda-\alpha}{k}-1}F_{A,k}^{(n)}(\lambda,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt \qquad (4.54)$$

$$(Re(\lambda)>Re(\alpha)>0)$$

$$F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{kB_k(\lambda,\gamma-\lambda)}$$
$$\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma-\lambda}{k}-1}F_{B,k}^{(n)}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt \qquad (4.55)$$

$$(Re(\gamma)>Re(\lambda)>0)$$

$$F_{C,k}^{(n)}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=\frac{1}{kB_k(\alpha,\lambda-\alpha)}$$
$$\cdot\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\lambda-\alpha}{k}-1}F_{C,k}^{(n)}(\lambda,\beta;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt \qquad (4.56)$$

$$(Re(\lambda)>Re(\alpha)>0)$$

$$F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=\frac{1}{kB_k(\lambda,\gamma-\lambda)}$$
$$\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma-\lambda}{k}-1}F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt \qquad (4.57)$$

$$(Re(\gamma)>Re(\lambda)>0)$$

şeklindedir.

**İspat**. Yukarıdaki integral gösterimlerinin ispatı için birinci yol olarak (4.1), (4.2), (4.3) ve (4.4) deki $k$-Lauricella fonksiyonlarının seri tanımlarının pay ve paydalarını $(\lambda)_{m_1+\cdots+m_n,k}$ ile çarptıktan sonra (4.49) özelliğinin Teorem3.6. nin ispatındaki gibi kullanılması yeterlidir.

Ayrıca bu ispatlar için ikinci bir yol olarak da (4.5), (4.6), (4.7), (4.8) ilişkilerinin ve Teorem3.6. integral gösterimlerinin ve de (2.11) ilişkisinin kullanılmasına dayanan daha kolay ve kısa bir yöntem seçilebilir. Şimdi bu yöntemin nasıl uygulandığını görelim:

(4.54) integral gösterimi, (4.5), (3.45), (2.11) ve (4.9) eşitliklerinin dikkate alınmasıyla

$$
\begin{aligned}
&F^{(n)}_{A,k}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F^{(n)}_{A}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\alpha}{k},\frac{\lambda-\alpha}{k})}\\
&\cdot\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\lambda-\alpha}{k}-1}F^{(n)}_{A}(\frac{\lambda}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};ktx_1,\ldots,ktx_n)dt\\
&=\frac{1}{kB_k(\alpha,\lambda-\alpha)}\\
&\cdot\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\lambda-\alpha}{k}-1}F^{(n)}_{A,k}(\lambda,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt
\end{aligned}
$$

şeklinde bulunur.

(4.55) integral gösterimi, (4.6), (3.46), (2.11) ve (4.10) eşitliklerinin dikkate alınmasıyla

$$
\begin{aligned}
&F^{(n)}_{B,k}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F^{(n)}_{B}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\lambda}{k},\frac{\gamma-\lambda}{k})}\\
&\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma-\lambda}{k}-1}F^{(n)}_{B}(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\lambda}{k};ktx_1,\ldots,ktx_n)dt\\
&=\frac{1}{kB_k(\lambda,\gamma-\lambda)}\\
&\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma-\lambda}{k}-1}F^{(n)}_{B,k}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt
\end{aligned}
$$

şeklinde elde edilir.

(4.56) integral gösterimi, (4.7), (3.47), (2.11) ve (4.11) eşitliklerinin kullanılmasıyla

$$
\begin{aligned}
&F^{(n)}_{C,k}(\alpha,\beta;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F^{(n)}_{C}(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\alpha}{k},\frac{\lambda-\alpha}{k})}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\lambda-\alpha}{k}-1}F^{(n)}_{C}(\frac{\lambda}{k},\frac{\beta}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};ktx_1,\ldots,ktx_n)dt\\
&=\frac{1}{kB_k(\alpha,\lambda-\alpha)}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\lambda-\alpha}{k}-1}F^{(n)}_{C,k}(\lambda,\beta;\gamma_1,\ldots,\gamma_n;tx_1,\ldots,tx_n)dt
\end{aligned}
$$

şeklinde bulunur.

(4.57) integral gösterimi, (4.8), (3.48), (2.11) ve (4.12) eşitliklerinin dikkate alınmasıyla

$$
\begin{aligned}
&F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)\\
&=F_D^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\ldots,kx_n)\\
&=\frac{1}{B(\frac{\lambda}{k},\frac{\gamma-\lambda}{k})}\\
&\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma-\lambda}{k}-1}F_D^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\lambda}{k};ktx_1,\ldots,ktx_n)dt\\
&=\frac{1}{kB_k(\lambda,\gamma-\lambda)}\\
&\cdot\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\gamma-\lambda}{k}-1}F_{D,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\lambda;tx_1,\ldots,tx_n)dt
\end{aligned}
$$

şeklinde elde edilir. ■

## 4.7. $F_{A,k}^{(n)}$ Fonksiyonu için Dönüşüm Formülleri

**Teorem 4.10.** $F_{A,k}^{(n)}$ için $\binom{n}{1}=n$ tane dönüşüm formülü aşağıdadır.

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1)^{-\frac{\alpha}{k}}\\
&\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{kx_1-1},\frac{x_2}{1-kx_1},\ldots,\frac{x_n}{1-kx_1})
\end{aligned}
$$

