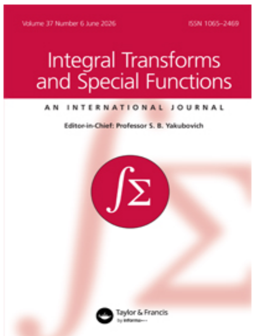

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Research Article

Commutators of maximal operator with Lipschitz functions on local Morrey-Lorentz spaces

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ABSTRACT
In this paper, we give necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and the commutators of the maximal operator $[b, M]$ in the local Morrey-Lorentz spaces $M^{\text{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n)$, whereby some new characterizations for certain subclasses of $\dot{\Lambda}_\beta(\mathbb{R}^n)$ spaces are obtained.

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1. Introduction

Let $0 < p, q \le \infty$ and let $0 \le \lambda \le 1$. We define the local Morrey-Lorentz spaces as the spaces of all measurable functions with finite quasinorm

$$\|f\|_{M^{\text{loc}}_{p,r;\lambda}} := \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{L_r(0,t)}.$$

The purpose of this paper is to give necessary and sufficient conditions for the boundedness of the maximal commutators M_b and the commutators of the maximal operator $[b, M]$ on the local Morrey-Lorentz spaces $M^{\text{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $\dot{\Lambda}_\beta(\mathbb{R}^n)$. Local Morrey-Lorentz spaces $M^{\text{loc}}_{p,r;\lambda}(\mathbb{R}^n)$, which are natural generalizations of the Lorentz spaces $L_{p,q}(\mathbb{R}^n) \equiv M^{\text{loc}}_{p,q;0}(\mathbb{R}^n)$ and the classical Lorentz spaces $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n) \equiv M^{\text{loc}}_{p,q;1}(\mathbb{R}^n)$, were introduced and their main properties were obtained in [1], see also [2–4]. For $0 < q \le p < \infty$ and $0 < \lambda \le \frac{q}{p}$, the local Morrey-Lorentz spaces $M^{\text{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ are equal to weak Lebesgue spaces $WL_{\frac{1}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$. In [1] the basic properties of $M^{\text{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ were given and the boundedness of the maximal operator was proved. Generally speaking, local Morrey spaces were also introduced separately by Guliyev [5] and Garcia-Cuerva and Herrero [6] (see also [7]).

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For $f \in L_1^{\rm loc}(\mathbb{R}^n)$, the maximal operator M is defined by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r)$ is the ball of radius r centred at $x \in \mathbb{R}^n$, ${}^{\complement}B(x,r)$ is its complement and $|B(x,r)| = v_n r^n$, $v_n = |B(0,1)|$, here $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$.

The maximal commutator generated by the operator M and $b \in L_1^{\rm loc}(\mathbb{R}^n)$ is defined by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|\, dy.$$

The commutators generated by the operator M and a suitable function b is defined by

$$[b, M]f(x) = b(x)Mf(x) - M(bf)(x).$$

Obviously, the operators M_b and $[b, M]$ essentially differ from each other since M_b is positive and sublinear and $[b, M]$ is neither positive nor sublinear. The operators M, $[b, M]$ and M_b play an important role in real and harmonic analysis and applications (see, for instance [8–10]).

The commutator estimates have many important applications, for example, in studying the regularity and boundedness of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and in characterizing certain function spaces (see, for instance [11–15]). The nonlinear commutator of maximal function $[b, M]$ can be used in tudying the product of a function in H_1 and a function in BMO (see [16] for instance). Note that, the boundedness of the operator M_b on $L_p(\mathbb{R}^n)$ spaces was proved by Garcia-Cuerva et al. [9]. In [8] by Bastero et al. studied the necessary and sufficient condition for the boundedness of $[b, M]$ on $L_p(\mathbb{R}^n)$ spaces.

In [17,18] was obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and commutators of maximal operator $[b, M]$ on the Lorentz spaces $L_{p,q}$, see also [19].

The structure of the paper is as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we obtain necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $M_{p,r;\lambda}^{\rm loc}(\mathbb{R}^n)$ spaces. In Section 4 we give necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on $M_{p,r;\lambda}^{\rm loc}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and some basic properties

We start with the definition of Lorentz spaces, see [20]. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L_p spaces, also they are appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on $\mathbb{R}^n$, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\},$$

where

$$d_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad s > 0.$$

The Lorentz space $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$, $0 < p,\ q \le \infty$ is the collection of all measurable functions f on $\mathbb{R}^n$ such the quantity

$$\|f\|_{L_{p,q}} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,\infty)} \tag{2.1}$$

is finite. Clearly $L_{p,p} \equiv L_p$ and $L_{p,\infty} \equiv WL_p$. The functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if either $1 \le q \le p$ or $p = q = \infty$.

