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GENERALIZED LOCAL MIXED MORREY ESTIMATES FOR LINEAR ELLIPTIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT

We consider the 2*b*-order linear elliptic systems

$$L(x,D)u := \sum_{|\alpha|=2b} A_\alpha(x) D^\alpha u(x) = f(x)$$

in the generalized local mixed Morrey spaces $M^{\{x_0\}}_{\vec{p},\varphi}(R^n)$ and generalized mixed Morrey spaces $M_{\vec{p},\varphi}(R^n)$, where the principal coefficients $A\alpha$ are functions with vanishing mean oscillation. We obtain local regularity results for the strong solutions to $L(x, D)$ in the spaces $M^{\{x_0\}}_{\vec{p},\varphi}(R^n)$ and $M_{\vec{p},\varphi}(R^n)$. Solutions to the linear elliptic systems with discontinuous coefficients are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. We can apply this local regularity results in generalized local mixed Morrey spaces to study the regularity in generalized local mixed Morrey spaces of of the Navier-Stokes equations. Solutions to the Navier-Stokes equations are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. In particular, solutions of the Navier-Stokes equations often include turbulence, which remains one of the greatest unsolved problems in physics, despite its immense importance in science and engineering. The possibilities nowadays to exploit supercomputers and highly developed numerical methods for nonlinear partial differential equations allow us to determine even the general solutions to the Navier-Stokes equations. However the difficulties become greater with increasing Reynolds number. This has to do with the particular structure of the solutions at high Reynolds numbers. Note that in the limiting case of high Reynolds numbers, most of these exact solutions have a boundary-layer character.

Keywords: elliptic systems; generalized local Morrey space; vanishing mean oscillation.

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1. Introduction and main result

The theory of regularity has been a central line of research in elliptic PDE since the mid-20th century. Regularity is a topic of the mathematical study of partial differential equations PDE about the integrability and differentiability of weak solutions. It is often difficult to construct a classical solution satisfying the PDE in regular sense. Its goal is to determine whether all solutions of smooth elliptic variational PDEs are smooth.

In this paper we deal with local regularity results and derive interior a priori estimates in the generalized local mixed Morrey spaces $M^{\{x_0\}}_{\vec{p},\varphi}(R^n)$ and generalized mixed Morrey spaces $M_{\vec{p},\phi}(R^n)$ for the strong solutions of the uniformly elliptic system

$$L(x,D)u := \sum_{|\alpha|=2b} A_\alpha(x) D^\alpha u(x) = f(x)$$

where the principal coefficients A_α are functions with vanishing mean oscillation (VMO). In [1-4] a Calderon-Zygmund type theory in the framework of the classical Morrey spaces $L_{p,\lambda}$ has been developed for linear and quasi-linear elliptic and parabolic systems with VMO principal coefficients. On the other hand, in the recent years an exhaustive Calderon-Zygmund theory has been elaborated both for elliptic and parabolic equations/systems in divergence form with VMO-coefficients in the framework of the various types of Morrey spaces (cf. [3, 4, 5-15]). Last generalizations of the spaces allows finer control on the local oscillation properties of a function near its singular points and that is why regularity results in generalized Morrey spaces $M_{p,\varphi}$ of solutions to PDEs with discontinuous coefficients are of great importance in the applications to differential geometry, stochastic control, nonlinear optimization, adaptive discontinuous Galerkin FEMs, etc.

Guliyev [16], Mizuhara and Nakai [17-19] introduced generalized Morrey spaces $M_{p,\varphi}$ independently (see, also

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[16, 20]). In general, Garcia-Cuerva and Herrero [21] and Guliyev [17] also introduced independently (see also [22]) the local Morrey spaces. Note that, in [17] Guliyev introduced the local Morrey-type spaces including the generalized local Morrey spaces and studied integral operators in these spaces. In 2019, Nogayama [23] considered a new Morrey space, with the L_p norm replaced by the mixed Lebesgue norm $L_{\vec{p}}(R^n)$, which is call mixed Morrey spaces.

For $x\in R^n$ and $r>0$ let $B(x,r)$ denote the open ball centered at x of radius r. Let $\Omega\subset R^n$, $n\geq 3$ be a bounded domain, we denote by $|\Omega|$ the Lebesgue measure of Ω and $\Omega(x,r)=\Omega\cap B(x,r)$. $S^{n-1}=\left\{\xi\in R^n:\left|\xi\right|=1\right\}$ is the unit sphere in R^n. We first recall the definition of mixed Lebesgue space defined in [24]. Let the letter $\vec{p}$ denote n-tuples of the numbers in $(0,\infty]$, $n\geq 1$, $\vec{p}=\left(p_1,p_{2,\cdots},p_n\right)\in\left(0,\infty\right]^n$, $1\leq\vec{p}<\infty$ means $1\leq p_i<\infty$ for each i. For $1\leq\vec{p}<\infty$, we denote $\vec{p}'=\left(p_1',_{\cdots,}p_n'\right)$, where p_i' satisfies $1/p_i+1/p_i'=1$. The mixed Lebesgue norm $\left\|\cdot\right\|_{L_{\vec{p}}}$ or $\left\|\cdot\right\|_{L_{(p_1,\cdots,p_n)}}$ is defined by

$$\left\|f\right\|_{L_{\vec{p}}}\equiv\left\|f\right\|_{L_{(p_1,\cdots,p_n)}}=$$

$$=\left(\int_R\cdots\left(\int_R\left(\int_R\left|f\left(x_1,x_2,\ldots,x_n\right)\right|^{p_1}dx_1\right)^{\frac{p_2}{p_1}}dx_2\right)^{\frac{p_3}{p_2}}\ldots dx_n\right)^{\frac{1}{p_n}},$$

where $f:R^n\to R$ is a measurable function. If $p_j=\infty$ for some $j=1,\ldots,n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L_{\vec{p}}\left(R^n\right)=L_{(p_1,\ldots,p_n)}$ to be the set of all $f\in L_0(R^n)$ with $\left\|f\right\|_{L_{\vec{p}}}<\infty$, where $L_0(R^n)$ denotes the set of measureable functions on R^n.

