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Full Paper

$\mathcal{I}_{ss}^{\oplus}$ −Supplemented modules

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Abstract: We describe $\mathcal{I}_{ss}^{\oplus}-$ supplemented modules as a proper generalisation of $\oplus_{ss}$ −supplemented modules. We show that each direct summand of a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module satisfying condition (D_3) is also $\mathcal{I}_{ss}^{\oplus}$ −supplemented. Then we prove that the finite direct sum of $\mathcal{I}_{ss}^{\oplus}-$supplemented submodules as a duo module is $\mathcal{I}_{ss}^{\oplus}-$supplemented. Moreover, we have given some types of rings whose modules are $\mathcal{I}_{ss}^{\oplus}$ −supplemented.

Keywords: $ss-$supplement, $\mathcal{I}_{ss}^{\oplus}$ −supplemented module, endomorphism ring

INTRODUCTION

All along the present text, whole modules will be considered as unital left R −modules where R denotes an associative ring having identity element. Let S be such a module. The notations $L \leq S$ and $L \leq_{\oplus} S$ notify that L is a submodule of S and L is a direct summand of S respectively. $End_R(S)$ indicates the endomorphism ring of an R −module S. A submodule L of S is named *small* in S, denoted as $L \ll S$, if $S \neq L + K$ for each proper submodule K of S. $Rad(S)$ signifies the intersection of whole maximal submodules of S, equivalently the sum of whole small submodules of S. Moreover, $Soc(S)$ stands for the socle of a module S, i.e. the sum of whole simple submodules of S. Explicitly for a module S, $Soc(S)$ is the largest semi-simple submodule of S. A submodule L of a module S is named *fully invariant* if $\vartheta(L)$ is included in L for each endomorphism ϑ of S [1]. A module S is referred to as *dual Rickart*, shortly *d-Rickart* if, for each $\vartheta \in End_R(S)$, $Im(\vartheta) \leq_{\oplus} S$. Also, a module S is referred to as *T−dual Rickart*, shortly *T-d-Rickart* if, for each homomorphism $\vartheta : S \to T$, $Im(\vartheta) \leq_{\oplus} T$ [2].

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Let S be a module and $K, H \leq S$. H is named a *supplement* of K in S, provided $S = K + H$ and $K \cap H \ll H$. A module S is named *supplemented* if each submodule of S possesses a supplement in S. A module S is named *amply supplemented* if, for any submodules L and K of S with $S = L + K$, there is a supplement of L in S that is contained in K [1]. A module S is named *lifting* if, for each $K \leq S$, there is a $D \leq_{\oplus} S$ such that $D \leq K$ and $K/D \ll S/D$ [3].

Mohamed and Müller [4] generalised the notion of the lifting modules to the notion of $\oplus -$supplemented modules as follows. A module S is $\oplus -$*supplemented* if each submodule of S possesses a supplement which is a direct summand of S. Many generalisations of $\oplus -$supplemented modules have been defined and examined in several studies [5-7].

S is named a $\mathcal{I}-$*lifting module*, provided there is a $K \leq_{\oplus} S$ such that $K \leq Im(\vartheta)$ and $Im(\vartheta)/K \ll S/K$, for each $\vartheta \in End_R(S)$ [8]. In the same paper a module S is named $\mathcal{I}-$*supplemented*, provided $Im(\vartheta)$ possesses a supplement in S for each $\vartheta \in End_R(S)$, and a module S is named *amply* $\mathcal{I}-$*supplemented*, provided $Im(\vartheta)$ possesses ample supplements in S for each $\vartheta \in End_R(S)$. The sum of whole simple submodules of S which are small in S is named $Soc_s(S)$ for a module S, i.e. $Soc_s(S) = \{L \ll S | L \text{ is simple}\}$ [9].