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_2)^{-\frac{\alpha}{k}}\\
&\cdot F_{A,k}^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\beta_3,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-kx_2},\frac{x_2}{kx_2-1},\frac{x_3}{1-kx_2},\ldots,\frac{x_n}{1-kx_2})\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_n)^{-\frac{\alpha}{k}}\\
&\cdot F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_{n-1},\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-kx_n},\ldots,\frac{x_{n-1}}{1-kx_n},\frac{x_n}{kx_n-1})
\end{aligned}
$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için birinci yol, Teoerem 3.7. deki ispat yöntemine benzer şekilde,

$$
\begin{aligned}
&t_1=1-u_1,t_2=u_2,\ldots,t_n=u_n\\
&t_1=u_1,t_2=1-u_2,t_3=u_3,\ldots,t_n=u_n\\
&\vdots\\
&t_1=u_1,\ldots,t_{n-1}=u_{n-1},t_n=1-u_n
\end{aligned}
$$

$\binom{n}{1}$ olası dönüşümün (4.47) integraline sırasıyla uygulanmasıdır.

Ayrıca yukarıdaki dönüşüm formüllerinin ispatı için alternatif ikinci bir yol da, (4.5) ilişkisi ve Teorem 3.7. deki dönüşüm formüllerinin kullanılmasına dayalı olan daha kolay ve kısa bir yöntemin kullanılmasıdır. Gerçekten de ilk dönüşüm formülü sırasıyla (4.5) ilişkisi, Teorem 3.7. deki ilk dönüşüm formülü ve (4.9) ilişkisinin dikkate alınmasıyla

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&=F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&=(1-kx_1)^{-\frac{\alpha}{k}}\\
&\cdot F_A^{(n)}(\frac{\alpha}{k},\frac{\gamma_1-\beta_1}{k},\frac{\beta_2}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};\frac{kx_1}{kx_1-1},\frac{kx_2}{1-kx_1},\ldots,\frac{kx_n}{1-kx_1})\\
&=(1-kx_1)^{-\frac{\alpha}{k}}F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{kx_1-1},\frac{x_2}{1-kx_1},\ldots,\frac{x_n}{1-kx_1})
\end{aligned}$$

şeklinde elde edilir. Diğerleri de aynı yöntemle ispatlanabilir. ■

**Teorem 4.11.** $F_{A,k}^{(n)}$ in bir diğer $\binom{n}{2}$ tane dönüşüm formülü aşağıdadır.

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1-kx_2)^{-\frac{\alpha}{k}}\\
&\quad\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\beta_3,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{kx_1+kx_2-1},\\
&\quad\frac{x_2}{kx_1+kx_2-1},\frac{x_3}{1-kx_1-kx_2},\ldots,\frac{x_n}{1-kx_1-kx_2})
\end{aligned}$$

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1-kx_3)^{-\frac{\alpha}{k}}\\
&\quad\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{kx_1+kx_3-1},\\
&\quad\frac{x_2}{1-kx_1-kx_3},\frac{x_3}{kx_1+kx_3-1},\frac{x_4}{1-kx_1-kx_3},\ldots,\frac{x_n}{1-kx_1-kx_3})\\
&\quad\vdots
\end{aligned}$$

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1-kx_n)^{-\frac{\alpha}{k}}\\
&\quad\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\beta_2,\ldots,\beta_{n-1},\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{kx_1+kx_n-1},\\
&\quad\frac{x_2}{1-kx_1-kx_n},\ldots,\frac{x_{n-1}}{1-kx_1-kx_n},\frac{x_n}{kx_1+kx_n-1})
\end{aligned}$$

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_2-kx_3)^{-\frac{\alpha}{k}}\\
&\quad\cdot F_{A,k}^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-kx_2-kx_3},\\
&\quad\frac{x_2}{kx_2+kx_3-1},\frac{x_3}{kx_2+kx_3-1},\frac{x_4}{1-kx_2-kx_3},\ldots,\frac{x_n}{1-kx_2-kx_3})\\
&\quad\vdots
\end{aligned}$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için birinci yol, Teorem 3.8. deki ispat yöntemine benzer şekilde,

$$
\begin{aligned}
&t_1 = 1-u_1, t_2 = 1-u_2, t_3 = u_3, \ldots, t_n = u_n\\
&t_1 = 1-u_1, t_2 = u_2, t_3 = 1-u_3, t_4 = u_4, \ldots, t_n = u_n\\
&\vdots\\
&t_1 = 1-u_1, t_2 = u_2, \ldots, t_{n-1} = u_{n-1}, t_n = 1-u_n\\
&t_1 = u_1, t_2 = 1-u_2, t_3 = 1-u_3, t_4 = u_4, \ldots, t_n = u_n\\
&\vdots
\end{aligned}
$$

$\binom{n}{2}$ olası dönüşümün (4.47) integraline sırasıyla uygulanmasıdır.

