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THE SOLUTIONS OF FRACTIONAL PDE'S WITH ALTERNATIVE VARIATIONAL ITERATION METHOD

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Abstract: In [Z. Odibat, A study on the convergence of variational iteration method, *Math. and Comput. Model.*, **51**, No-s: 9-10 (2010), 1181-1192], Odibat proposed an alternative approach of variational iteration method (VIM). In this paper, we applied alternative VIM to solve some class of fractional partial differential equations (PDE). We also compare the results with homotopy analysis method (HAM).

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1. Introduction

Differential equations involving derivatives of non-integer order have proved to be valuable tools to the modeling of many physical phenomena. Several methods have been used to solve fractional PDE's, such as Adomian decomposition method, homotopy analysis method, variational iteration method and so on.

VIM is one of the powerful methods by which the exact and appropriate analytical solutions for nonlinear equations can be obtained. In [5], Odibat presented an alternative approach of VIM and studied the convergence of the method for nonlinear differential equations.

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In this work, we applied alternative VIM to solve fractional KdV, K(2, 2), mKdV and BBM-Burger's equations. We have to point out that, these equations are successfully solved by HAM in [2]. We also compare the results of VIM with HAM.

2. Basic Definitions

In this section, we give some definitions and properties of the fractional calculus.

Definition 1. A real function $f(t), t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n, if and only if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (\alpha > 0), \\ J^0 f(t) &= f(t). \end{aligned}$$

Properties of the operator (J^α) can be found in [4], [7], [3], we mention only the following:

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [1].

Definition 3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

for $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$.

Also, we need here two of its basic properties.

Lemma 4. *If* $n-1<\alpha\leq n$, $n\in\mathbb{N}$ *and* $f\in C_{\mu}^{n}$, $\mu\geq -1$ *then*

$$(D^{\alpha}J^{\alpha})f(t)=f(t)$$

and

$$(J^{\alpha}D^{\alpha})f(t)=f(t)-\sum_{k=0}^{n-1}f^{k}(0^{+})\frac{t^{k}}{k!},\quad t>0.$$

Definition 5. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha>0$, is defined as

$$D_t^{\alpha}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\left[(t-s)^{n-\alpha-1}\frac{\partial^n}{\partial s^n}u(x,s)\right]ds, & n-1<\alpha<n,\\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha=n\in\mathbb{N}.\end{cases}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult [8].

3. Alternative Approach of VIM

Consider the fractional differential equation,

$$\mathfrak{L}u(x,t)=g(x,t),\qquad t>0,\tag{1}$$

where

$$\mathfrak{L}u(x,t)=D_t^{\alpha}u(x,t)+Lu(x,t)+Nu(x,t),$$

$m-1<\alpha\leq m$, $m\in\mathbb{N}$, L is a linear and N is a nonlinear operator, $g(x,t)$ is a known analytic function and D_t^{α} is the Caputo fractional derivative of order α. The initial conditions for problem (1) are given in terms of the field variables with their integer order as,

$$\frac{\partial^k u(x,0)}{\partial t^k}=f_k(x),\quad k=0,1,\dots,m-1,$$

where f_k's are real functions.

In [6], a general framework of the variational iteration method is presented for analytical treatment of fractional partial differential equations. Following this framework, the solution $u(x,t)=\lim_{k\to\infty}u_n(x,t)$ of problem (1) can be derived from the iteration formula,

$$\begin{cases} u_{k+1}(x,t) = u_k(x,t) - J^{\alpha}\left[\mathfrak{L}u(x,t) - g(x,t)\right], \quad 0 < \alpha \leq 1, \\ u_{k+1}(x,t) = u_k(x,t) - (\alpha-1)J^{\alpha}\left[\mathfrak{L}u(x,t) - g(x,t)\right], \quad 1 < \alpha \leq 2, \\ \vdots \\ u_{k+1}(x,t) = u_k(x,t) - \frac{(\alpha-1)\cdots(\alpha-m+1)}{(m-1)!}J^{\alpha}\left[\mathfrak{L}u(x,t) - g(x,t)\right], \ m-1 < \alpha \leq m, \end{cases}$$

where J^{α} is Riemann-Liouville fractional integral operator of order $\alpha > 0$, which is defined as in Definiton 2, and Caputo fractional derivative, D_t^{α} of order $\alpha > 0$ is defined as Definition 3.