In the following we give the definitions of the generalized local mixed Morrey space $M_{\vec{p},\varphi}^{\{x_0\}}(\Omega)$ and the generalized mixed Morrey space $M_{\vec{p},\varphi}(\Omega)$, and their weak versions.

Definition 1.1. ([16, 25]) Let $x_0\in\Omega$, $1\leq\vec{p}<\infty$ and $\varphi:\Omega\times(0,\infty)\to(0,\infty)$ be a measurable function, where $\Omega\subset R^n$, $n\geq 3$ is a bounded domain. We denote by $M_{\vec{p},\varphi}^{\{x_0\}}(\Omega)$ and $M_{\vec{p},\varphi}(\Omega)$ the generalized local mixed Morrey space, the generalized mixed Morrey space respectively, the spaces of all functions $f\in L_{\vec{p}}(\Omega)$ with finite norms

$$\left\|f\right\|_{M_{\vec{p},\varphi}^{\{x_0\}}(\Omega)}=\sup_{r>0}\varphi\left(x_0,r\right)^{-1}\left|B\left(x_0,r\right)\right|^{-\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}\left\|f\right\|_{L_{\vec{p}}(\Omega(x_0,r))}$$

and

$$\left\|f\right\|_{M_{\vec{p},\varphi}(\Omega)}=\sup_{x\in\Omega,r>0}\varphi\left(x,r\right)^{-1}\left|B\left(x_0,r\right)\right|^{-\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}\left\|f\right\|_{L_{\vec{p}}(\Omega(x,r))}$$

respectively, where $L_{\vec{p}}\left(\Omega(x,r)\right)$-norm is given by

$$\left\|f\right\|_{L_{\vec{p}}(\Omega(x,r))}=\left(\int_{\Omega(x,r)}\left|f\left(y\right)\right|^p dy\right)^{\frac{1}{\vec{p}}}.$$

Further, $WM_{\vec{p},\varphi}^{\{x_0\}}\left(\Omega\right)$ and $WM_{\vec{p},\varphi}\left(\Omega\right)$ are the weak local generalized mixed Morrey space and weak generalized Morrey space respectively of all functions $f\in WL_p(\Omega)$ for which,

$$\left\|f\right\|_{WM_{\vec{p},\varphi}^{\{x_0\}}(\Omega)}=\sup_{r>0}\varphi\left(x_0,r\right)^{-1}\left|B\left(x_0,r\right)\right|^{-\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}\left\|f\right\|_{WL_p(\Omega(x_0,r))}<\infty$$

and

$$f_{WM_{p,\varphi}(\Omega)}=\sup_{x\in\Omega,r>0}\varphi\left(x,r\right)^{-1}\left|B\left(x,r\right)\right|^{-\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}f_{L_p(\Omega(x,r))}<\infty$$

respectively, where $WL_p(\Omega(x,r))$-norm is given by

$$\left\|f\right\|_{WL_p(\Omega(x,r))}=\sup_{t>0}t\cdot\left|\left\{y\in\Omega\left(x,r\right):\left|f\left(y\right)\right|>t\right\}\right|^{1/p}.$$

For $u=\left(u_1,\ldots,u_m\right):\Omega\to R^m$ we write $\left|u\right|^2=\sum_{j\leq m}\left|u_j\right|^2$. For any function f and any domain D with $f:D\to R$ we write

$$f_D=\frac{1}{\left|D\right|}\int_D f\left(y\right)dy,\ \left\|f\right\|_{p,D}^p=\left\|f\right\|_{L_p(D)}^p=\int_D\left|f\left(y\right)\right|^p dy.$$

Definition 1.2. ([14]) The generalized local mixed Sobolev-Morrey space $WM_{\vec{p},\varphi}^{2b,(x_0)}(\Omega)$ and generalized mixed Sobolev-Morrey space $WM_{\vec{p},\varphi}^{2b}(\Omega)$ consists of all functions $u\in L_{\vec{p}}(\Omega)$ with generalized derivatives $D^\alpha u$, $\left|\alpha\right|\leq 2b$, belonging to $M_{\vec{p},\varphi}^{(x_0)}(\Omega)$ and $M_{\vec{p},\varphi}(\Omega)$, and endowed with the norm

$$\left\|u\right\|_{W_{\vec{p},\varphi}^{2b,\{x_0\}}(\Omega)}=\sum_{s=0}^{2b}\sum_{\left|\alpha\right|=s}\left\|D^\alpha u\right\|_{M_{\vec{p},\phi}^{\{x_0\}}(\Omega)}$$

and

$$\left\|u\right\|_{W_{\vec{p},\varphi}^{2b}(\Omega)}=\sum_{s=0}^{2b}\sum_{\left|\alpha\right|=s}\left\|D^\alpha u\right\|_{M_{\vec{p},\phi}(\Omega)},$$

respectively.