Ayrıca yukarıdaki dönüşüm formüllerinin ispatı için alternatif ikinci bir yol da (4.5) ilişkisi ve Teorem 3.8. deki dönüşüm formüllerinin kullanılmasına dayalı daha kolay ve kısa bir yöntemin kullanılmasıdır. Gerçekten de (4.5) ilişkisi ve Teorem 3.8. deki ilk dönüşüm formülü kullanılırsa

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n)\\
&= F_A^{(n)}(\frac{\alpha}{k}, \frac{\beta_1}{k}, \ldots, \frac{\beta_n}{k}; \frac{\gamma_1}{k}, \ldots, \frac{\gamma_n}{k}; kx_1, \ldots, kx_n)\\
&= (1-kx_1-kx_2)^{-\frac{\alpha}{k}}\\
&\cdot F_A^{(n)}(\frac{\alpha}{k}, \frac{\gamma_1-\beta_1}{k}, \frac{\gamma_2-\beta_2}{k}, \frac{\beta_3}{k}, \ldots, \frac{\beta_n}{k}; \frac{\gamma_1}{k}, \ldots, \frac{\gamma_n}{k}; \frac{kx_1}{kx_1+kx_2-1},\\
&\frac{kx_2}{kx_1+kx_2-1}, \frac{kx_3}{1-kx_1-kx_2}, \ldots, \frac{kx_n}{1-kx_1-kx_2})
\end{aligned}
$$

olup, (4.9) ilişkisinin de dikkate alınmasıyla

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n)\\
&= (1-kx_1-kx_2)^{-\frac{\alpha}{k}}\\
&\cdot F_{A,k}^{(n)}(\alpha, \gamma_1-\beta_1, \gamma_2-\beta_2, \beta_3, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \frac{x_1}{kx_1+kx_2-1},\\
&\frac{x_2}{kx_1+kx_2-1}, \frac{x_3}{1-kx_1-kx_2}, \ldots, \frac{x_n}{1-kx_1-kx_2})
\end{aligned}
$$

teoremdeki ilk dönüşüm formülü elde edilir. Diğer dönüşüm formülleri de aynı yöntemle elde edilebilir. ■

**Teorem 4.12.** $F_{A,k}^{(n)}$ in bir diğer $\binom{n}{3}$ tane dönüşüm formülü aşağıda listelenmiştir,

$$F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1-kx_2-kx_3)^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\beta_4,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;$$
$$\frac{x_1}{kx_1+kx_2+kx_3-1},\frac{x_2}{kx_1+kx_2+kx_3-1},\frac{x_3}{kx_1+kx_2+kx_3-1},$$
$$\frac{x_4}{1-kx_1-kx_2-kx_3},\ldots,\frac{x_n}{1-kx_1-kx_2-kx_3})$$

$$F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1-kx_2-kx_4)^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\beta_3,\gamma_4-\beta_4,\beta_5,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;$$
$$\frac{x_1}{kx_1+kx_2+kx_4-1},\frac{x_2}{kx_1+kx_2+kx_4-1},\frac{x_3}{1-kx_1-kx_2-kx_4},\frac{x_4}{kx_1+kx_2+kx_4-1},$$
$$\frac{x_5}{1-kx_1-kx_2-kx_4},\ldots,\frac{x_n}{1-kx_1-kx_2-kx_4})$$

$$\vdots$$

$$F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_1-kx_2-kx_n)^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\gamma_2-\beta_2,\beta_3,\ldots,\beta_{n-1},\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{kx_1+kx_2+kx_n-1},$$
$$\frac{x_2}{kx_1+kx_2+kx_n-1},\frac{x_3}{1-kx_1-kx_2-kx_n},\ldots,$$
$$\frac{x_{n-1}}{1-kx_1-kx_2-kx_n},\frac{x_n}{kx_1+kx_2+kx_n-1})$$

$$F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)=(1-kx_2-kx_3-kx_4)^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\gamma_3-\beta_3,\gamma_4-\beta_4,\beta_5,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-kx_2-kx_3-kx_4},$$
$$\frac{x_2}{kx_2+kx_3+kx_4-1},\frac{x_3}{kx_2+kx_3+kx_4-1},\frac{x_4}{kx_2+kx_3+kx_4-1},\frac{x_5}{1-kx_2-kx_3-kx_4},$$
$$\ldots,\frac{x_n}{1-kx_2-kx_3-kx_4})$$

$$\vdots$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için birinci yol, Teorem 3.9. daki ispat yönteminin benzer şekilde

$$t_1 = 1 - u_1, t_2 = 1 - u_2, t_3 = 1 - u_3, t_4 = u_4, \ldots, t_n = u_n$$
$$t_1 = 1 - u_1, t_2 = 1 - u_2, t_3 = u_3, t_4 = 1 - u_4, t_5 = u_5, \ldots, t_n = u_n$$
$$\vdots$$
$$t_1 = 1 - u_1, t_2 = 1 - u_2, t_3 = u_3, \ldots, t_{n-1} = u_{n-1}, t_n = 1 - u_n$$
$$t_1 = u_1, t_2 = 1 - u_2, t_3 = 1 - u_3, t_4 = 1 - u_4, t_5 = u_5, \ldots, t_n = u_n$$
$$\vdots$$

$\binom{n}{3}$ olası dönüşümün (4.47) integraline sırasıyla uygulanmasıdır.