Definition 2.1 ([1]): Let $0 < p, r \le \infty$ and $0 \le \lambda \le 1$. We denote by $M^{\rm loc}_{p,r;\lambda} \equiv M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$ the local Morrey-Lorentz space, the space of all measurable functions with finite quasinorm

$$\|f\|_{M^{\rm loc}_{p,r;\lambda}} := \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{L_q(0,r)}.$$

In the cases $\lambda < 0$ or $\lambda > 1$, we have $M^{\rm loc}_{p,r;\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Also $M^{\rm loc}_{p,r;0} = L_{p,r}$ and $M^{\rm loc}_{p,p;\lambda} \equiv M^{\rm loc}_{p;\lambda}$. In the limiting case $\lambda = 1$ the space $M^{\rm loc}_{p,r;1}$ is the classical Lorentz space $\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{r}}}$. For $0 < r \le p < \infty$ and $0 < \lambda \le \frac{r}{p}$, the local Morrey-Lorentz spaces $M^{\rm loc}_{p,r;\lambda}$ are equal to weak Lebesgue spaces $WL_{\frac{1}{p}-\frac{\lambda}{r}}$. Note that, in the case $q = \infty$ we have $M^{\rm loc}_{p,\infty;\lambda} = \Lambda_{\infty, t^{\frac{1}{p}}} = WL_p$.

We denote by $WM^{\rm loc}_{p,r;\lambda}$ the weak local Morrey-Lorentz space of all measurable functions with finite quasinorm

$$\|f\|_{WM^{\rm loc}_{p,r;\lambda}} := \sup_{t>0} t^{-\frac{\lambda}{r}} \|\tau^{\frac{1}{p}-\frac{1}{r}} f^*(\tau)\|_{WL_r(0,t)}.$$

Lemma 2.1 ([1]): *Let* $0 < r \le p < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\lambda}{r}$ *and* $0 < \lambda \le \frac{r}{p}$. *Then*

$$\left(\frac{r}{p}\right)^{-\frac{1}{r}} \|f\|_{WL_s} \le \|f\|_{M^{\rm loc}_{p,r;\lambda}} \le \lambda^{-\frac{1}{r}} \|f\|_{WL_s}.$$

In particular, $\|f\|_{WL_\infty} = \|f\|_{M^{\rm loc}_{\frac{r}{\lambda},r;\lambda}}$.

Lemma 2.2: *The inequalities*

$$(f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2)$$
$$(fg)^*(t_1+t_2) \le f^*(t_1) g^*(t_2)$$

holds for all $t_1, t_2 \ge 0$. *In particular, the inequalities*

$$(f+g)^*(t) \le f^*(t/2) + g^*(t/2)$$

$$(fg)^*(t) \le f^*(t/2)g^*(t/2)$$

holds for all $t \ge 0$.

Lemma 2.3 ([21, Lemma 2.4]): *Let* $0 < p, p_1, p_2, r, r_1, r_2 < \infty$, $0 \le \lambda \le 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. *Suppose that* $f \in M^{\rm loc}_{p_1,r_1;\lambda}(\mathbb{R}^n)$ *and* $f \in M^{\rm loc}_{p_2,r_2;\lambda}(\mathbb{R}^n)$. *Then*

$$\|fg\|_{M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)} \le 2^{\frac{1}{p}-\frac{1}{r}} \|f\|_{M^{\rm loc}_{p_1,r_1;\lambda}(\mathbb{R}^n)} \|g\|_{M^{\rm loc}_{p_2,r_2;\lambda}(\mathbb{R}^n)}.$$

Corollary 2.1 ([21, Corollary 2.2]): *Let* $0 \le \lambda < 1$, $1 < p, p', r, r' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{r} + \frac{1}{r'} = 1$. *Suppose that* $f \in M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$. *Then*

$$\|f\|_{L_1(B)} \le \|f\|_{M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)} |B|^{\frac{1}{p'}+\frac{\lambda}{r}}.$$

The following theorem is the boundedness of the maximal operator in local Morrey-Lorentz spaces $M^{\rm loc}_{p,r;\lambda}$.