For $u\in L_{\vec{p}}(\Omega;R^m)$ we write $\left\|u\right\|_{\vec{p},\Omega}$ instead of $\left\|u\right\|_{L\vec{p}\left(\Omega;R^m\right)}$. Analogously, $u=(u_1,\ldots,u_m)\in WM_{\vec{p},\varphi}^{2b,\{x_0\}}(\Omega;R^m)$ and $u=(u_1,\ldots,u_m)\in WM_{\vec{p},\varphi}^{2b}(\Omega;R^m)$ means $u_k\in W_{\vec{p},\varphi}^{2b,(x_0)}(\Omega)$ and $u_k\in W_{\vec{p},\varphi}^{2b}(\Omega)$, and the norm $\left\|u\right\|_{WM_{\vec{p},\phi}^{2b,(x_0)}\left(\Omega;R^m\right)}$ and $\left\|u\right\|_{WM_{\vec{p},\phi}^{2b}\left(\Omega;R^m\right)}$ is given by $\sum_{k=1}^m\left\|u_k\right\|_{W_{\vec{p},\phi}^{2b,\{x_0\}}(\Omega)}$ and $\sum_{k=1}^m\left\|u_k\right\|_{WM_{\vec{p},\phi}^{2b}(\Omega)}$, respectively.

We will use also a localized version $W_{\vec{p},\varphi,loc}^{2b,\{x_0\}}(\Omega;R^m)$ and $W_{\vec{p},\varphi,loc}^{2b}(\Omega;R^m)$ of $W_{\vec{p},\varphi}^{2b,\{x_0\}}(\Omega;R^m)$ and $W_{\vec{p},\varphi}^{2b}(\Omega;R^m)$, respectively, consisting of all functions u that belong to $u\in W_{\vec{p},\varphi}^{2b,\{x_0\}}(\Omega';R^m)$ and $u\in W_{\vec{p},\varphi}^{2b}(\Omega';R^m)$ for each $\Omega'\subset\Omega$.

Remark 1.1. (1) If $\varphi(x,r)=r^{\frac{\lambda-n}{n}\sum_{i=1}^n\frac{1}{p_i}}$ with $0<\lambda<n$, then $M_{\vec{p},\varphi}^{(x_0)}(\Omega)=L_{\vec{p},\lambda}^{\{x_0\}}(\Omega)$ is the local mixed Morrey space, $M_{\vec{p},\varphi}(\Omega)=L_{\vec{p},\lambda}(\Omega)$ is the mixed Morrey space and $WM_{\vec{p},\varphi}^{\{x_0\}}(\Omega)=WL_{\vec{p},\lambda}^{\{x_0\}}(\Omega)$ is the weak local mixed Morrey space, $WM_{\vec{p},\varphi}(\Omega)=WL_{\vec{p},\lambda}(\Omega)$ is the weak mixed Morrey space.

(2) If $\varphi(x,r)=\left|B(x,r)\right|^{-\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}$, then $M_{\vec{p},\varphi}^{\{x_0\}}(\Omega)=M_{\vec{p},\varphi}(\Omega)=$ $=L_{\vec{p},\lambda}(R^n)$ is the mixed Lebesgue space and $WM_{\vec{p},\varphi}^{\{x_0\}}(\Omega)=$ $=WM_{\vec{p},\varphi}(\Omega)=WL_{\vec{p}}(\Omega)$ is the weak mixed Lebesgue space.

(3) It is clear that if $\varphi(x,r)=r^{\frac{\lambda-n}{n}\sum_{i=1}^n\frac{1}{p_i}}$ with $0<\lambda<n$, then $M_{\vec{p},\varphi}$ gives rise to the classical Morrey space $L_{\vec{p},\lambda}$, while $L_{\vec{p},1}\equiv L_{\vec{p}}$ and $W_{\vec{p},1}^{2b}$ reduces to the classical parabolic Sobolev space $W_{\vec{p}}^{2b}$ (cf. [4]) when $\varphi\equiv 1$.

We give our main result in the following.

Theorem 1.1. Assume that (3.4) is provided and $A_\alpha=\{a_\alpha^{jk}\}\in VMO(\Omega)\cap L^\infty(\Omega)$. Let $x_0\in\Omega, u\in W_{\vec{p},loc}^{2b,\{x_0\}}(\Omega;R^m)$ be a strong solution to (3.1) with $\vec{p}\in(1,\infty)$. Let $f\in M_{\vec{p},\varphi}^{\{x_0\}}(\Omega;R^m)$ such that W is satisfy (2.3). Then $u\in W_{\vec{p},\varphi,loc}^{2b,\{x_0\}}(\Omega;R^m)$ and

$$\left\|u\right\|_{W_{\vec{p},\varphi}^{2b,\{x_0\}}\left(\Omega',R^m\right)}\leq C\left(\left\|f\right\|_{M_{\vec{p},\varphi}^{\{x_0\}}(\Omega)}+\left\|u\right\|_{M_{\vec{p},\varphi}^{\{x_0\}}(\Omega^*)}\right)\qquad(1.1)$$

for all $\Omega'\subset\Omega''\subset\Omega$, where the constant C depends on n, $\vec{p}$, m, b, ω, $\left\|A_\alpha\right\|_{\infty;\Omega}$, the VMO-moduli γ_{A_α} of the coefficients and on $dist(\Omega',\partial\Omega'')$.