Ayrıca yukarıdaki dönüşüm formüllerinin ispatı için alternatif ikinci bir yol da (4.5) ilişkisi ve Teorem 3.9. deki dönüşüm formüllerinin kullanılmasına dayalı daha kolay ve kısa bir yöntemin kullanılmasıdır. Gerçekten de ilk dönüşüm formülü sırasıyla (4.5) ilişkisi, Teorem 3.9. deki ilk dönüşüm formülü ve (4.9) ilişkisinin dikkate alınmasıyla,

$$\begin{aligned}
&F_{A,k}^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n) \\
&= F_A^{(n)}(\frac{\alpha}{k}, \frac{\beta_1}{k}, \ldots, \frac{\beta_n}{k}; \frac{\gamma_1}{k}, \ldots, \frac{\gamma_n}{k}; kx_1, \ldots, kx_n) \\
&= (1 - kx_1 - kx_2 - kx_3)^{-\frac{\alpha}{k}} \\
&\cdot F_A^{(n)}(\frac{\alpha}{k}, \frac{\gamma_1 - \beta_1}{k}, \frac{\gamma_2 - \beta_2}{k}, \frac{\gamma_3 - \beta_3}{k}, \frac{\beta_4}{k}, \ldots, \frac{\beta_n}{k}; \frac{\gamma_1}{k}, \ldots, \frac{\gamma_n}{k}; \\
&\frac{kx_1}{kx_1 + kx_2 + kx_3 - 1}, \frac{kx_2}{kx_1 + kx_2 + kx_3 - 1}, \frac{kx_3}{kx_1 + kx_2 + kx_3 - 1}, \\
&\frac{kx_4}{1 - kx_1 - kx_2 - kx_3}, \ldots, \frac{kx_n}{1 - kx_1 - kx_2 - kx_3}) \\
&= (1 - kx_1 - kx_2 - kx_3)^{-\frac{\alpha}{k}} \\
&\cdot F_{A,k}^{(n)}(\alpha, \gamma_1 - \beta_1, \gamma_2 - \beta_2, \gamma_3 - \beta_3, \beta_4, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; \\
&\frac{x_1}{kx_1 + kx_2 + kx_3 - 1}, \frac{x_2}{kx_1 + kx_2 + kx_3 - 1}, \frac{x_3}{kx_1 + kx_2 + kx_3 - 1}, \\
&\frac{x_4}{1 - kx_1 - kx_2 - kx_3}, \ldots, \frac{x_n}{1 - kx_1 - kx_2 - kx_3})
\end{aligned}$$

şeklinde elde edilir. Diğerleri de aynı yöntemle bulunabilir. ■

**Teorem 4.13.** $F_{A,k}^{(n)}$ in bir diğer $\binom{n}{n-1} = n$ tane dönüşüm formülü,

$$F_{A,k}^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) = (1 - kx_2 - \cdots - kx_n)^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha, \beta_1, \gamma_2 - \beta_2, \dots, \gamma_n - \beta_n; \gamma_1, \dots, \gamma_n; \frac{x_1}{1 - kx_2 - \cdots - kx_n},$$
$$\frac{x_2}{kx_2 + \cdots + kx_n - 1}, \dots, \frac{x_n}{kx_2 + \cdots + kx_n - 1})$$

$$F_{A,k}^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) = (1 - kx_1 - kx_3 - \cdots - kx_n)^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha, \gamma_1 - \beta_1, \beta_2, \gamma_3 - \beta_3, \dots, \gamma_n - \beta_n; \gamma_1, \dots, \gamma_n; \frac{x_1}{kx_1 + kx_3 + \cdots + kx_n - 1},$$
$$\frac{x_2}{1 - kx_1 - kx_3 - \cdots - kx_n}, \frac{x_3}{kx_1 + kx_3 + \cdots + kx_n - 1}, \dots, \frac{x_n}{kx_1 + kx_3 + \cdots + kx_n - 1})$$
$$\vdots$$

$$F_{A,k}^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) = (1 - kx_1 - \cdots - kx_{n-1})^{-\frac{\alpha}{k}}$$
$$\cdot F_{A,k}^{(n)}(\alpha, \gamma_1 - \beta_1, \dots, \gamma_{n-1} - \beta_{n-1}, \beta_n; \gamma_1, \dots, \gamma_n; \frac{x_1}{kx_1 + \cdots + kx_{n-1} - 1}, \dots,$$
$$\frac{x_{n-1}}{kx_1 + \cdots + kx_{n-1} - 1}, \frac{x_n}{1 - kx_1 - \cdots - kx_{n-1}})$$

dır [8].

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için birinci yol, Teorem 3.10. daki ispat yöntemine benzer şekilde,

$$t_1 = u_1, t_2 = 1 - u_2, t_3 = 1 - u_3, \dots, t_n = 1 - u_n$$
$$t_1 = 1 - u_1, t_2 = u_2, t_3 = 1 - u_3, \dots, t_n = 1 - u_n$$
$$\vdots$$
$$t_1 = 1 - u_1, t_2 = 1 - u_2, \dots, t_{n-1} = 1 - u_{n-1}, t_n = u_n$$

$\binom{n}{n-1}$ olası dönüşümün (4.47) integraline sırasıyla uygulanmasıdır.