According to alternative approach of VIM in [5], one can show that the variational iteration solution $u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$ obtained using the iteration formula,

$$\begin{cases} v_0 = \displaystyle\sum_{k=0}^{m-1} \frac{f_k(x)}{k!} t^k, \\ v_{k+1} = -\dfrac{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}{(m-1)!} J^{\alpha}\left(\mathfrak{L}\left[\displaystyle\sum_{i=0}^{k} v_i(x,t)\right] - g(x,t)\right), \end{cases} \tag{2}$$

converges to a solution of (1) if $\exists\ 0 < \gamma < 1$ such that $\|v_{k+1}\| \leq \gamma \|v_k\|$, $\forall k \in \mathbb{N} \cup \{0\}$.

4. Test Problems

In this section, we present several examples of [2] to illustrate the applicability of alternative approach of VIM to solve fractional partial differential equations. The reader can compare the results with the HAM solutions in [2] and the exact solution for $\alpha = 1$ in [9]. Throughout this section, we use J^{α} as Riemann-Liouville fractional integral operator, D_t^{α} as Caputo fractional differential operator of order $0 < \alpha \leq 1$ with respect to time variable t and $v(x,t) = \sum_{i=0}^{k} v_i(x,t)$.

Example 6. We first consider the fractional KdV equation

$$D_t^{\alpha} u - 3(u^2)_x + u_{xxx} = 0, \tag{3}$$

with initial conditions $u(x,0) = 6x$, see [2], [9].

From (2), the iteration formula for problem (3) can be constructed as,

$$\begin{cases} v_0 = 6x, \\ v_{k+1} = -J^{\alpha}\left\{D_t^{\alpha} v(x,t) - 3(v^2)_x(x,t) + v_{xxx}(x,t)\right\}. \end{cases}$$

Using the above iteration formula, we obtain the following successive approxi-

mations,

$$\left\{\begin{aligned}
&v_1=\frac{6^3x}{\Gamma(\alpha+1)}t^{\alpha},\\
&v_2=\frac{2\cdot 6^5x}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{6^7x}{\Gamma(\alpha+1)^2}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha},\\
&v_3=\frac{4\cdot 6^7x}{\Gamma(3\alpha+1)}t^{3\alpha}\\
&+\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\frac{2\cdot 6^9x}{\Gamma(\alpha+1)^2}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}+\frac{4\cdot 6^9x}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]t^{4\alpha}\\
&+\frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}\left[\frac{4\cdot 6^{11}x}{\Gamma(2\alpha+1)^2}+\frac{2\cdot 6^{11}x}{\Gamma(\alpha+1)^3}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right]t^{5\alpha}\\
&+\frac{4\cdot 6^{13}x}{\Gamma(\alpha+1)^2}\frac{\Gamma(5\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}t^{6\alpha}+\frac{6^{15}x}{\Gamma(\alpha+1)^4}\frac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)}t^{7\alpha},\\
&\quad\vdots
\end{aligned}\right.$$

For $\alpha=1$, we get the approximate solution as

$$u(x,t)\approx\sum_{k=0}^{3}v_k(x,t)=6x\left(1+36t+36^2t^2+36^3t^3+\text{small terms}\right)$$

after three steps and the exact solution of (3) is

$$u(x,t)=\sum_{k=0}^{\infty}v_k(x,t)=6x\left(1+36t+36^2t^2+36^3t^3+\ldots\right)=\frac{6x}{1-36t}.$$

Example 7. We next consider the following fractional K(2, 2) equation

$$D_t^{\alpha}u+(u^2)_x+(u^2)_{xxx}=0, \tag{4}$$

with initial conditions $u(x,0)=x$, see [2], [9].

The iteration formula for problem (4) can be constructed as,

$$\left\{\begin{aligned}
&v_0=x,\\
&v_{k+1}=-J^{\alpha}\left\{D_t^{\alpha}v(x,t)+(v^2)_x(x,t)+(v^2)_{xxx}(x,t)\right\}.
\end{aligned}\right.$$

Using the above iteration formula, we obtain the following successive approxi-

mations,

$$\begin{cases} v_1 = -\dfrac{2x}{\Gamma(\alpha+1)}t^{\alpha}, \\ v_2 = \dfrac{2^3x}{\Gamma(2\alpha+1)}t^{2\alpha} - \dfrac{2^3x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}, \\ v_3 = -\dfrac{2^5x}{\Gamma(3\alpha+1)}t^{3\alpha} \\ +\dfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\dfrac{2^5x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} + \dfrac{2^6x}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]t^{4\alpha} \\ -\dfrac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}\left[\dfrac{2^7x}{\Gamma(2\alpha+1)^2} + \dfrac{2^6x}{\Gamma(\alpha+1)^3}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right]t^{5\alpha} \\ +\dfrac{2^8x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(5\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}t^{6\alpha} \\ -\dfrac{2^7x}{\Gamma(\alpha+1)^4}\dfrac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)}t^{7\alpha}, \\ \quad\vdots \end{cases}$$