Corollary 1.1. Assume that (3.4) is provided and $A_\alpha=\{a_\alpha^{jk}\}\in VMO(\Omega)\cap L^\infty(\Omega)$. Let $u\in W^{2b}_{\vec{p},loc}(\Omega;R^m)$ be a strong solution to (3.1) with $\vec{p}\in(1,\infty)$. Let $f\in M_{\vec{p},\varphi}(\Omega;R^m)$ such that ω is satisfy (2.4). Then $u\in W^{2b}_{\vec{p},\varphi,loc}(\Omega;R^m)$ and

$$\|u\|_{W^{2b}_{\vec{p},\varphi}(\Omega';R^m)}\le C\left(\|f\|_{M_{\vec{p},\varphi}(\Omega)}+\|u\|_{M_{\vec{p},\varphi}(\Omega'')}\right)$$

for all $\Omega'\subset\Omega''\subset\Omega$, where the constant C depends on n, $\vec{p}$, m, b, ω, $\|A_\alpha\|_{\infty;\Omega}$, the VMO-moduli γ_{A_α} of the coefficients and on $dist(\Omega',\partial\Omega'')$.

From Corollary 1.1 it is easy to obtain the following result.

Corollary 1.2. Assume that (3.4) is provided and $A_\alpha=\{a_\alpha^{jk}\}\in VMO(\Omega)\cap L^\infty(\Omega)$. Let $u\in W^{2b}_{\vec{p},loc}(\Omega;R^m)$ be a strong solution to (3.1) with $p\in(1,\infty)$. Let $f\in M_{\vec{p},\varphi}(\Omega;R^m)$ such that ω is satisfy (2.2). Then $u\in W^{2b}_{\vec{p},\varphi,loc}(\Omega;R^m)$ and

$$\|u\|_{W^{2b}_{\vec{p},\varphi}(\Omega';R^m)}\le C\left(\|f\|_{W_{\vec{p},\varphi}(\Omega)}+\|u\|_{M_{\vec{p},\varphi}(\Omega^*)}\right)$$

for all $\Omega'\subset\Omega''\subset\Omega$, where the constant C depends on n, $\vec{p}$, m, b, ω, $\|A_\alpha\|_{\infty;\Omega}$, the VMO-moduli γ_{A_α} of the coefficients and on $dist(\Omega',\partial\Omega'')$.

Remark 1.2. Corollary 1.2 was proved in [14]. Note that condition (2.4) in Corollary 2.1 is weaker than condition (2.2) in Proposition 2.2. Indeed, if condition (2.2) holds, then

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{ess\inf_{t<s<\infty}\varphi(x,s)s^{\sum_{i=1}^n\frac{1}{p_i}}}{t^{\sum_{i=1}^n\frac{1}{p_i}+1}}dt\le\int_r^\infty\left(1+\ln\frac{t}{r}\right)\varphi(x,t)\frac{dt}{t},\quad r\in(0,\infty),$$

so condition (2.4) holds.

On the other hand the function

$$\varphi(x,r)=r^{\beta-\sum_{i=1}^n\frac{1}{p_i}}\left|\sin\left(\max\left\{1,\frac{\pi}{r}\right\}\right)\right|,\ 0<\beta<\sum_{i=1}^n\frac{1}{p_i}$$

satisfy condition (2.4) in the case $r\in(0,1)$, $ess\inf_{r<s<\infty}\varphi(x,s)s^{\sum_{i=1}^n\frac{1}{p_i}}=0$ and,

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{ess\inf_{t<s<\infty}\varphi(x,s)s^{\sum_{i=1}^n\frac{1}{p_i}}}{t^{\sum_{i=1}^n\frac{1}{p_i}+1}}dt\approx\begin{cases}o & if\ \ r\in(0,1)\\ r^{\beta-\sum_{i=1}^n\frac{1}{p_i}} & if\ \ r\in(1,\infty)\end{cases}\lesssim\varphi(x,r),\ r\in(0,\infty),$$

but do not satisfy condition (2.2).

2. Preliminaries

Mixed Lebesgue spaces are widely used in the study of partial differential equations. It was shown that mixed Lebesgue spaces share similar properties with the ordinary Lebesgue spaces. Many classical operators are bounded on these spaces. The following analogue of the Holder's inequality for $L_{\vec{p}}(\Omega)$ is well known (see, for example, [26]).

Theorem 2.1. Let $\Omega\subset R^n$ be a measurable set, $1\le\vec{p}\le\infty$ and $\frac{1}{\vec{p}}+\frac{1}{\vec{p}'}=1$. Then for any $f\in L_{\vec{p}}(\Omega)$ and $g\in L_{\vec{p}'}(\Omega)$, the following inequality is valid

$$\int_\Omega|f(x)g(x)|dx\le\|f\|_{L_{\vec{p}}(\Omega)}\|g\|_{L_{\vec{p}'}(\Omega)}.$$

The following property is valid.