Theorem 2.1 ([3, Theorem 1.1]): *Let* $1 \le r \le \infty$, $0 \le \lambda < 1$ *and* $\frac{r}{q+\lambda} \le p < \infty$.

(i) *If* $\frac{r}{r+\lambda} < p < \frac{r}{\lambda}$, *then the operator* M *is bounded in the local Morrey-Lorentz space* $M^{\rm loc}_{p,r;\lambda}$.
(ii) *If* $p = \frac{r}{r+\lambda}$, *then the operator* M *is bounded from* $M^{\rm loc}_{p,r;\lambda}$ *to the weak space* $WM^{\rm loc}_{p,r;\lambda}$.

The following theorem is the boundedness of the fractional maximal operator in local Morrey-Lorentz spaces $M^{\rm loc}_{p,r;\lambda}$.

Theorem 2.2 ([4, Theorem 3.1]): *Let* $0 \le \lambda < 1$, $0 \le \alpha < n$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$, $\frac{r}{r+\lambda} \le p \le (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$ *and* $f \in M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$.

(i) *If* $\frac{r}{r+\lambda} < p < (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$, *then the operator* M_α *is bounded from the space* $M^{\rm loc}_{p,r;\lambda}$ *to* $M^{\rm loc}_{q,s;\lambda}$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\alpha}{n}$.
(ii) *If* $p = \frac{r}{r+\lambda}$, *then the operator* M_α *is bounded from the space* $M^{\rm loc}_{p,r;\lambda}$ *to* $WM^{\rm loc}_{q,s;\lambda}$ *if and only if* $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$.
(iii) *If* $p = (\frac{\lambda}{r} + \frac{\alpha}{n})^{-1}$, *then the operator* M_α *is bounded from the space* $M^{\rm loc}_{p,r;\lambda}$ *to* $M^{\rm loc}_{q,s;\lambda}$.

3. $M^{\rm loc}_{p,r;\lambda}$-boundedness of the maximal commutator operator M_b

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator M_b from the space $M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$ to $M^{\rm loc}_{q,s;\lambda}(\mathbb{R}^n)$.

Definition 3.1: Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [22].

Lemma 3.1: *Let* $0 < \beta < 1$, *we have*

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| \, \mathrm{d}x,$$

where $f_B = \frac{1}{|B|} \int_B f(y) \, \mathrm{d}y$.

The following lemma is valid.

Lemma 3.2: *Let* $0 < \beta < 1$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, *then the following pointwise estimate holds:*

$$M_b f(x) \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_\beta f(x).$$

Proof: If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then

$$\begin{aligned} M_b(f)(x) &\approx \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)| \, \mathrm{d}y \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{B \ni x} |B|^{-1+\frac{\beta}{n}} \int_B |f(y)| \, \mathrm{d}y \\ &\approx \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_\beta f(x). \end{aligned}$$

■

In this section we obtain some new characteristics for the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$, which is one of the main theorems of this paper.

Theorem 3.1: *Let* $0 \leq \lambda < 1$, $0 < \beta < 1$, $1 \leq r \leq s \leq \infty$, $1 \leq q \leq \infty$, $\frac{r}{r+\lambda} \leq p \leq (\frac{\lambda}{r} + \frac{\beta}{n})^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\beta}{n}$. *The following assertions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
(ii) *The operator* M_b *is bounded from the space* $M^{\mathrm{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ *to* $M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)$.
(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)}} \leq C. \tag{3.1}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \le C. \tag{3.2}$$

Proof: (*i*) $\Rightarrow$ (*ii*). Suppose that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Combining Theorem 2.1 and Lemma 3.2, we get

$$\begin{aligned} \|M_b f\|_{M^{\rm loc}_{q,s;\lambda}} &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{M^{\rm loc}_{q,s;\lambda}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)}. \end{aligned}$$

(*ii*) $\Rightarrow$ (*i*). Assume that M_b is bounded from the space $M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$ to $M^{\rm loc}_{q,s;\lambda}(\mathbb{R}^n)$. Let $B = B(x, r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\|\chi_B\|_{M^{\rm loc}_{q,s;\lambda}} \approx |B|^{\frac{1}{q} - \frac{\lambda}{s}}. \tag{3.3}$$

