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A NEW GENERALIZATION ON ABSOLUTE MATRIX SUMMABILITY FACTORS OF FOURIER SERIES

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Abstract. In this paper, two theorems for $|A, p_n; \delta|_k$ summability which generalize recent theorems on $|A, p_n|_k$ summability of Fourier series have been proved. This work also reveals many factor theorems for other summability methods.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v \tag{1.1}$$

where

$$A_n^\alpha = O(n^\alpha),\ \alpha > -1,\ A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [9])

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_n^\alpha|^k < \infty. \tag{1.3}$$

If we take $\delta = 0$, then we get $|C, \alpha|_k$ summability (see [8],[11]).
Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.4}$$

The sequence–to–sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.5}$$

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defines the sequence (σ_n) of the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [10]).
The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\Delta \sigma_{n-1}|^k < \infty, \tag{1.6}$$

where

$$\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

In the special case, when $p_n = 1$ for all values of n, (*resp.* $\delta = 0$), $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ (*resp.* $|\bar{N}, p_n|_k$) summability (see [1]). Also, if we take $\delta = 0$, $k = 1$ and $p_n = \frac{1}{n+1}$ (*resp.* $k = 1$ and $\delta = 0$) summability $|\bar{N}, p_n; \delta|_k$ becomes $|R, log n, 1|$ (*resp.* $|\bar{N}, p_n|$) summability.
Let f be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without loss of generality we may assume that the constant term in the Fourier series of f is zero, so that

$$\int_{-\pi}^{\pi} f(t) dt = 0,$$

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t), \tag{1.7}$$

where (a_n) and (b_n) denote the Fourier coefficients.
We write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \quad \text{and} \quad \varphi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \varphi(u) du, \quad (\alpha > 0). \tag{1.8}$$

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{1.9}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.10}$$

Then A defines the sequence–to–sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.11}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series–to–sequence and series–to–series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv}$$
$$= \sum_{i=0}^{n} a_i \bar{a}_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} a_v. \quad (1.12)$$

Since $\quad \bar{a}_{n-1,n} = \sum_{i=1}^{n-1} a_{n-1,i} = 0,$

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \bar{a}_{nv}a_v - \sum_{v=0}^{n-1} \bar{a}_{n-1,v}a_v$$
$$= \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v})a_v + \bar{a}_{n-1,n}a_n = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \quad (1.13)$$

The series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [13])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\bar{\Delta}A_n(s)|^k < \infty, \quad (1.14)$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s). \quad (1.15)$$

In the special case, if we set $\delta = 0$, then we obtain $|A, p_n|_k$ summability (see [15]). If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\delta = 0$ and $p_n = 1$ for all n, $|A, p_n; \delta|_k$ summability is the same as $|A|_k$ summability (see [16]) and if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A|_k$ summability is the same as $|R, p_n|_k$ summability (see [3]). Finally, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n; \delta|_k$ summability.

2. The known results

Several authors have studied on absolute summability factors of Fourier series (see [5]-[7], [14], [17]-[19]). Recently, in [17], Yıldız has generalized two theorems of Bor (see [2]) for the $|A, p_n|_k$ summability method in the following form;

Theorem 2.1. *Let there be sequences* (p_n) *and* (λ_n) *such that*

$$P_n = O(np_n) \quad (2.1)$$

$$P_n \Delta p_n = O(p_n p_{n+1}) \quad (2.2)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n|^k < \infty \quad (2.3)$$

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty, \quad (2.4)$$

and let $A=(a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{no}=1,\ n=0,1,..., \tag{2.5}$$

$$a_{n-1,v}\geq a_{nv},\ \ for\ n\geq v+1, \tag{2.6}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right), \tag{2.7}$$

$$\hat{a}_{n,v+1}=O(v|\Delta_v(\hat{a}_{nv})|). \tag{2.8}$$

If $\varphi_1(t)$ is of bounded variation in $(0,\pi)$, then the series $\sum C_n(t)\frac{\lambda_n P_n}{np_n}$ is summable $|A,p_n|_k,\ k\geq 1$.

Theorem 2.2. *If the conditions* (2.1)–(2.8) *of Theorem 2.1 and*

$$B_n\equiv\sum_{v=1}^{n} va_v=O(n),\quad n\to\infty, \tag{2.9}$$

are satisfied, then the series $\sum a_n\frac{\lambda_n P_n}{np_n}$ *is summable* $|A,p_n|_k,\ k\geq 1$.

3. The main results

The aim of this paper is to generalize Theorem 2.1 and Theorem 2.2 for the $|A,p_n;\delta|_k$ summability method. Now, we shall prove the following theorems.