Kaynar et al. [10] introduced $ss-$supplement submodules as a generalisation of direct summands. Let $H, K \leq S$. Then H is named an $ss-$*supplement* of K in S if $S = H + K$ and $H \cap K \leq Soc_s(H)$. It is proved that for a module S, $Soc_s(S) = Rad(S) \cap Soc(S)$. In the same paper it is shown that H is an $ss-$supplement of K in S if and only if $S = H + K, H \cap K \ll H$ and $H \cap K$ is semi-simple if and only if $S = H + K, H \cap K \leq Rad(H)$ and $H \cap K$ is semi-simple. It is defined that a module S is $ss-$*supplemented*, provided each submodule of S possesses an $ss-$supplement in S, and a module S is named *amply* $ss-$*supplemented*, provided each submodule of S possesses ample $ss-$supplements in S, that is for any submodules L and K of S with $S = L + K$, there is an $ss-$supplement of L in S that is contained in K [10].

A module S is defined as $ss-$*lifting*, provided that for each submodule U of S, S possesses a decomposition $S = S_1 \oplus S_2$ such that $S_1 \leq U, U \cap S_2 \leq Soc_s(S_2)$ [11]. It is shown in Theorem 1 [11] that S is an $ss-$lifting module if and only if S is an amply $ss-$supplemented module and each $ss-$supplement submodule of S is a direct summand.

A module S is defined as $\oplus_{ss}-$*supplemented*, provided each submodule of S possesses an $ss-$supplement H such that $H \leq_{\oplus} S$ [5]. It is explicit that each $ss-$lifting module is a $\oplus_{ss}-$supplemented module.

A module S is named $\mathcal{I}_{ss}-$*lifting*, provided that for each $\vartheta \in End_R(S)$, there is $H \leq_{\oplus} S$ such that $H \leq Im(\vartheta)$, $Im(\vartheta)/H \ll S/H$ and $Im(\vartheta)/H$ is semi-simple [12]. In the same paper a module S is named $\mathcal{I}_{ss}-$*supplemented*, provided $Im(\vartheta)$ possesses an $ss-$supplement in S for each $\vartheta \in End_R(S)$, and a module S is named *amply* $\mathcal{I}_{ss}-$*supplemented*, provided $Im(\vartheta)$ possesses ample $ss-$supplements in S for each $\vartheta \in End_R(S)$.

Building upon these definitions, this study describes the concept of $\mathcal{I}_{ss}^{\oplus}-$supplemented modules as a proper generalisation of $\oplus_{ss}-$supplemented modules. We establish the equivalence between a module S with $Soc_s(S) = 0$ being $\mathcal{I}_{ss}^{\oplus}-$supplemented and S being d-Rickart. As an immediate result, we demonstrate that an $R-$module S is $\mathcal{I}_{ss}^{\oplus}-$supplemented over a left $V-$ring R if and only if S is d-Rickart. We provide that each finite direct sum of copies of d-Rickart module S is $\mathcal{I}_{ss}^{\oplus}-$supplemented. We show that for the modules S_1 and S_2 which are fully invariant in $S_1 \oplus S_2, S_1 \oplus S_2$ is a $\mathcal{I}_{ss}^{\oplus}-$supplemented module if and only if S_λ is $\mathcal{I}_{ss}^{\oplus}-$supplemented for each $\lambda =$

1, 2. We prove that a projective module S with $Rad(S) \leq Soc(S)$ is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if $S/Im(\vartheta)$ possesses a projective cover for each $\vartheta \in End_R(S)$. Moreover, when a ring R is left perfect with $Rad(R) \leq Soc({}_RR)$, each projective left R −module is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module, that is each free left R −module is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module.

RESULTS AND DISCUSSION

Definition 1. We name a module S as $\mathcal{I}_{ss}^{\oplus}$ −*supplemented*, provided that for each $\vartheta \in End_R(S)$, there is $D \leq_{\oplus} S$ such that $S = Im(\vartheta) + D$, $Im(\vartheta) \cap D \ll D$ and $Im(\vartheta) \cap D$ is semi-simple.

The example below shows that although each ss −lifting module is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module, the converse of this fact is not correct.