On the other hand, for all $x \in B$ we have

$$\begin{aligned} |b(x) - b_B| &\le \frac{1}{|B|} \int_B |b(x) - b(y)|\, \mathrm{d}y \\ &= \frac{1}{|B|} \int_B |b(x) - b(y)| \chi_B(y)\, \mathrm{d}y \\ &\le M_b(\chi_B)(x). \end{aligned}$$

Since M_b is bounded from the space $M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$ to $M^{\rm loc}_{q,s;\lambda}(\mathbb{R}^n)$, then by (3.3) we obtain

$$\begin{aligned} \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - b_B)\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}} &\le \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|M_b(\chi_B)\|_{M^{\rm loc}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}} \\ &\lesssim \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|\chi_B\|_{M^{\rm loc}_{p,r;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}} \approx |B|^{-\frac{\beta}{n} + \frac{1}{p} - \frac{\lambda}{r} - \frac{1}{q} + \frac{\lambda}{s}} = 1, \end{aligned} \tag{3.4}$$

which implies that (3.1) holds since the ball $B \subset \mathbb{R}^n$ is arbitrary.

(*iii*) $\Rightarrow$ (*iv*). Assume that (3.1) holds, we will prove (3.2). For any fixed ball B, by Corollary 2.1, inequalities (3.1) and (3.3), it is easy to see

$$\begin{aligned} \frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_B|\, \mathrm{d}y &\lesssim \frac{1}{|B|^{1+\frac{\beta}{n}}} \|(b - b_B)\chi_B\|_{M^{\rm loc}_{q,s;\lambda}} |B|^{\frac{1}{q'} + \frac{\lambda}{s}}. \\ &\approx \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - b_B)\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}} \\ &\lesssim \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|\chi_B\|_{M^{\rm loc}_{p,r;\lambda}}}{\|\chi_B\|_{M^{\rm loc}_{q,s;\lambda}}} \approx |B|^{-\frac{\beta}{n} + \frac{1}{p} - \frac{\lambda}{r} - \frac{1}{q} + \frac{\lambda}{s}} = 1. \end{aligned}$$

$(iv) \Rightarrow (i)$. For any fixed ball B, we have

$$\begin{aligned}\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-b_B|\,\mathrm{d}y &= \frac{1}{|B|^{\frac{\beta}{n}}}\frac{\|(b-b_B)\chi_B\|_{L_1}}{|B|}\\ &\le \sup_B \frac{1}{|B|^{\frac{\beta}{n}}}\frac{\|(b-b_B)\chi_B\|_{L_1}}{|B|}\\ &\lesssim 1,\end{aligned}$$

which implies that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ by using Lemma 3.1. Thus the proof of the theorem is completed. ∎

In the case $r = p$, $s = q$ from Theorem 3.1 we get the following new corollary.

Corollary 3.1: *Let* $0 \le \lambda < 1$, $0 < \beta < 1$, $1 \le p \le \frac{\beta}{n(1-\lambda)}$, $1 \le q \le \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n(1-\lambda)}$. *The following assertions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
(ii) *The operator* M_b *is bounded from the space* $M^{\mathrm{loc}}_{p;\lambda}(\mathbb{R}^n)$ *to* $M^{\mathrm{loc}}_{q;\lambda}(\mathbb{R}^n)$.
(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}}\frac{\|(b(\cdot)-b_B)\chi_B\|_{M^{\mathrm{loc}}_{q;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q;\lambda}(\mathbb{R}^n)}} \le C.$$

(iv) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}}\frac{\|(b(\cdot)-b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \le C.$$

In the case $\lambda = 0$ from Theorem 3.1 we get the following corollary.

Corollary 3.2 ([19, Theorem 4.1]): *Let* $0 < \beta < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$, $1 \le p \le \frac{n}{\beta}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. *The following assertions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
(ii) *The operator* M_b *is bounded from* $L_{p,r}(\mathbb{R}^n)$ *to* $L_{q,s}(\mathbb{R}^n)$.
(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}}\frac{\|(b(\cdot)-b_B)\chi_B\|_{L_{q,s}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{q,s}(\mathbb{R}^n)}} \le C.$$

(iv) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}}\frac{\|(b(\cdot)-b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \le C.$$

4. $M^{loc}_{p,q;\lambda}$-boundedness of the commutator of maximal operator $[b,M]$