Theorem 3.1. *Let* $\varphi_1(t)$ *be of bounded variation in* $(0,\pi)$. *If the conditions* (2.1)-(2.2) *of Theorem 2.1 are satisfied, and if* $A=(a_{nv})$ *is a positive normal matrix satisfying conditions* (2.5)–(2.8) *of Theorem 2.1, and also*

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|=O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\}\quad as\quad m\to\infty, \tag{3.1}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|=O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\}\quad as\quad m\to\infty. \tag{3.2}$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|\lambda_n|^k}{n}=O(1)\quad as\quad m\to\infty, \tag{3.3}$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta\lambda_n|=O(1)\quad as\quad m\to\infty, \tag{3.4}$$

then the series $\sum C_n(t)\frac{\lambda_n P_n}{np_n}$ *is summable* $|A,p_n;\delta|_k$ *for* $k\geq 1$ *and* $0\leq\delta<1/k$.

Remark. It should be noted that if we take $\delta=0$ in this theorem, then the conditions (3.3) and (3.4) are satisfied by a hypotheses of the Theorem 2.1. Also in this case condition (3.1) and (3.2) are obvious.

Theorem 3.2. *If the conditions* (2.1)–(2.2) *and* (2.5)–(2.8) *of Theorem 2.1, and condition* (2.9) *of Theorem 2.2 and also conditions* (3.1)–(3.4) *of Theorem 3.1 are satisfied, then the series* $\sum a_n\frac{\lambda_n P_n}{np_n}$ *is summable* $|A,p_n;\delta|_k$ *for* $k\geq 1$ *and* $0\leq\delta<1/k$.

We need the following lemmas for the proof of Theorem 3.1 and Theorem 3.2.

Lemma 3.3. [12] *If $\varphi_1(t)$ is of bounded variation in $(0,\pi)$ for any $x \in (-\pi,\pi)$, then*

$$\sum_{v=1}^{n} vC_v(x) = O(n) \quad as \quad n \to \infty. \tag{3.5}$$

Lemma 3.4. [2] *If the sequence (p_n) such that conditions (2.1) and (2.2) of Theorem 2.1 are satisfied, then*

$$\Delta\left\{\frac{P_n}{p_n n^2}\right\} = O\left(\frac{1}{n^2}\right). \tag{3.6}$$

4. Proof of Theorem 3.2

Proof. Let (V_n) denotes the A-transform of the series $\sum a_n P_n \lambda_n (np_n)^{-1}$. Then, by (1.12) and (1.13)

$$V_n = \sum_{v=0}^{n} \bar{a}_{nv} a_v P_v \lambda_v (vp_v)^{-1}$$

$$\bar{\Delta} V_n = V_n - V_{n-1} = \sum_{v=0}^{n} \hat{a}_{nv} a_v P_v \lambda_v (vp_v)^{-1}.$$

and since $\hat{a}_{n0} = \bar{a}_{n0} - \bar{a}_{n-1,0} = 0$ we get

$$\bar{\Delta} V_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v (vp_v)^{-1}.$$

Applying Abel's transformation to this sum and by condition (2.9), we have

$$\begin{aligned}
\bar{\Delta} V_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v (vp_v)^{-1} = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \left\{ \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{a}_{nv}) P_v \lambda_v}{v^2 p_v} + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{v^2 p_v} \Delta\lambda_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta_v \left(\frac{P_v}{v^2 p_v}\right) \right\} \sum_{r=1}^{v} r a_r \\
&+ \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} B_n + \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{a}_{nv}) P_v \lambda_v}{v^2 p_v} B_v \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta_v \left(\frac{P_v}{v^2 p_v}\right) B_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{v^2 p_v} \Delta\lambda_v B_v \\
&= V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.2, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |V_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{4.1}$$

Firstly, since $a_{nn} = O\left(\frac{p_n}{P_n}\right)$, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,1}|^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\frac{a_{nn}\lambda_n P_n}{n^2 p_n}B_n\right|^k\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\lambda_n|^k|B_n|^k\frac{1}{n^{2k}} = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|\lambda_n|^k}{n^k}n^{k-1}\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|\lambda_n|^k}{n} = O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by conditions (2.9) and (3.3). Now, applying Hölder's inequality, we have that

$$\begin{aligned}&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,2}|^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv})\frac{P_v\lambda_v}{v^2 p_v}B_v\right|^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\left(\frac{P_v}{v^2p_v}\right)^k|\lambda_v|^k|B_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\end{aligned}$$

