A matrix application of quasi-monotone sequences to Fourier series

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Abstract. In this paper, we have generalized a main theorem dealing with weighted mean summability method for absolute matrix summability method which plays a vital role in summability theory and applications to the other sciences by using quasi-monotone sequences. The main result in this paper extends the results in [**8**].

1. Introduction

A sequence (d_n) is said to be δ-quasi-monotone, if $d_n \to 0$, $d_n > 0$ ultimately, and $\Delta d_n \geqslant -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and δ=(δ_n) is a sequence of positive numbers (see [**1**]). For any sequence (λ_n) we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in BV$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [**9**]),

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \ \texttt{and}\ t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \ ({t_n}^1 = t_n) \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geqslant 1$, if (see [**11**], [**13**])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid^k < \infty. \tag{1.3}$$

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If we set α=1, then we have $\mid C,1\mid_k$ summability. Let (p_n) be a sequence of positive number such that

$$P_n=\sum_{v=0}^{\infty}p_v\to\infty \quad as \quad n\to\infty, \quad (P_{-i}=p_{-i}=0,\ i\geqslant 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$w_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v s_v \tag{1.5}$$

defines the sequence (w_n) of the Riesz mean or simply the $\left(\bar{N},p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [**12**]). The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k\geqslant 1$, if (see [**2**])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|w_n-w_{n-1}|^k<\infty. \tag{1.6}$$

In the special case when $p_n=1$ for all values of n (respect. $k=1$), then $|\bar{N},p_n|_k$ summability is the same as $|C,1|_k$ (respect. $|\bar{N},p_n|$) summability. We write $X_n=\sum_{v=1}^{n}\frac{p_v}{P_v}$, then (X_n) is a positive increasing sequence tending to infinity with n.

1.1. An application to trigonometric Fourier series. Let f be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. Without any loss of generality the constant term in the Fourier series of f can be taken to be zero, so that

$$f(t)\sim\sum_{n=1}^{\infty}(a_n cosnt+b_n sinnt)=\sum_{n=1}^{\infty}C_n(t). \tag{1.7}$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt, \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)cos(nt)dt, \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)sin(nt)dt.$$

We write

$$\phi(t)=\frac{1}{2}\left\{f(x+t)+f(x-t)\right\},\phi_\alpha(t)=\frac{\alpha}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)\,du, \quad (\alpha>0). \tag{1.8}$$

It is well known that if $\phi(t)\in\mathcal{BV}(0,\pi)$, then $z_n(x)=O(1)$, where $z_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [**10**]).

$$z_n(x)=\frac{1}{n+1}\sum_{v=1}^{n}vC_v(x). \tag{1.9}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.10}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geqslant 1$, if (see [**22**])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.11}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{1.12}$$

If we take $p_n = 1$, for all n, $|A, p_n|_k$ summability is the same as $|A|_k$ summability (see [**23**]). And also if we take $a_{nv} = \frac{p_v}{P_n}$, then we have $\left|\bar{N}, p_n\right|_k$ summability.

Theorem 1.1. [**8**] *Let $\lambda_n \to 0$ as $n \to \infty$ and let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \text{ as } n \to \infty. \tag{1.13}$$

Suppose that there exists a sequence of numbers (A_n) which is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent, and $|\Delta\lambda_n| \leqslant |A_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n{}^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{1.14}$$

satisfies, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

Bor has obtained the following result dealing with Fourier series.

Theorem 1.2. [**8**] *If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, and the sequences (A_n), (λ_n), and (X_n) satisfy the conditions of Theorem 1.1, then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.*

2. MAIN RESULTS

The aim of this paper is to generalize Theorem 1.2 for $|A, p_n|_k$ summability factors of Fourier series.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \bar{\Delta} a_{nv} = a_{nv} - a_{n-1,v}, \quad a_{-1,0} = 0 \tag{2.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv}, \quad n = 1, 2, ... \tag{2.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{2.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{2.4}$$

THEOREM 2.1. *Let (p_n) be a sequence of positive numbers such that $P_n = O(np_n)$ as $n \to \infty$, if $\phi_1(t) \in \mathcal{BV}(0, \pi)$, and the sequences (A_n), (λ_n), and (X_n) satisfy the conditions of Theorem 1.2. Let $k \geqslant 1$ and if $A = (a_{nv})$ is a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, ..., \tag{2.5}$$

$$a_{n-1,v} \geqslant a_{nv}, \text{ for } n \geqslant v + 1, \tag{2.6}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{2.7}$$

$$\hat{a}_{n,v+1} = O(v|\overline{\Delta} a_{nv}|) \tag{2.8}$$

then the series $\sum C_n(x)\lambda_n$ is summable $|A, p_n|_k$, $k \geqslant 1$.