In this section we obtain necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b,M]$ from the space $M^{loc}_{p,r;\lambda}(\mathbb{R}^n)$ to $M^{loc}_{q,s;\lambda}(\mathbb{R}^n)$.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b,M]$ and M_b are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L_1^{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} |[b,M]f(x)| &= |b(x)Mf(x) - M(bf)(x)| \\ &= |M(b(x)f)(x) - M(bf)(x)| \le M(|b(x) - b|f)(x) = M_bf(x). \end{aligned}$$

Denote by M_Bf the local maximal function of f:

$$M_Bf(x) := \sup_{B' \ni x : B' \subset B} \frac{1}{|B'|} \int_{B'} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

Applying Theorem 3.1, we obtain the following result.

Theorem 4.1: *Let* $0 \le \lambda < 1$, $0 < \beta < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$, $\frac{r}{r+\lambda} \le p \le (\frac{\lambda}{r} + \frac{\beta}{n})^{-1}$ *and* $\frac{1}{p} - \frac{1}{q} = \lambda(\frac{1}{r} - \frac{1}{s}) + \frac{\beta}{n}$. *The following assertions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \ge 0$ *a.e. in* $\mathbb{R}^n$.

(ii) *The operator* $[b,M]$ *is bounded from the space* $M^{loc}_{p,r;\lambda}(\mathbb{R}^n)$ *to* $M^{loc}_{q,s;\lambda}(\mathbb{R}^n)$.

(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - M_B(b)(\cdot))\chi_B\|_{M^{loc}_{q,s;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{loc}_{q,s;\lambda}(\mathbb{R}^n)}} \le C. \tag{4.1}$$

(iv) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - M_B(b)(\cdot))\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \le C. \tag{4.2}$$

Proof: *(i)* $\Rightarrow$ *(ii)*. Suppose that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ a.e. in $\mathbb{R}^n$. Combining Lemma 3.2 and Theorem 3.1, we get

$$\begin{aligned} \|[b,M]f\|_{M^{loc}_{q,s;\lambda}(\mathbb{R}^n)} &\le \|M_bf\|_{M^{loc}_{q,s;\lambda}(\mathbb{R}^n)} \\ &\le \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{M^{loc}_{q,s;\lambda}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{M^{loc}_{p,r;\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we obtain that $[b, M]$ is bounded from the space $M^{\mathrm{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ to $M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)$.

(*ii*) $\Rightarrow$ (*iii*). Assume that $[b, M]$ is bounded from the space $M^{\mathrm{loc}}_{p,r;\lambda}(\mathbb{R}^n)$ to $M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)$. Let $B = B(x, r)$ be a fixed ball. Since

$$M(b\chi_B)\chi_B = M_B(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned}|M_B(b) - b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B| \\ &\le |M(b\chi_B) - bM(\chi_B)| = |[b, M]\chi_B|.\end{aligned}$$

Hence

$$\|M_B(b) - b\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)} \le \|[b, M]\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}(\mathbb{R}^n)}.$$

Thus we get

$$\begin{aligned}\frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - M_B(b))\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}} &\le \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|[b, M](\chi_B)\|_{M^{\mathrm{loc}}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}} \\ &\lesssim \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|\chi_B\|_{M^{\mathrm{loc}}_{p,r;\lambda}}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}} \approx |B|^{-\frac{\beta}{n}+\frac{1}{p}-\frac{\lambda}{r}-\frac{1}{q}+\frac{\lambda}{s}} = 1,\end{aligned}$$

which deduces that (*iii*).

(*iii*) $\Rightarrow$ (*iv*). Assume that (4.1) holds, then for any fixed ball B, by Corollary 2.1, we conclude that

$$\begin{aligned}\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - M_B(b)(x)|dx &\lesssim \frac{1}{|B|^{1+\frac{\beta}{n}}} \|(b - M_B(b))\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}} |B|^{\frac{1}{q'}+\frac{\lambda}{s}} \\ &\approx \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - M_B(b))\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}} \\ &\le \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|[b, M](\chi_B)\|_{M^{\mathrm{loc}}_{q,s;\lambda}}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}} \\ &\lesssim \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|\chi_B\|_{M^{\mathrm{loc}}_{p,r;\lambda}}}{\|\chi_B\|_{M^{\mathrm{loc}}_{q,s;\lambda}}} \approx |B|^{-\frac{\beta}{n}+\frac{1}{p}-\frac{\lambda}{r}-\frac{1}{q}+\frac{\lambda}{s}} = 1.\end{aligned}$$

(*iv*) $\Rightarrow$ (*i*). Assume that (4.2) holds, we will prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ a.e. in $\mathbb{R}^n$.