Lemma 2.1. Let $0<\vec{p}<\infty$ and B be a balls in R^n. Then

$$\|\chi_B\|_{L_{\vec{p}}}=\|\chi_B\|_{WL_{\vec{p}}}=|B|^{\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}.$$

From Theorem 2.1 and Lemma 2.1 we get the following estimate.

Lemma 2.2. For $1\le\vec{p}<\infty$ and for the balls $B=B(x,r)$ the following inequality is valid:

$$\int_B|f(y)|dy\le|B|^{\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}}\|f\|_{L_{\vec{p}}(B)}.$$

The following lemma shows the Lebesgue differential theorem in mixed norm Lebesgue spaces as follows.

Lemma 2.3. [26, Lemma 2.4] Let $f\in L_1^{loc}(R^n)$ and $0<\vec{p}<\infty$, then

$$\lim_{r\to0}\|\chi_{B(x,r)}\|_{L_{\vec{p}}}^{-1}\|f\|_{L_{\vec{p}}(B(x,r))}=|f(x)|\ \text{ a.e. } x\in R^n.$$

Definition 2.1. Let $K(x,\xi):R^n\times(R^n\setminus\{0\})\to R$ be a variable Calderon-Zygmund kernel, i.e. :

1. for each fixed $x\in R^n$, $K(x)$ is a Calderon-Zygmund kernel

(a) $K(x;\cdot)\in C^\infty(R^n\setminus\{0\})$

(b) $K(x;\mu\xi)=\mu^{-n}K(x,\xi),\ \ \forall\mu>0$

(c) $\int_{S^{n-1}}K(x;\xi)d\sigma_\xi=0,\ \ \int_{S^{n-1}}|K(x,\xi)|d\sigma_\xi<\infty$;

2. for every multi-index $\beta:\sup_{\xi\in S^{n-1}}|D_\xi^\beta K(x;\xi)|\le C(\beta)$ independently of x, where S^{n-1} is the unit sphere in R^n.

Given a function $f\in L_1(\Omega)$, define the singular integral operator

$$Tf(x):=p.v.\int_{R^n}K(x;x-y)f(y)dy$$

and its commutator with multiplication by a function $a\in L_\infty(R^n)$ as

$$C[a,f](x):=p.v.\int_{R^n}K(x;x-y)[a(y)-a(x)]f(y)dy=T(af)(x)-a(x)Tf(x).$$

In [27] the authors show boundedness of the maximal operator in $L_{p,\lambda}(R^n)$ that allows them to prove continuity of fractional and classical Calderon-Zygmund operators in Morrey spaces. Thus the boundedness of the singular integrals implies regularity of the solutions in the corresponding spaces. In [18] Mizuhara studied the boundedness in generalized Morrey spaces $M_{p,\varphi}(R^n)$ of some classical integral operators. Nakai later extends the results of Chiarenza and Frasca in $M_{p,\varphi}(R^n)$ by imposing certain integral and doubling conditions on φ (see [19]). Taking a weight $\omega(x,r)=\varphi(x\,r)r^n$ the conditions of Mizuhara-Nakai become

$$\int_r^\infty\frac{\varphi(x,s)}{s}ds\le C\varphi(x,r),C^{-1}\le\frac{\varphi(x,t)}{\varphi(x,r)}\le C,\forall0<r\le t\le2r\quad(2.1)$$

where the constants do not depend on t, r and $x\in R^n$.

In a series of papers, Guliyev studied the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators, such as Calderon-Zygmund, Riesz, etc. (see [16, 17]). Later this result is extended on spaces with weaker condition on the weight pair (φ_1,φ_2) (see [28]). For more recent results on boundedness and continuity of singular integral operators in generalized

Morrey and new function spaces and their applications in the differential equations theory, see [3-6, 12-14, 25] and the references there in.

Proposition 2.1. Let $1 \le \vec{p} < \infty$ and (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \frac{\varphi_1(x,t)}{t} dt \le C\varphi_2(x,r)$$

where C does not depend on x and r. Then the Calderon-Zygmund integral operators T are bounded from $M_{\vec{p},\varphi_1}(R^n)$ to $M_{\vec{p},\varphi_2}(R^n)$ for $p>1$ and from $M_{1,\varphi_1}(R^n)$ to the weak space $WM_{1,\varphi_1}(R^n)$.

Proposition 2.2. Let $1 < \vec{p} < \infty$, $a \in BMO$, φ be a weight satisfying: there exist positive constant $C>1$ such that

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) \frac{\varphi(x,s)}{s} ds \le C\phi(x,r),$$

$$C^{-1} \le \frac{\varphi(x,t)}{\varphi(x,r)} \le C, \forall 0 < r \le t \le 2r \qquad (2.2)$$

where the constants do not depend on t, r and $x \in \Omega$.

Then there exists a positive constant $C = C(\vec{p}, \varphi, K)$ such that

$$\|Tf\|_{M_{\vec{p},\varphi}(\Omega)} \le C\|f\|_{M_{\vec{p},\varphi}(\Omega)}$$

and

$$\|C[a,f]\|_{M_{\vec{p},\varphi}(\Omega)} \le C\|a\|_* \|f\|_{M_{\vec{p},\varphi}(\Omega)}.$$

In addition, if $a \in VMO$, then for each $\varepsilon>1$ there exists $r_0 = r_0(\varepsilon, \gamma_a) > 0$ such that for any $r \in (0, r_0)$ and any ball B_r the following inequality holds:

$$\|C[a,f]\|_{M_{\vec{p},\varphi}(B_r)} \le C\varepsilon \|f\|_{M_{\vec{p},\varphi}(\Omega)}$$

The L_p and $M_{p,\varphi}$ -boundedness of the operators T and $C[a,\cdot]$ have been obtained in [11, 29] and [12, 16, 17], respectively. For the sake of completeness, we summarize these results here.