We need the following lemma for the proof of Theorem 2.1.
Lemma 3 [**3**] Under the conditions of Theorem 1.1, we have that

$$|\lambda_n| X_n = O(1) \ as \ n \to \infty, \tag{2.9}$$

$$n X_n |A_n| = O(1) \ as \ n \to \infty, \tag{2.10}$$

$$\sum_{n=1}^{\infty} n X_n |\Delta A_n| < \infty. \tag{2.11}$$

3. Conclusion

In this paper, the concept of absolute matrix summability is investigated. In this investigation, we proved interesting theorem related to $|A, p_n|_k$. We also obtain applications to Fourier series. One may expect this investigation to be a useful tool in the field of analysis in modeling various problems occurring in many areas of science, dynamical systems, computer science, information theory, economical science and biological science.

We can apply Theorem 2.1 to weighted mean $A = (a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$ when $0 \leqslant v \leqslant n$, where $P_n = p_0 + p_1 + ... + p_n$. We have that,

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad and \quad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}} \tag{3.1}$$

(see [**14**]-[**21**] and [**24**]- [**29**] for the related bibliography).

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References

[1] R.P. Boas (Jr.), *Quasi-positive sequences and trigonometric series*, Proc. Lond. Math. Soc. 14A, 38-46 (1965)
[2] H. Bor, *On two summability methods*, Math. Proc. Camb. Philos. Soc. 97, 147-149, (1985)
[3] H. Bor, *On quasi-monotone sequences and their applications*, Bull. Austral. Math. Soc. 43 no. 2, 187–192, (1991)
[4] H. Bor, *Quasi-monotone and almost increasing sequences and their new applications*, Abstr. Appl. Anal. Art. ID. 793548, 6PP., (2012)
[5] H. Bor, *On absolute weighted mean summability of infinite series and Fourier series*, Filomat 30 2803–2807, (2016)
[6] H. Bor, *Some new results on absolute Riesz summablity of infinite series and Fourier series*, Positivity 20 599-605, (2016)
[7] H. Bor, *An Application of power increasing sequences to infinite series and Fourier series*, Filomat 31 1543–1547, (2017)
[8] H. Bor, *An application of quasi-monotone sequences to infinite series and Fourier series*, Anal. Math. Phys pp. DOI 10.1007/s133240170164x (2017).
[9] E. Cesàro , *Sur la multiplication des sries*, Bull. Sci. Math. 14, 114-120 (1890)
[10] K. K. Chen, *Functions of bounded variation and Cesàro means of Fourier series*, Acad. Sin. Sci. Record 1, 283-289 (1954)
[11] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc. 7, 113-141 (1957)
[12] G. H. Hardy, Clarendon Press, Oxford Univ, 1949.
[13] E. Kogbetliantz, *Sur les sries absolument sommables par la mthode des moyennes arithmtiques*, Bull. Sci. Math. 49, 234-256 (1925)
[14] Mursaleen, *Infinite matrices and absolute almost convergence*, Internat. J. Math. Math. Sci. 6 no. 3, 503–510, (1983)
[15] H. S. Özarslan, and T. Kandefer, *On the relative strength of two absolute summability methods*, J. Comput. Anal. Appl. 11 no. 3, 576–583, (2009)
[16] H. S. Özarslan, S. Yıldız, *On the local property of summability of factored Fourier series*, Int. J. Pure Math. 3 1–5, (2016)
[17] H. S. Özarslan, S. Yıldız, *A new study on the absolute summability factors of Fourier series.* J. Math. Anal. 7, 31–36, (2016)
[18] H. S. Özarslan, S. Yıldız, *A new theorem on the localization of factored Fourier Series*, Adv. Math. And Comp. Sci. Appl. 17, 144–146, (2016)
[19] M. A. Sarıgöl, *On two absolute Riesz summability factors of infinite series*, Proc. Amer. Math. Soc., 118, 485–488 (1993)
[20] M. A. Sarıgöl, *On the local properties of factored Fourier series*, Appl. Math. Comp. 216, 3386–3390, (2010)
[21] M. A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait Journal of Science, 42 (2) 28–35, (2015)

[22] W. T. Sulaiman, *Inclusion theorems for absolute matrix summability methods of an infinite series*, IV. Indian J. Pure Appl. Math. 34 11 (2003) 1547–1557.
[23] N. Tanovič-Miller, *On strong summability*, Glas. Mat. Ser III 14 (34) (1979), 87–97.
[24] S. Yıldız, *A new theorem on local properties of factored Fourier series*, Bull. Math. Anal. App . 8 (2), 1–8, (2016)
[25] S. Yıldız, *A new note on local property of factored Fourier series*, Bull. Math. Anal. Appl. 8 (4), 91–97, (2016)
[26] S. Yıldız, *On Riesz summability factors of Fourier series*, Trans. A. Razmadze Math. Inst. 171, 328–331, (2017)
[27] S. Yıldız, *A new generalization on absolute matrix summability factors of Fourier series*, J. Inequal. Spec. Funct. 8 (2), 65–73, (2017)
[28] S. Yıldız, *On Absolute Matrix Summability Factors of infinite Series and Fourier Series*, Gazi Univ. J. Sci. 30 (1), 363–370, (2017)
[29] S. Yıldız, *On Application of Matrix Summability to Fourier Series*, Math. Methods Appl. Sci., DOI: 10.1002/mma.4635, (2017)

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