Ayrıca yukarıdaki dönüşüm formüllerinin ispatı için alternatif ikinci bir yol da (4.5) ilişkisi ve Teorem 3.10. daki dönüşüm formüllerinin kullanılmasına dayalı daha kolay ve kısa bir yöntemin kullanılmasıdır. Gerçekten de ilk dönüşüm formülü sırasıyla (4.5) ilişkisi, Teorem 3.10. daki ilk dönüşüm formülü ve (4.9) ilişkisinin dikkate alınmasıyla,

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&= F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&= (1-kx_2-\cdots-kx_n)^{-\frac{\alpha}{k}}\\
&\cdot F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\gamma_2-\beta_2}{k},\ldots,\frac{\gamma_n-\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};\frac{kx_1}{1-kx_2-\cdots-kx_n},\\
&\quad\frac{kx_2}{kx_2+\cdots+kx_n-1},\ldots,\frac{kx_n}{kx_2+\cdots+kx_n-1})\\
&= (1-kx_2-\cdots-kx_n)^{-\frac{\alpha}{k}}\\
&\cdot F_{A,k}^{(n)}(\alpha,\beta_1,\gamma_2-\beta_2,\ldots,\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{1-kx_2-\cdots-kx_n},\\
&\quad\frac{x_2}{kx_2+\cdots+kx_n-1},\ldots,\frac{x_n}{kx_2+\cdots+kx_n-1})
\end{aligned}
$$

şeklinde bulunur. Diğerleri de aynı yöntemle ispatlanabilir. ■

**Teorem 4.14.** $F_{A,k}^{(n)}$ in diğer $\binom{n}{n}=1$ tane dönüşüm formülü aşağıdadır.

$$
\begin{aligned}
F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) &= (1-kx_1-\cdots-kx_n)^{-\frac{\alpha}{k}}\\
&\cdot F_{A,k}^{(n)}(\alpha,\gamma_1-\beta_1,\ldots,\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\\
&\qquad\frac{x_1}{kx_1+\cdots+kx_n-1},\ldots,\frac{x_n}{kx_1+\cdots+kx_n-1})
\end{aligned}
$$

**İspat**. İspat için birinci yol, Teorem 3.11. deki ispat yöntemine benzer şekilde,

$$t_1 = 1-u_1,\ldots,t_n = 1-u_n$$

dönüşümünün (4.47) integraline uygulanmasıdır.

Ayrıca ispat için ikinci bir yol da şudur: (4.5) ilişkisinden ve Teorem 3.11. den

$$
\begin{aligned}
&F_{A,k}^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
&= F_A^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\ldots,\frac{\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};kx_1,\ldots,kx_n)\\
&= (1-kx_1-\cdots-kx_n)^{-\frac{\alpha}{k}}\\
&\quad\cdot F_A^{(n)}(\frac{\alpha}{k},\frac{\gamma_1-\beta_1}{k},\ldots,\frac{\gamma_n-\beta_n}{k};\frac{\gamma_1}{k},\ldots,\frac{\gamma_n}{k};\\
&\qquad\frac{kx_1}{kx_1+\cdots+kx_n-1},\ldots,\frac{kx_n}{kx_1+\cdots+kx_n-1})
\end{aligned}
$$

olup, (4.9) ilişkisinin göz önüne alınmasıyla da

$$
\begin{aligned}
&F^{(n)}_{A,k}(\alpha,\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n;x_1,\dots,x_n)\\
&=(1-kx_1-\cdots-kx_n)^{-\frac{\alpha}{k}}\\
&\quad\cdot F^{(n)}_{A,k}(\alpha,\gamma_1-\beta_1,\dots,\gamma_n-\beta_n;\gamma_1,\dots,\gamma_n;\\
&\qquad\qquad\frac{x_1}{kx_1+\cdots+kx_n-1},\dots,\frac{x_n}{kx_1+\cdots+kx_n-1})
\end{aligned}
$$

elde edilir ki bu da istenilen dönüşüm formülüdür. ■

**Sonuç 4.15.** Bu kısımda $F^{(n)}_{A,k}$ için verilen toplam dönüşüm sayısı

$$\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n-1$$

dir.

## 4.8. $F^{(n)}_{D,k}$ Fonksiyonu için Dönüşüm Formülleri

**Teorem 4.16.** $F^{(n)}_{D,k}$ için bir dönüşüm formülü

$$
\begin{aligned}
&F^{(n)}_{D,k}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_1)^{-\frac{\beta_1}{k}}\cdots(1-kx_n)^{-\frac{\beta_n}{k}}\\
&\qquad\cdot F^{(n)}_{D,k}(\gamma-\alpha,\beta_1,\dots,\beta_n;\gamma;\frac{x_1}{kx_1-1},\dots,\frac{x_n}{kx_n-1})
\end{aligned}
$$

şeklindedir.