By the definitions of $\bar{A}$ and $\hat{A}$ matrices of series–to–sequence and series–to–series transformations

$$\begin{aligned}\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1}\\ &= \sum_{i=v}^{n}a_{ni} - \sum_{i=v}^{n-1}a_{n-1,i} - \sum_{i=v+1}^{n}a_{ni} + \sum_{i=v+1}^{n-1}a_{n-1,i} = a_{nv} - a_{n-1,v},\end{aligned} \tag{4.2}$$

and since $a_{n-1,v} \geq a_{nv}$, $\bar{a}_{n0} = 1$ we get

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}(a_{n-1,v} - a_{nv}) = 1 - a_{n-1,0} - 1 + a_{n0} + a_{nn} \leq a_{nn}. \tag{4.3}$$

By using conditions $(2.7), (3.1), (4.2), (4.3)$ we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\left(\frac{P_v}{v^2p_v}\right)^k|\lambda_v|^k|B_v|^k\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^k|B_v|^k\left(\frac{P_v}{v^2p_v}\right)^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|,\end{aligned}$$

and using condition (2.9) and (3.3), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,2}|^k &= O(1)\sum_{v=1}^{m}|\lambda_v|^k|B_v|^k\left(\frac{P_v}{v^2p_v}\right)^k\left(\frac{P_v}{p_v}\right)^{\delta k-1}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|\lambda_v|^k}{v} = O(1) \quad \text{as} \quad m\to\infty.\end{aligned}$$

On the other hand, since $\Delta\left\{\frac{P_v}{v^2 p_v}\right\} = O\left(\frac{1}{v^2}\right)$ by Lemma 3.4, we obtain

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,3}|^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\Delta_v\left(\frac{P_v}{v^2 p_v}\right)B_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{1}{v}\right\}\times\left\{\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,v+1}|}{v}\right\}^{k-1}$$

By using $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$, $\hat{a}_{n,v+1} = \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1}$ and virtue of the hypotheses of Theorem 2.1 we get, $\hat{a}_{n,v+1} = O(v|\Delta_v(\hat{a}_{nv})|)$. By using this condition, and condition (4.3), we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{1}{v}\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|,$$

and conditions (3.2) and (3.3), we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,3}|^k = O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|\lambda_{v+1}|^k}{v} = O(1) \quad \text{as} \quad m\to\infty$$

Finally, using condition (2.1) we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|V_{n,4}|^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}P_v}{v^2 p_v}\Delta\lambda_v B_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}}{v}\Delta\lambda_v B_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\frac{|B_v|^k}{v^k}\right\}\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right\}^{k-1}$$

By using conditions (2.5), (2.6), we get, for $1\le v\le n-1$,

$$\hat{a}_{n,v+1} = \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} = \sum_{i=v+1}^{n}a_{ni} - \sum_{i=v+1}^{n-1}a_{n-1,i}$$

$$= 1 - \sum_{i=0}^{v}a_{ni} - 1 + \sum_{i=0}^{v}a_{n-1,i} = \sum_{i=0}^{v}(a_{n-1,i}-a_{ni}) \le \sum_{i=0}^{n-1}(a_{n-1,i}-a_{ni}) = 1 - 1 + a_{nn} = a_{nn}$$

where

$$\sum_{i=0}^{v}(a_{n-1,i}-a_{ni}) \ge 0 \tag{4.4}$$

By condition (2.5), $\bar{a}_{n0} = 1$, and

$$\hat{a}_{n,v+1} = \sum_{i=0}^{v}(a_{n-1,i}-a_{ni}) \le \sum_{i=0}^{n-1}(a_{n-1,i}-a_{ni}) = 1 - 1 + a_{nn} = a_{nn},$$

we have

$$\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v| = O(a_{nn}). \tag{4.5}$$

By using condition (2.7), (2.9), (3.2), (3.4), (4.4), (4.5), also we have

$$\begin{aligned}\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |V_{n,4}|^k &= O(1)\sum_{v=1}^{m} |\Delta\lambda_v| \frac{|B_v|^k}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}|\hat{a}_{n,v+1}| \\ &= O(1)\sum_{v=1}^{m} |\Delta\lambda_v| \frac{|B_v|^k}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| \\ &= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} |\Delta\lambda_v| = O(1) \quad \text{as} \quad m \to \infty.\end{aligned}$$

This completes the proof of Theorem 3.2. □

5. Proof of Theorem 3.1

Theorem 3.1 is a direct consequence of Theorem 3.2 and Lemma 3.3.

Corollary 5.1. *If we take $\delta = 0$ in Theorem 3.1 and Theorem 3.2, then we get two theorems dealing with $|A, p_n|_k$ summability factors of Fourier series.*

Corollary 5.2. *If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n (resp. $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$) in Theorem 3.1 and Theorem 3.2, then we get new results concerning the $|C, 1; \delta|_k$ (resp. $|\bar{N}, p_n|_k$) summability method of Fourier series.*

Corollary 5.3. *If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1 and Theorem 3.2, then we obtain a new theorem dealing with $|\bar{N}, p_n; \delta|_k$ summability methods of Fourier series.*

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