For $\alpha = 1$, we get the approximate solution as

$$u(x,t) \approx \sum_{k=0}^{3} v_k(x,t) = x\left(1 - 2t + 4t^2 - 8t^3 + \text{small terms}\right)$$

after three steps and the exact solution of (4) is

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t) = x\left(1 - 2t + 4t^2 - 8t^3 + \ldots\right) = \frac{x}{1+2t}.$$

Example 8. We consider the modified fractional KdV (mKdV) equation

$$D_t^{\alpha}u + \frac{1}{2}(u^2)_x - u_{xx} = 0, \tag{5}$$

with initial conditions $u(x,0) = x$, [2], [9].

The iteration formula for problem (5) can be constructed as,

$$\begin{cases} v_0 = x, \\ v_{k+1} = -J^{\alpha}\left\{D_t^{\alpha}v(x,t) + \frac{1}{2}(v^2)_x(x,t) - v_{xx}(x,t)\right\}. \end{cases}$$

Using the above iteration formula, we obtain the following successive approxi-

mations,

$$\begin{cases} v_1 = -\dfrac{x}{\Gamma(\alpha+1)}t^{\alpha}, \\ v_2 = \dfrac{2x}{\Gamma(2\alpha+1)}t^{2\alpha} - \dfrac{x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}, \\ v_3 = -\dfrac{2^2x}{\Gamma(3\alpha+1)}t^{3\alpha} \\ +\dfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\dfrac{2x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} + \dfrac{2^2x}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]t^{4\alpha} \\ -\dfrac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}\left[\dfrac{2^2x}{\Gamma(2\alpha+1)^2} + \dfrac{2x}{\Gamma(\alpha+1)^3}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right]t^{5\alpha} \\ +\dfrac{2^2x}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(5\alpha+1)}{\Gamma(3\alpha+1)\Gamma(6\alpha+1)}t^{6\alpha} \\ -\dfrac{x}{\Gamma(\alpha+1)^4}\dfrac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)}{\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)}t^{7\alpha}, \\ \quad\vdots \end{cases}$$

For $\alpha = 1$, we get the approximate solution as

$$u(x,t) \approx \sum_{k=0}^{3} v_k(x,t) = x\left(1 - t + t^2 - t^3 + \text{small terms}\right)$$

after three steps and the exact solution of (5) is

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t) = x\left(1 - t + t^2 - t^3 + \ldots\right) = \frac{x}{1+t}.$$

Example 9. We consider the fractional BBM-Burger's equation [2]

$$D_t^{\alpha}u + u_x + uu_x - au_{xx} - bu_{xxt} = 0,$$

with initial conditions $u(x,0) = x^2, t > 0$.

Using the above iteration formula, we obtain the following successive approximations,

$$\begin{cases} v_1 = \dfrac{-2x^3 - 2x + 2a}{\Gamma(\alpha+1)}t^{\alpha}, \\ v_2 = -\dfrac{12bx}{\Gamma(2\alpha)}t^{2\alpha-1} + \dfrac{10x^4 + 12x^2 - 16ax + 2}{\Gamma(2\alpha+1)}t^{2\alpha} \\ -\dfrac{12x^5 + 16x^3 - 12ax^2 + 4x - 4a}{\Gamma(\alpha+1)^2}\dfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}, \\ \quad\vdots \end{cases}$$