Example 1. Consider a local Dedekind domain, say R, and let Q be its quotient field. Consider $S = {}_RQ$. Then S is a hollow module and so it is an indecomposable module. Since $Rad(S) = S$ and $Soc(S) = 0$, S is not an ss −lifting module by Example 1 [11]. On the other hand, since S is a $\mathcal{I}_{ss}$ −lifting module by Example 2.2 [12], then S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module.

Definition 2. Let S and T be modules. We name a module S as $T - \mathcal{I}_{ss}^{\oplus}$ −*supplemented*, provided that for each homomorphism $\vartheta : S \to T$, there is a $D \leq_{\oplus} T$ such that $T = Im(\vartheta) + D$, $Im(\vartheta) \cap D \ll D$ and $Im(\vartheta) \cap D$ is semi-simple.

According to given definitions, a module S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if S is an $S - \mathcal{I}_{ss}^{\oplus}$ −supplemented module. It is explicit that if S is a T-d-Rickart module, then S is a $T - \mathcal{I}_{ss}^{\oplus}$ − supplemented module. Moreover, when T is a semi-simple module, S is a $T - \mathcal{I}_{ss}^{\oplus}$ −supplemented module for any R −module S.

Theorem 1. Let S and T be modules. Then S is a $T - \mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if S' is a $T' - \mathcal{I}_{ss}^{\oplus}$ −supplemented module for each fully invariant $T' \leq_{\oplus} T$ and for each $S' \leq_{\oplus} S$.

Proof. For some $e^2 = e \in End_R(S)$, let $S' = eS$. Let $T' \leq_{\oplus} T$ be fully invariant and $\psi \in Hom(S', T')$. Since $\psi e(S) = \psi S' \leq T' \leq T$ and S is a $T - \mathcal{I}_{ss}^{\oplus}$ −supplemented module, there is a $L \leq_{\oplus} T$ such that $T = \psi e(S) + L$, $\psi e(S) \cap L \ll L$ and $\psi e(S) \cap L$ is semi-simple. Note here that $\psi e(S) \cap L \ll T'$ by Section 19.3 [1]. Then we derive that $\psi e(S) + (L \cap T') = T'$. Since T' is a fully invariant submodule of T, then $L \cap T' \leq_{\oplus} T'$ by Lemma 2.1 [13]. Thus, $\psi e(S) \cap (L \cap T') \ll L \cap T'$ by Section 19.3 [1]. Also, since $\psi e(S) \cap (L \cap T')$ is semi-simple by Section 8.1.5 [14], then S' is a $T' - \mathcal{I}_{ss}^{\oplus}$ −supplemented module. The rest of the proof is explicit.

Corollary 1. Let S be a module. Then the assertions below are equivalent:

1) S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module.
2) Each $D \leq_{\oplus} S$ is a $T - \mathcal{I}_{ss}^{\oplus}$ −supplemented module for any fully invariant $T \leq_{\oplus} S$.

If a module S is a d-Rickart module, then S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module. However, for instance, the $\mathbb{Z}$ −module $\mathbb{Z}_4$ is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module; nevertheless it is not d-Rickart.

Proposition 1. Let S be a module where $Soc_s(S) = 0$. Then S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if S is a d-Rickart module.

Proof. Suppose that S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module and let ϑ be an arbitrary endomorphism of S. Then there is an $L \leq_{\oplus} S$ such that $S = Im(\vartheta) + L$, $Im(\vartheta) \cap L \ll L$ and $Im(\vartheta) \cap L$ is semi-simple. Thus, $Im(\vartheta) \cap L \leq Soc_s(S)$. By the assumption that $Soc_s(S) = 0$, then we have $S = Im(\vartheta) \oplus L$. Hence S is a d-Rickart module. The other part of the proof is explicit.

If each simple left R −module is injective, then the ring R is called a left V −ring [1]. It is evident, based on the facts in Section 23.1 [1], that R is a left V −ring if and only if the radical of each left R −module S is zero. In the context of commutative rings, the equivalence between being a left V −ring and von Neumann regular is established in Section 23.5 [1].

Corollary 2. Let S be an R −module where R is a left V −ring. Then S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if S is d-Rickart.

Proof. By Section 23.1 [1], $Rad(S) = 0$ for each left R −module S. Thus, the claim stems from Proposition 1.