Proposition 2.3. Let $x_0 \in \Omega$, $1 < \vec{p} < \infty$, $a \in BMO$, φ be a weight satisfying

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{ess\inf_{t<s<\infty} \varphi(x_0,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\frac{n}{p}+1}} dt \le C\varphi(x_0,r) \qquad (2.3)$$

where C does not depend on r. Then there exists a positive constant $C=C(p,\varphi,K)$ such that

$$\|Tf\|_{M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)} \le C\|f\|_{M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)}$$

and

$$\|C[a,f]\|_{M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)} \le C\|a\|_* \|f\|_{M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)}.$$

In addition, if $a \in VMO$, then for each $\varepsilon>0$ there exists $r_0 = r_0(\varepsilon, \gamma_a) > 0$ such that for any $r \in (0, r_0)$ and B_r any ball the following inequality holds:

$$\|C[a,f]\|_{M^{\{x_0\}}_{\vec{p},\varphi}(B_r)} \le C\varepsilon \|f\|_{M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)}.$$

Corollary 2.1. Let $1 < \vec{p} < \infty$, $a \in BMO$, φ be a weight satisfying

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{ess\inf_{t<s<\infty} \varphi(x,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\frac{n}{p}+1}} dt \le C\varphi(x,r) \qquad (2.4)$$

where C does not depend on x and r. Then there exists a positive constant $C=C(p,\varphi,K)$ such that

$$\|Tf\|_{M_{\vec{p},\varphi}(\Omega)} \le C\|f\|_{M_{\vec{p},\varphi}(\Omega)}$$

and

$$\|C[a,f]\|_{M_{\vec{p},\varphi}(\Omega)} \le C\|a\|_* \|f\|_{M_{\vec{p},\varphi}(\Omega)}.$$

In addition, if $a \in VMO$, then for each $\varepsilon>0$ there exists $r_0 = r_0(\varepsilon, \gamma_a) > 0$ such that for any $r \in (0, r_0)$ and any ball B_r the following inequality holds:

$$\|C[a,f]\|_{M_{\vec{p},\varphi}(B_r)} \le C\varepsilon \|f\|_{M_{\vec{p},\varphi}(\Omega)}.$$

3. Proof of the Main Result

Definition 3.1. ([14]) For $a \in L^1_{loc}(R^n)$ and any $r>0$ set

$$\gamma_a(R) := \sup_{B_r, r \le R} \frac{1}{|B_r|} \int_{B_r} |a(y) - a_{B_r}| dy,$$

where B_r is any ball in R^n. We say that $a \in BMO$, if $\|a\|_* = \sup_{R>0} \gamma_a(R) < \infty$ and $a \in VMO$ with -modulus γ_a if $a \in BMO$ and $\lim_{R \to 0} \gamma_a(R) = 0$.

We deal with the operators with discontinuous coefficients a_α^{jk} belonging to the Sarason function class *VMO*. Let $M^{m\times m}$ be the set of $m \times m$-matrices. For a matrix-valued function $A \in M^{m\times m}$ with entries $a^{jk} \in VMO$ we define the *VMO*-modulus of A as $\gamma_A = \sum_{j,k=1}^m \gamma_{a^{jk}}$.

After this $u: \Omega \to R^m$, $m \ge 1$, stands for the unknown function, $f = (f_1, \dots, f_m): \Omega \to R^m$ is a given vector-valued function and the coefficient matrix $A_\alpha(x) \in M^{m\times m}$ has entries $\{a_\alpha^{jk}\}_{j,k=1}^m$, $a_\alpha^{jk}: \Omega \to R$, which are measurable functions. Fixed an integer $b \ge 1$, we deal with the $2b$-order linear system

$$L(x,D)u := \sum_{|\alpha|=2b} A_\alpha(x) D^\alpha u(x) = f(x) \quad a.e.\ in \quad \Omega, \qquad (3.1)$$

that is equivalent to the system of differential equations

$$\sum_{k=1}^m \sum_{|\alpha|=2b} a_\alpha^{jk} D^\alpha u_k = \sum_{k=1}^m l^{jk}(x,D) u_k = f^j(x), \ j = 1, \dots, m. \qquad (3.2)$$

The entries $l^{jk}(x,D)$ of the matrix differential operator $L(x,D)$ are homogeneous polynomials of degree $2b$, that is,

$$l^{jk}(x,\xi) := \sum_{|\alpha|=2b} a_\alpha^{jk}(x) \xi^\alpha, \ \xi \in R^n, \ \xi^\alpha := \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n} \qquad (3.3)$$

That is the characteristic determinant of $\mathscr{L}(x,\xi)$ is non-vanishing for a.e. $x \in \Omega$ and all $\xi \ne 0$ elliptic, in other words, the operator $\mathscr{L}(x,D)$ is uniformly elliptic. Due to the homogeneity of l^{jk} this condition can be written as

$$\exists \delta > 0 : \det\left\{\sum_{|\alpha|=2b} A_\alpha(x) \xi^\alpha\right\} \ge \delta |\xi|^{2bm} \qquad (3.4)$$

for almost all $x \in \Omega$ and all $\xi \in R^n$

Let's take into consideration

$$\mathscr{L}(x_0, D) := \sum_{|\alpha|=2b} A_\alpha(x_0) D^\alpha, \ x_0 \in \Omega.$$