Denote by

$$E := \{x \in B : b(x) \le b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\,\mathrm{d}t = \int_F |b(t) - b_B|\,\mathrm{d}t,$$

in view of the inequality $b(x) \le b_B \le M_B(b)$, $x \in E$, we get

$$\begin{aligned}\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b - b_B| &= \frac{2}{|B|}\int_E |b - b_B| \\ &\le \frac{2}{|B|^{1+\frac{\beta}{n}}}\int_E |b - M_B(b)| \\ &\le \frac{2}{|B|^{1+\frac{\beta}{n}}}\int_B |b - M_B(b)| \lesssim c.\end{aligned}$$

Consequently, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. In order to show that $b \ge 0$ a.e. in $\mathbb{R}^n$, it suffices to show $b^- = 0$ a.e. in $\mathbb{R}^n$. Observe that $0 \le b^-(y) \le |b(y)| \le M_B(b)(y)$ for $y \in B$, therefore, for any $y \in B$, there holds

$$0 \le b^-(y) = |b(y)| - b^+(y) \le M_B(b)(y) - b^+(y) + b^-(y) = M_B(b)(y) - b(y).$$

Then for any ball B, we have

$$\begin{aligned}\frac{1}{|B|}\int_B b^-(y)\,\mathrm{d}y &\le \frac{1}{|B|}\int_B (M_B(b)(y) - b(y))\,\mathrm{d}y \\ &= \frac{1}{|B|}\int_B |b(y) - M_B(b)(y)|\,\mathrm{d}y \\ &\le \frac{|B|^{\frac{\beta}{n}}}{|B|^{1+\frac{\beta}{n}}}\int_B |b(y) - M_B(b)(y)|\,\mathrm{d}y \le C|B|^{\frac{\beta}{n}}.\end{aligned}$$

Let $|B| \to 0$ with $x \in B$. Lebesgue's differentiation theorem assures that

$$0 \le b^-(x) = \lim_{|B|\to 0}\frac{1}{|B|}\int_B b^-(y)\,\mathrm{d}y = 0.$$

Thus $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ a.e. in $\mathbb{R}^n$. Thus the proof of the theorem is completed. ■

In the case $r = p$, $s = q$ from Theorem 4.1 we get the following new corollary.

Corollary 4.1: *Let $0 \le \lambda < 1$, $0 < \beta < 1$, $1 \le p \le \frac{\beta}{n(1-\lambda)}$, $1 \le q \le \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n(1-\lambda)}$. The following assertions are equivalent:*

(i) *$b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ a.e. in $\mathbb{R}^n$.*
(ii) *The operator M_b is bounded from the space $M^{\mathrm{loc}}_{p;\lambda}(\mathbb{R}^n)$ to $M^{\mathrm{loc}}_{q;\lambda}(\mathbb{R}^n)$.*

(iii) *There exist a constant C > 0 such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{M^{\rm loc}_{q;\lambda}(\mathbb{R}^n)}}{\|\chi_B\|_{M^{\rm loc}_{q;\lambda}(\mathbb{R}^n)}} \le C.$$

(iv) *There exist a constant C > 0 such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \le C.$$

In the case $\lambda = 0$ from Theorem 4.1 we get the following corollary.

Corollary 4.2 ([18]): *Let* $0 < \beta < 1$, $1 \le r \le s \le \infty$, $1 \le q \le \infty$, $1 \le p \le \frac{n}{\beta}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. *The following assertions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \ge 0$ *a.e. in* $\mathbb{R}^n$.
(ii) *The operator* $[b, M]$ *is bounded from the space* $M^{\rm loc}_{p,r;\lambda}(\mathbb{R}^n)$ *to* $M^{\rm loc}_{q,s;\lambda}(\mathbb{R}^n)$.
(iii) *There exist a constant C > 0 such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - M_B(b)(\cdot))\chi_B\|_{L_{q,s}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{q,s}(\mathbb{R}^n)}} \le C.$$

(iv) *There exist a constant C > 0 such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - M_B(b)(\cdot))\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \le C.$$

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