Corollary 3. Let S be an R −module where R is a commutative von Neumann regular ring. Then S is a d-Rickart module if and only if S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module.

Proof. The proof stems from Corollary 2 and Section 23.5 [1].

Corollary 4. Let R be a Dedekind domain which is not a field and S be a torsion-free R −module. Then S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if S is a d-Rickart module.

Proof. The proof stems from Proposition 1 and Proposition 2.1 [15].

Proposition 2. Let S be an indecomposable module. Assuming that if $\vartheta \in End_R(S)$ such that $Im(\vartheta) \leq Soc_s(S)$, then $\vartheta = 0$. S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module if and only if each non-zero endomorphism of S is an epimorphism.

Proof. Assume that $0 \neq \vartheta \in End_R(S)$. There is an $L \leq_{\oplus} S$ such that $S = Im(\vartheta) + L, Im(\vartheta) \cap L \ll L$ and $Im(\vartheta) \cap L$ is semi-simple as S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module by assumption. Since S is an indecomposable module, then $L = S$ or $L = 0$. If $L = S$, then $Im(\vartheta) \ll S$ and $Im(\vartheta)$ is semi-simple. Thus, $Im(\vartheta) \leq Soc_s(S)$. Then by the assumption, $\vartheta = 0$. This is a contradiction. Therefore $L = 0$, and so ϑ is an epimorphism. The rest of the proof is explicit.

Corollary 5. Let S be an indecomposable module. Then S is a d-Rickart module if and only if S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module, and for each $\vartheta \in End_R(S)$, $Im(\vartheta) \leq Soc_s(S)$ implies that $\vartheta = 0$.

Proof. ($\Longrightarrow$) This is explicit.

($\Longleftarrow$) By Proposition 2 each non-zero endomorphism of S is an epimorphism. Thus, by Proposition 4.4 [2], S is a d-Rickart module.

Recall that a module S is termed *retractable* if, for each $0 \neq N \leq S$, there is a non-zero endomorphism ϑ of S such that $\vartheta(S) \leq N$.

Corollary 6. Let S be an indecomposable retractable module. Suppose that $\vartheta \in End_R(S)$ with $Im(\vartheta) \leq Soc_s(S)$ implies that $\vartheta = 0$. If S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module, then S is a simple module.

Proof. Let indecomposable retractable module S be $\mathcal{I}_{ss}^{\oplus}$ −supplemented and N be any non-zero submodule of S. Since S is retractable, there is a non-zero endomorphism ϑ of S such that $\vartheta(S) \leq N$. Since ϑ is an epimorphism by Proposition 2, then we have $N = S$. Therefore S is a simple module.

Proposition 3. Let S be a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module. Let L be a submodule of S such that S/L is a projective module. Then S is an $L - \mathcal{I}_{ss}^{\oplus}$ −supplemented module.

Proof. Let ϑ be any homomorphism from S to L. Let us take the endomorphism $\iota\vartheta : S \to S$, where ι is the inclusion homomorphism from L to S. Since S is a $\mathcal{I}_{ss}^{\oplus}$ −supplemented module, there is a $K \leq_{\oplus} S$ such that $S = Im(\vartheta) + K, Im(\vartheta) \cap K \ll K$ and $Im(\vartheta) \cap K$ is semi-simple. Thus, $L = Im(\vartheta) + (L \cap K)$. Since S/L is projective, $L \cap K \leq_{\oplus} S$ by Lemma 2.3 [6]. Then $L \cap K \leq_{\oplus} L$ and

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$L \cap K \leq_{\oplus} K$. By Section 19.3 [1], $Im(\vartheta) \cap K \ll L \cap K$. Hence S is an $L - \mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Corollary 7. Let $S = S_1 \oplus S_2$ be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module and S_2 be a projective module. Then S_1 is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. By Proposition 3, S is an $S_1 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module. Thus, S_1 is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module by Theorem 1.