Then the $2bm$-order differential operator

$$L(x_0, D) := \det \mathscr{L}(x_0, D) = \det\left\{\sum_{|\alpha|=2b} a_\alpha^{jk}(x_0) D^\alpha\right\}_{j,k=1}^m \qquad (3.5)$$

is elliptic as it follows from (3.4), and let $\tilde{\Gamma}(x_0;x-y)$ be its fundamental solution. If the space dimension n is odd, then

$$\tilde{\Gamma}(x_0;x-y)=|x-y|^{2bm-n}P\left(x_0;\frac{x-y}{|x-y|}\right) \tag{3.6}$$

with $P(x_0;\xi)$ being a real analytic function of $\xi\in S^{n-1}$. If n is even, it is enough to introduce a fictitious new variable x_{n+1} and extend all functions as constants with respect to it (see [30]). Let $\{L_{jk}(x_0,\xi)\}_{j,k=1}^{m}$ be the cofactor matrix of $\{l^{jk}(x_0,\xi)\}_{j,k=1}^{m}$. Then $L_{jk}(x_0,D)$ are differential operators of order up to $2b(m-1)$ or identically zero. Since

$$\sum_{k=1}^{m}l^{ik}(x_0,\xi)L_{jk}(x_0,\xi)=\delta_{ij}L(x_0,\xi) \tag{3.7}$$

with the Kronecker symbol δ_{ij}, the fundamental matrix of $L(x_0,D)$ is given by

$$\Gamma(x_0;x)=\{\Gamma^{jk}(x_0;x)\}_{j,k=1}^{m}=\{L_{kj}(x_0,D)\tilde{\Gamma}(x_0;x)\}_{j,k=1}^{m}$$

Let $B_r\subset\Omega$ be such that $x_0\in B_r$, $v\in C_0^{\infty}(B_r)$ and let us write

$$\mathcal{L}(x_0,D)v(x)=(\mathcal{L}(x_0,D)-\mathcal{L}(x,D))v(x)+\mathcal{L}(x,D)v(x).$$

Using the standard approach [10, 11, 30] we obtain an explicit representation formula for v via *Newtonian potentials*

$$v(x)=\int_{B_r}\Gamma(x_0;x-y)\mathcal{L}v(y)dy+$$
$$+\int_{B_r}\Gamma(x_0;x-y)(\mathcal{L}(x_0,D)-\mathcal{L}(y,D))v(y)dy.$$

Taking the α-derivatives with $|\alpha|=2b$ and then unfreezing the coefficients putting $x_0=x$ we get

$$D^{\alpha}v(x)=p.v.\int_{B_r}D^{\alpha}\Gamma(x;x-y)\mathcal{L}v(y)dy+$$
$$+\sum_{|\alpha|=2b}p.v.\int_{B_r}D^{\alpha}\Gamma(x;x-y)(A_{\alpha'}(x)-A_{\alpha'}(y))D^{\alpha'}v(y)dy+$$
$$+\int_{S^{n-1}}D^{\beta^s}\Gamma(x;y)\upsilon_s\,d\sigma_y\,\mathcal{L}v(x)=$$
$$=:T_{\alpha}(\mathcal{L}v)(x)+\sum_{|\alpha|=2b}C_{\alpha}[A_{\alpha'},D^{\alpha'}v](x)+\mathcal{L}v(x)\mathcal{O}_{\beta}(x),$$

where the derivatives $D^{\alpha}\Gamma(\cdot,\cdot)$ and $D^{\beta^s}\Gamma(\cdot,\cdot)$ are taken with respect to the second variable, the multi-indices β^s are given by

$$\beta^s=(\alpha_1,\dots,\alpha_{s-1},\alpha_s,\alpha_{s+1},\dots,\alpha_n),\ |\beta^s|=2b-1$$

and $\nu=(\nu_1,\dots\nu_n)$ is the outer normal to S^{n-1}. Let us note that T_{α} are Calderon-Zygmund type singular integral operators, C_{α} are commutators of T_{α} with *VMO* functions, and Q_{β} are bounded integrals (cf. [3, 10, 11]).

The proof of our main result is based on some real analysis results on the boundedness of Calderon-Zygmund type singular integral operators and their commutators obtained in [12, 16, 17]. In order to estimate the semi norms we need the following interpolation inequality which follows from [3, 12].

Lemma 3.1. There is a constant C, independent of r, such that

$$\Theta_s\le\varepsilon\Theta_{2b}+\frac{C}{\varepsilon^{s/(2b-s)}}\Theta_0 \ \text{ for each } \varepsilon\in(0,2),$$

where $\Theta_s=\sup_{0<\theta<1}[\theta(1-\theta)r]^s\|D^su\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_r})}$.