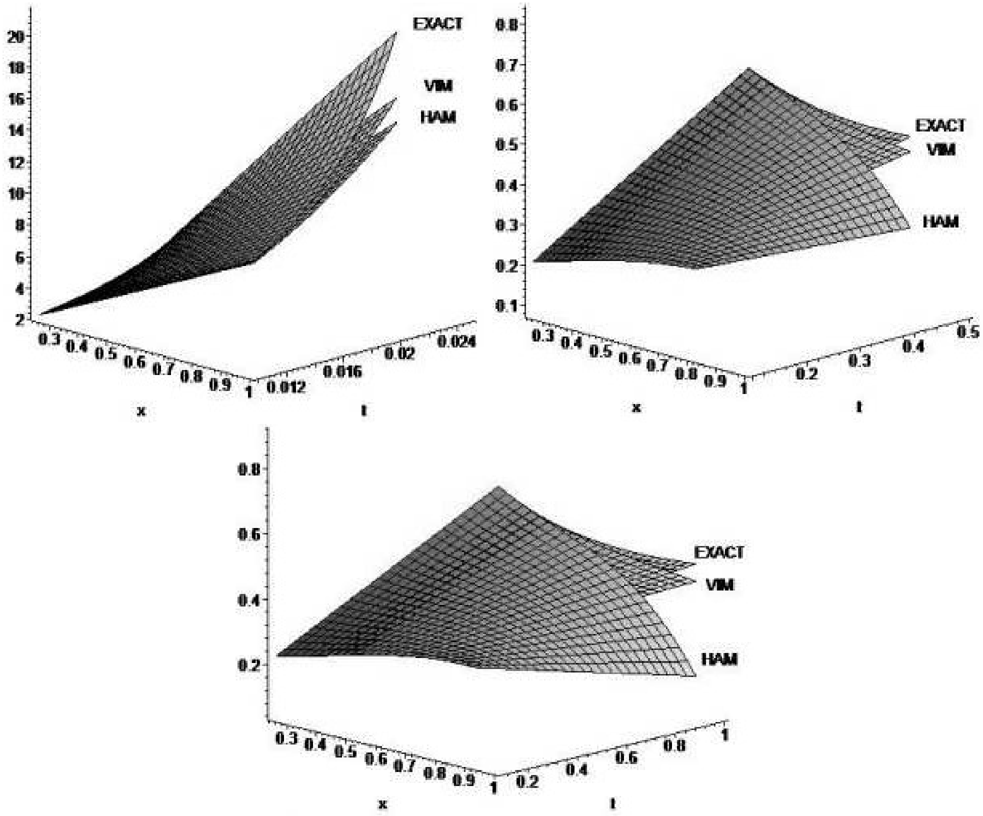

Figure 1: 3D surface graphics of the exact, VIM and HAM solutions of (3), (4) and (5) respectively

For $\alpha = a = b = 1$, after two steps, we get the approximate solution as

$$u(x,t) \approx \sum_{k=0}^{3} v_k(x,t) = x^2 + (-2x^3 - 14x + 2)t + (5x^4 + 6x^2 - 8x + 1)t^2 - \frac{1}{3}(12x^5 + 16x^3 - 12x^2 + 4x - 4)t^3.$$

The surface graphics of the exact, HAM and alternative VIM solutions of (3), (4) and (5) are plotted in Figure 1. The reader can see that the approximate solutions obtained by alternative VIM are closer than HAM to their exact solutions. Also Tables 1-3 present some computational results for the same examples.

	t=0.01			t=0.02		
x	VIM	HAM	Exact	VIM	HAM	Exact
0.25	2.3245850	2.3043840	2.3437500	4.3085637	3.9174720	5.3571428
0.50	4.6491698	4.6087680	4.6875000	8.6171274	7.8349440	10.714286
0.75	6.9737548	6.9131520	7.0312500	12.925692	11.752416	16.071428
1.00	9.2983396	9.2175360	9.3750000	17.234255	15.669888	21.428572

Table 1: Approximate solutions of equation (3) using three steps with $\alpha = 1$

	t=0.1			t=0.4		
x	VIM	HAM	Exact	VIM	HAM	Exact
0.25	0.2082417	0.2080000	0.2083333	0.1294096	0.8200000	0.1388889
0.50	0.4164835	0.4160000	0.4166667	0.2588192	0.1640000	0.2777778
0.75	0.6247251	0.6240000	0.6250000	0.3882287	0.2460000	0.4166667
1.00	0.8329669	0.8320000	0.8333333	0.5176383	0.3280000	0.5555556

Table 2: Approximate solutions of equation (4) using three steps with $\alpha = 1$

	t=0.1			t=0.9		
x	VIM	HAM	Exact	VIM	HAM	Exact
0.25	0.2272659	0.2272500	0.2272727	0.1182567	0.0452500	0.1315789
0.50	0.4545317	0.4545000	0.4545455	0.2365135	0.0905000	0.2631579
0.75	0.6817976	0.6817500	0.6818182	0.3547703	0.1357500	0.3947368
1.00	0.9090635	0.9090000	0.9090909	0.4730270	0.1810000	0.5263158

Table 3: Approximate solutions of equation (5) using three steps with $\alpha = 1$

5. Conclusion

In this paper, the alternative variational iteration method has been successfully employed to obtain the approximate solution of some class of fractional differential equations. The main advantage of the method is its fast convergence to the solution. The numerical results obtained here, conform to its high degree of accuracy. Moreover, it avoids the volume of calculations required by the homotopy analysis method. This work shows that alternative variational iteration method is a very powerful and efficient tool for solving the fractional partial differential equations.

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