Theorem 2. Let S_1, S_2 and T be modules. Assuming that T is an $S_\lambda - \mathcal{J}_{ss}^{\oplus}$ −supplemented module for each $\lambda = 1, 2$, then T is an $S_1 \oplus S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module if, for each homomorphism ψ from T to $S_1 \oplus S_2$ and any projection π of $S_1 \oplus S_2$, we have $\pi(Im(\psi)) = Im(\psi) \cap Im(\pi)$. The converse is correct if, for each $\lambda = 1, 2$, S_λ is fully invariant in $S_1 \oplus S_2$.

Proof. Suppose that T is an $S_\lambda - \mathcal{J}_{ss}^{\oplus}$ −supplemented module for each $\lambda = 1, 2$. To prove that T is an $S_1 \oplus S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module, let $\vartheta = (\pi_1\vartheta, \pi_2\vartheta)$ be any homomorphism from T to $S_1 \oplus S_2$, where each $\pi_\lambda : S_1 \oplus S_2 \to S_\lambda$ for each $\lambda = 1, 2$ is the canonical projection. Since T is an $S_\lambda - \mathcal{J}_{ss}^{\oplus}$ −supplemented module, there is a $K_\lambda \leq_{\oplus} S_\lambda$ such that $S_\lambda = (\pi_\lambda\vartheta)(T) + K_\lambda$, $(\pi_\lambda\vartheta)(T) \cap K_\lambda \ll K_\lambda$ and $(\pi_\lambda\vartheta)(T) \cap K_\lambda$ is semi-simple for each $\lambda = 1, 2$. If $K = K_1 \oplus K_2$, obviously $K \leq_{\oplus} S_1 \oplus S_2$. As $(\pi_1\vartheta)(T) = \pi_1(\vartheta(T) + S_2) = (\vartheta(T) + S_2) \cap S_1$ and $(\pi_2\vartheta)(T) = \pi_2(\vartheta(T) + K_1) = (\vartheta(T) + K_1) \cap S_2$, then we obtain $S_1 \leq \vartheta(T) + S_2 + K_1$ and $S_2 \leq \vartheta(T) + K_1 + K_2$. Thus, $S_1 \oplus S_2 = \vartheta(T) + K_1 + K_2 = \vartheta(T) + K$. Moreover, $S_1 \oplus S_2 = (\pi_1\vartheta)(T) + (\pi_2\vartheta)(T) + K_1 + K_2 = \vartheta(T) + K$. Since $\vartheta(T) \cap (K_1 + K_2) \leq (\vartheta(T) + K_1) \cap K_2 + (\vartheta(T) + K_2) \cap K_1$, we have $\vartheta(T) \cap (K_1 + K_2) \leq (\vartheta(T) + S_1) \cap K_2 + (\vartheta(T) + S_2) \cap K_1$. Since $\vartheta(T) + S_1 = (\pi_2\vartheta)(T) \oplus S_1$ and $\vartheta(T) + S_2 = (\pi_1\vartheta)(T) \oplus S_2$, then $\vartheta(T) \cap K \leq ((\pi_2\vartheta)(T) \cap K_2) + ((\pi_1\vartheta)(T) \cap K_1$. As $(\pi_\lambda\vartheta)(T) \cap K_\lambda \ll K_\lambda$ for each $\lambda = 1, 2$, $\vartheta(T) \cap K \ll K_1 + K_2 = K$ by Section 19.3 [1]. Also, since $((\pi_\lambda\vartheta)(T) \cap K_\lambda$ is a semi-simple module for each $\lambda = 1, 2$, $\vartheta(T) \cap K$ is semi-simple by Section 8.1.5 [14]. Hence T is an $S_1 \oplus S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module. The rest of the proof is explicit by Theorem 1.

Corollary 8. Let $S_1, S_2, \dots, S_m$ be left R −modules. Let $\oplus_{\lambda=1}^{m} S_\lambda$ be an $S_\eta - \mathcal{J}_{ss}^{\oplus}$ −supplemented module for $\eta = 1, 2, \dots, m$. Then $\oplus_{\lambda=1}^{m} S_\lambda$ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module. The converse is correct if each S_λ is fully invariant in $\oplus_{\lambda=1}^{m} S_\lambda$.