**İspat**. Yukarıdaki dönüşüm formülünün ispatı için birinci yol, Teorem 3.13. deki ispat yöntemine benzer şekilde,

$$t=1-u\Longrightarrow dt=-du$$

değişken değiştirmesinin (4.48) integraline uygulanmasıdır.

Ayrıca yukarıdaki dönüşüm formülünün ispatı için ikinci bir yol da (4.8) ilişkisi ve Teorem 3.13. deki dönüşüm formülü ve (4.12) ilişkisi dikkate alınarak

$$
\begin{aligned}
&F^{(n)}_{D,k}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)\\
&=F^{(n)}_{D}(\frac{\alpha}{k},\frac{\beta_1}{k},\dots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\dots,kx_n)\\
&=(1-kx_1)^{-\frac{\beta_1}{k}}\cdots(1-kx_n)^{-\frac{\beta_n}{k}}F^{(n)}_{D}(\frac{\gamma-\alpha}{k},\frac{\beta_1}{k},\dots,\frac{\beta_n}{k};\frac{\gamma}{k};\frac{kx_1}{kx_1-1},\dots,\frac{kx_n}{kx_n-1})\\
&=(1-kx_1)^{-\frac{\beta_1}{k}}\cdots(1-kx_n)^{-\frac{\beta_n}{k}}F^{(n)}_{D,k}(\gamma-\alpha,\beta_1,\dots,\beta_n;\gamma;\frac{x_1}{kx_1-1},\dots,\frac{x_n}{kx_n-1})
\end{aligned}
$$

şeklinde verilebilir. ■

**Teorem 4.17.** $F_{D,k}^{(n)}$ için $n$ tane dönüşüm formülü aşağıdadır.

$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_1)^{-\frac{\alpha}{k}}$$
$$\cdot F_{D,k}^{(n)}(\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\dots,\beta_n;\gamma;\frac{x_1}{kx_1-1},\frac{x_1-x_2}{kx_1-1},\dots,\frac{x_1-x_n}{kx_1-1})$$
$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_2)^{-\frac{\alpha}{k}}$$
$$\cdot F_{D,k}^{(n)}(\alpha,\beta_1,\gamma-\beta_1-\cdots-\beta_n,\beta_3,\dots,\beta_n;\gamma;\frac{x_2-x_1}{kx_2-1},\frac{x_2}{kx_2-1},\frac{x_2-x_3}{kx_2-1},\dots,\frac{x_2-x_n}{kx_2-1})$$
$$\vdots$$
$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_n)^{-\frac{\alpha}{k}}$$
$$\cdot F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_{n-1},\gamma-\beta_1-\cdots-\beta_n;\gamma;\frac{x_n-x_1}{kx_n-1},\dots,\frac{x_n-x_{n-1}}{kx_n-1},\frac{x_n}{kx_n-1})$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için birinci yol, Teorem 3.14. in ispatına benzer şekilde,

$$t=\frac{u}{1-kx_1+kx_1u},\dots,t=\frac{u}{1-kx_n+kx_nu}$$

değişken değiştirmelerinin (4.48) integraline sırasıyla uygulanmasıdır.
Ayrıca yukarıdaki dönüşüm formüllerinin ispatı için ikinci bir yol da (4.8) ilişkisinin ve Teorem 3.14. deki dönüşüm formüllerinin kullanılmasına dayalı olan daha kolay ve kısa bir yöntemin kullanılmasıdır. Gerçekten de ilk dönüşüm formülü sırasıyla (4.8) ilişkisi, Teorem 3.14. deki ilk dönüşüm formülü ve (4.12) ilişkisinin dikkate alınmasıyla

$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)$$
$$=F_D^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\dots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\dots,kx_n)$$
$$=(1-kx_1)^{-\frac{\alpha}{k}}$$
$$\cdot F_D^{(n)}(\frac{\alpha}{k},\frac{\gamma-\beta_1-\cdots-\beta_n}{k},\frac{\beta_2}{k},\dots,\frac{\beta_n}{k};\frac{\gamma}{k};\frac{kx_1}{kx_1-1},\frac{kx_1-kx_2}{kx_1-1},\dots,\frac{kx_1-kx_n}{kx_1-1})$$
$$=(1-kx_1)^{-\frac{\alpha}{k}}$$
$$\cdot F_{D,k}^{(n)}(\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\dots,\beta_n;\gamma;\frac{x_1}{kx_1-1},\frac{x_1-x_2}{kx_1-1},\dots,\frac{x_1-x_n}{kx_1-1})$$