Proof. Let $\theta_0\in(0,2)$ be such that

$$\Theta_s\le 2[\theta_0(1-\theta_0)r]^s\|D^su\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_0 r})}^s.$$

By interpolation and scaling arguments we obtain

$$\|D^su\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_0 r})}\le\delta^{2b-s}\|D^{2b}u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_0 r})}+\frac{C'}{\delta^s}\|u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_0 r})}$$

for $\delta\in(0,r)$ and hence

$$\Theta_s\le 2[\theta_0(1-\theta_0)r]^s\delta^{2b-s}\|D^{2b}u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_0 r})}+$$
$$+\frac{2C'[\theta_0(1-\theta_0)r]^s}{\delta^s}\|u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_0 r})}.$$

The assertion follows choosing

$$\delta=(\varepsilon/2)^{\frac{1}{2b-s}}[\theta_0(1-\theta_0)r]<\theta_0 r \ \text{ for any } \varepsilon\in(0,2).$$

In the following we give the proof of our main result.

Proof of Theorem 1.1. Let us fix an arbitrary $x_0\in supp\ u$ and let $B_r\equiv B(x_0,r)\subset\subset\Omega$. If we take $v\in W_{0,\vec{p}}^{2b,(x_0)}(B(x_0,r))$ (the closure of $C_0^{\infty}(B(x_0,r))$ with respect to the norm in $W_{\vec{p}}^{2b,(x_0)}(B(x_0,r))$ with supp $v\subset B(x_0,r)$, then (3.9), Proposition 2.1 and $A_{\alpha}\in BMO(\Omega)$ imply that for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\gamma_{A\alpha})$ such that

$$\|D^{2b}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_r)}\le C\left(\|\mathcal{L}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_r)}+\varepsilon\|D^{2b}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_r)}\right)$$

when ever $r<r_0$. If we choose ε small enough, then we obtain

$$\|D^{2b}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_r)}\le C\|\mathcal{L}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_r)}. \tag{3.11}$$

Let $\theta\in(0,1)$, $\theta'=\theta(3-\theta)/2>0$ and define the cut-off function $C_0^{\infty}(B_r)$ such that

$$\varphi(x)=\begin{cases}1 & x\in B_{\theta r}(x_0)\\ 0 & x\notin B_{\theta' r}(x_0)\end{cases}$$

Since $\theta'-\theta=\theta(1-\theta)/2$, direct calculations give

$$|D^s\varphi|\le C(s)[\theta(1-\theta)r]^s,\quad \forall s=1,2,\dots,2b.$$

Setting $v=\varphi u$ in (3.11) we obtain

$$\|D^{2b}u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta r})}\le\|D^{2b}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}\le C\|\mathcal{L}v\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}\le$$
$$\le C\left(\|f\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}\right)+\sum_{s=1}^{2b-1}\frac{\|D^{2b-s}u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}}{[\theta(1-\theta)r]^s}+\frac{\|u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}}{[\theta(1-\theta)r]^{2b}}$$

Because of the choice of θ' we have $\theta(1-\theta)\le 2\theta'(1-\theta')$ that implies

$$[\theta(1-\theta)r]^{2b}\|D^{2b}u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta r})}\le C\left([\theta'(1-\theta')r]^{2b}\|f\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta r})}+\right.$$
$$\left.+\sum_{s=1}^{2b-1}[\theta'(1-\theta')r]^s\|D^su\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}+\|u\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta' r})}\right).$$

Setting $\Theta_s=\sup_{0<\theta<1}[\theta(1-\theta)r]^s\|D^su\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta r})}$ we can rewrite (3.12) as

$$\Theta_{2b}\le C\left(r^{2b}\|f\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_{\theta_r})}+\sum_{s=1}^{2b-1}\Theta_s+\Theta_0\right) \tag{3.13}$$

Choosing suitable $\varepsilon\in(0,2)$, and applying (3.10) we get

$$\Theta_{2b}\le C\left(r^{2b}\|f\|_{M_{\vec{p},\varphi}^{\{x_0\}}(B_r)}+\Theta_0\right)$$

In Lemma 3.1 fixing $\theta=1/2$ at the semi norm Θ_s we obtain

the following Caccioppoli-type estimate:

$$\left\|D^{2b}u\right\|_{M^{\{x_0\}}_{\vec{p},\varphi}(B_{r/2})} \le C\left(\left\|f\right\|_{M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)} + r^{-2b}\left\|u\right\|_{M^{\{x_0\}}_{\vec{p},\varphi}(B_r)}\right). \quad (3.14)$$

Then the estimate (1.1) follows now by means of standard covering arguments with balls $B_{r/2}$ for $r < dist(\Omega', \partial\Omega'')$ and partition of unity over Ω' subordinated to this covering. Thus the proof of the theorem is completed.

Conclusions

In this paper, we study the boundedness of the singular integral operators T and their commutators $C[a, f]$ with $b \in BMO$ on generalized local mixed Morrey spaces $M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)$ and generalized mixed Morrey spaces $M_{\vec{p},\varphi}(\Omega)$. We obtain local regularity results for the strong solutions to $2b$-order linear elliptic systems $L(x, D)$ in the generalized local mixed Morrey spaces $M^{\{x_0\}}_{\vec{p},\varphi}(\Omega)$ and generalized mixed Morrey spaces $M_{\vec{p},\varphi}(\Omega)$. Solutions to the linear elliptic systems with discontinuous coefficients are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. We can apply this local regularity results in generalized local mixed Morrey spaces to study the regularity in generalized local mixed Morrey spaces of of the Navier-Stokes equations.

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