şeklinde elde edilir. Diğerleri de aynı yöntemle ispatlanabilir. ■

**Teorem 4.18.** $F_{D,k}^{(n)}$ için $n$ tane dönüşüm formülü aşağıdadır.

$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_1)^{\frac{\gamma-\alpha-\beta_1}{k}}(1-kx_2)^{-\frac{\beta_2}{k}}\cdots(1-kx_n)^{-\frac{\beta_n}{k}}$$
$$\cdot F_{D,k}^{(n)}(\gamma-\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\dots,\beta_n;\gamma;x_1,\frac{x_2-x_1}{kx_2-1},\dots,\frac{x_n-x_1}{kx_n-1})$$
$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_1)^{-\frac{\beta_1}{k}}(1-kx_2)^{\frac{\gamma-\alpha-\beta_2}{k}}(1-kx_3)^{-\frac{\beta_3}{k}}\cdots(1-kx_n)^{\frac{-\beta_n}{k}}$$
$$\cdot F_{D,k}^{(n)}(\gamma-\alpha,\beta_1,\gamma-\beta_1-\cdots-\beta_n,\beta_3,\dots,\beta_n;\gamma;\frac{x_1-x_2}{kx_1-1},x_2,\frac{x_3-x_2}{kx_3-1},\dots,\frac{x_n-x_2}{kx_n-1})$$
$$\vdots$$
$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=(1-kx_1)^{-\frac{\beta_1}{k}}\cdots(1-kx_{n-1})^{-\frac{\beta_{n-1}}{k}}(1-x_n)^{\frac{\gamma-\alpha-\beta_n}{k}}$$
$$\cdot F_{D,k}^{(n)}(\gamma-\alpha,\beta_1,\dots,\beta_{n-1},\gamma-\beta_1-\cdots-\beta_n;\gamma;\frac{x_1-x_n}{kx_1-1},\dots,\frac{x_{n-1}-x_n}{kx_{n-1}-1},x_n)$$

**İspat**. Yukarıdaki dönüşüm formüllerinin ispatı için birinci yol, Teorem 3.15. in ispatına benzer şekilde,

$$t=\frac{1-u}{1-kux_1},\dots,t=\frac{1-u}{1-kux_n}$$

değişken değiştirmelerinin (4.48) integraline sırasıyla uygulanmasıdır.
Ayrıca bu dönüşüm formüllerinin ispatı için ikinci bir yol da (4.8) ilişkisinin, Teorem 3.15. deki dönüşüm formüllerinin ve (4.12) ilişkisinin kullanılmasına dayalı olan daha kolay ve kısa bir yöntemin kullanılmasıdır. Gerçekten de bu bilginin ışığı altında ilk dönüşüm formülü

$$F_{D,k}^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)$$
$$=F_{D}^{(n)}(\frac{\alpha}{k},\frac{\beta_1}{k},\dots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\dots,kx_n)$$
$$=(1-kx_1)^{\frac{\gamma-\alpha-\beta_1}{k}}(1-kx_2)^{-\frac{\beta_2}{k}}\cdots(1-kx_n)^{-\frac{\beta_n}{k}}$$
$$\cdot F_{D}^{(n)}(\frac{\gamma-\alpha}{k},\frac{\gamma-\beta_1-\cdots-\beta_n}{k},\frac{\beta_2}{k},\dots,\frac{\beta_n}{k};\frac{\gamma}{k};kx_1,\frac{kx_2-kx_1}{kx_2-1},\dots,\frac{kx_n-kx_1}{kx_n-1})$$
$$=(1-kx_1)^{\frac{\gamma-\alpha-\beta_1}{k}}(1-kx_2)^{-\frac{\beta_2}{k}}\cdots(1-kx_n)^{-\frac{\beta_n}{k}}$$
$$\cdot F_{D,k}^{(n)}(\gamma-\alpha,\gamma-\beta_1-\cdots-\beta_n,\beta_2,\dots,\beta_n;\gamma;x_1,\frac{x_2-x_1}{kx_2-1},\dots,\frac{x_n-x_1}{kx_n-1})$$

şeklinde bulunur. Diğerleri de aynı yöntemle ispatlanabilir. ■

**Sonuç 4.19.** Bu kısımda $F_{D,k}^{(n)}$ için verilen toplam dönüşüm sayısı $2n+1$ dir.

# 5. SONUÇ VE ÖNERİLER

Bu tezde $k$-Lauricella hipergeometrik fonksiyonları ve bunların konfluent formları, Pochhammer $k$-sembolü kullanılarak tanımlanmıştır. Ayrıca $k$-Lauricella fonksiyonlarının tek ve çok katlı Laplace tipi integralleri sunulmuştur. Son olarak da $F_{A,k}^{(n)}$ ve $F_{D,k}^{(n)}$ fonksiyonları için birer Euler tipi integral formülü ve bazı dönüşüm formülleri elde edilmiştir.

Bu çalışma boyunca $k$-Lauricella fonksiyonlarının bahsi geçen özelliklerinin ispatında klasikteki uzun ispatlara gerek duyulmadan daha kolay ve kısa bir yol takip edilebileceğine de vurgu yapılmıştır.

Dördüncü bölümdeki $k$-Lauricella fonksiyonları için elde edilen sonuçlar, $k = 1$ alınması durumunda klasik Lauricella fonksiyonları için üçüncü bölümdeki bilinen sonuçlar ile çakışır.

İlerleyen çalışmalarda, $k$-Lauricella fonksiyonları için yeni integral gösterimleri, yineleme formülleri, dönüşüm formülleri ve türev formülleri de elde edilebilir. Ayrıca bu çalışmanın, henüz çalışılmamış olan özel fonksiyonların $k$-genelleştirmeleri üzerine yapılacak olan araştırmalara da katkı sağlayacağı düşünülmektedir. Dahası bu tez Lauricella fonksiyonları ile ilgili çalışmalar için de bir rehber niteliğindedir.

## KAYNAKLAR


[1] Altın, A. (2011). *Uygulamalı Matematik*. Gazi Kitapevi, Ankara.

[2] Appell, P. (1882). Sur les fonctions hypergéométriques de deux variables. *Journal de Mathématiques Pures et Appliquées*, 8, 173–216.

[3] Brychkov, Yu. A., & Savischenko, N. V. (2021). Applications of multiple hypergeometric functions in communication theory: A new multiple-wave model with diffuse power. *Lobachevskii Journal of Mathematics* , 42(8), 1807–1828.

[4] Chamayou, J.F., & Wesolowski, J. (2009). Lauricella and Humbert functions through probabilistic tools. *Integral Transforms and Special Functions*, 20(7), 529–538.

[5] Çatak, C. (2023). *$k$-Horn hipergeometrik fonksiyonları $G_1^k$, $G_2^k$, $G_3^k$ ve özellikleri*. [Yüksek Lisans Tezi, Kırıkkale Üniversitesi Fen Bilimleri Enstitüsü, Kırıkkale].

[6] Çatak, C., Şahin, R., Olgun, A., & Yağcı, O. (2023). Some $k$-Horn hypergeometric functions and their properties. *Journal of New Results in Science*, 12(2), 97-107.

[7] Diaz, R., & Pariguan, E. (2007). On hypergeometric functions and Pochhammer $k$-symbol. *Divulgaciones Matematicas*, 15(2), 179–192.

[8] Exton, H. (1976). *Multiple hypergeometric functions and applications*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, London, Sydney and Toronto.

[9] Gürel Yılmaz, Ö., Aktaş, R., & Taşdelen, F. (2020). On some formulas for the $k$-analogue of Appell functions and generating relations via $k$-fractional derivative. *Fractal and Fractional*, 4, 48.

[10] Halıcı, S. (2022). *$k$-Srivastava hipergeometrik fonksiyonları*. [Yüksek Lisans Tezi, Kırşehir Ahi Evran Üniversitesi Fen Bilimleri Enstitüsü, Kırşehir].

[11] Halıcı, S., & Çetinkaya, A. (2024). $k$-Srivastava hypergeometric functions and their integral representations. *Miskolc Mathematical Notes*, 25(2), 693–706.

[12] Hasanov, A., & Srivastava, H. M. (2006). Some decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions. *Applied Mathematics Letters*, 19(2), 113-121.

[13] Hasanov, A., & Srivastava, H. M. (2007). Decomposition formulas associated with the Lauricella multivariable hypergeometric functions. *Computers and Mathematics with Applications*, 53(7), 1119-1128.

[14] Kıymaz, I. O., Çetinkaya, A., & Agarwal, P. (2017). A study on the $k$-generalizations of some known functions and fractional operators. *Journal of Inequalities and Special Functions*, 8(4), 31-41.

[15] Lauricella, G. (1893). Sulle funzioni ipergeometriche a più variabili. *Rendiconti del Circolo Matematico di Palermo*, 7, 111-158.

[16] Mubeen, S., & Habibullah, G. M. (2012). An integral representation of some $k$-hypergeometric functions. *International Mathematical Forum* , 7(4), 203-207.

[17] Mubeen, S., Iqbal, S., & Rahman, G. (2015). Contiguous function relations and an integral representation for Appell $k$-series $F_{1,k}$. *International Journal of Mathematical Research*, 4(2), 53-63.

[18] Mubeen, S., & Rehman, A. (2014). A note on $k$-Gamma function and Pochhammer $k$-symbol. *Journal of Informatics and Mathematical Sciences*, 6(2), 93-107.

[19] Srivastava, H. M. (1985). A class of generalised multiple hypergeometric series arising in physical and quantum chemical applications. *Journal of Physics A: Mathematical and General*, 18(5), L227.

[20] Srivastava, H. M., & Karlsson, P. W. (1985). *Multiple gaussian hypergeometric series*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.

[21] Srivastava, H. M., & Manocha, H. L. (1984). *A treatise on generating functions*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto.

**EKLER**

**EK-1**

Kongre Katılım Belgesi

**ÖZGEÇMİŞ**

| KİŞİSEL BİLGİLER | |
|---|---|
| Adı Soyadı | Ayşegül Kızılarslan |
| Uyruğu | T.C. |
| Orcid Numarası | 0009-0002-0471-6893 |

| EĞİTİM BİLGİLERİ | |
|---|---|
| **Lisans** | |
| Üniversite | Kırşehir Ahi Evran |
| Fakülte | Fen Edebiyat |
| Bölüm | Matematik |
| Mezuniyet Yılı | 2022 |
| **Yüksek Lisans** | |
| Üniversite | Kırşehir Ahi Evran |
| Enstitü | Fen Bilimleri |
| Anabilim Dalı | Matematik |
| Mezuniyet Yılı | 2025 |

| Tezden Üretilen Makaleler ve Bildiriler |
|---|
| Kızılarslan, A., & Çetinkaya, A. (2025). *$k$-Lauricella hypergeometric functions*. The ninth international conference on computational mathematics and engineering sciences, 17-19 May 2025, Diyarbakır-Türkiye. |