Corollary 9. Let $S_1, S_2, \dots, S_m$ be left R −modules. Let S_λ be an S_η-d-Rickart module for whole $\lambda, \eta \in I = \{1, 2, \dots, m\}$. Then $\oplus_{\lambda \in I} S_\lambda$ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. By Corollary 5.4 [2], $\oplus_{\lambda \in I} S_\lambda$ is an S_η-d-Rickart module for whole $\eta \in I = \{1, 2, \dots, m\}$. Thus, $\oplus_{\lambda \in I} S_\lambda$ is an $S_\eta - \mathcal{J}_{ss}^{\oplus}$ − supplemented module. Thus, by Corollary 8, $\oplus_{\lambda \in I} S_\lambda$ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Corollary 10. Each finite direct sum of copies of d-Rickart module S is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Theorem 3. Let S_1 and S_2 be modules. Suppose that for each $\lambda = 1, 2$, S_λ is fully invariant in $S_1 \oplus S_2$. Then $S_1 \oplus S_2$ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module if and only if S_λ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module for each $\lambda = 1, 2$.

Proof. Let S_λ be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module for each $\lambda = 1, 2$. Let $\vartheta = (\vartheta_{\lambda\eta})_{\lambda,\eta} \in End_R(S_1 \oplus S_2)$, where $\vartheta_{\lambda\eta} \in Hom(S_\eta, S_\lambda)$. Since S_λ is fully invariant in $S_1 \oplus S_2$ for each $\lambda = 1, 2$, then $Im(\vartheta) = Im(\vartheta_{11}) \oplus Im(\vartheta_{22})$. As S_λ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module for each $\lambda = 1, 2$, there is a $K_\lambda \leq_{\oplus} S_\lambda$ such that $S_\lambda = Im(\vartheta_{\lambda\lambda}) + K_\lambda$, $Im(\vartheta_{\lambda\lambda}) \cap K_\lambda \ll K_\lambda$ and $Im(\vartheta_{\lambda\lambda}) \cap K_\lambda$ is semi-simple. If $K =$

$K_1 \oplus K_2$, then $K \leq_{\oplus} S_1 \oplus S_2$. Also, $S_1 \oplus S_2 = Im(\vartheta_{11}) \oplus Im(\vartheta_{22}) + (K_1 \oplus K_2)$. Since $Im(\vartheta) \cap (K_1 \oplus K_2) \leq [(Im(\vartheta) + K_1) \cap K_2] + [(Im(\vartheta) + K_2) \cap K_1]$, we obtain $Im(\vartheta) \cap (K_1 \oplus K_2) \leq (Im(\vartheta_{11}) \cap K_1) + (Im(\vartheta_{22}) \cap K_2) \ll K_1 \oplus K_2$. Moreover, since $Im(\vartheta_{\lambda\lambda}) \cap K_\lambda$ is a semi-simple module for each $\lambda = 1, 2$, then $Im(\vartheta) \cap (K_1 \oplus K_2)$ is semi-simple by Section 8.1.5 [14]. Hence $S_1 \oplus S_2$ is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module. The necessity stems from Theorem 1.

As a reminder, a module S is named *duo module* when each submodule of S is a fully invariant submodule [13].

Corollary 11. Let $S = S_1 \oplus S_2$ be a duo module. Then S is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module if and only if S_1 and S_2 are $\mathcal{J}_{ss}^{\oplus}$ −supplemented modules.

Lemma 1. Let $S = S_1 \oplus S_2$. Then S is an $S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module if and only if, for each $\vartheta \in End_R(S)$ with $S_1 \leq Im(\vartheta)$, there is a $K \leq_{\oplus} S$ such that $K \leq S_2$, $S = K + Im(\vartheta), K \cap Im(\vartheta) \ll S$ and $K \cap Im(\vartheta)$ is semi-simple.

Proof. Let S be an $S_2 - \mathcal{J}_{ss}^{\oplus} -$ supplemented module. Assume that $\vartheta = (\pi_1\vartheta, \pi_2\vartheta)$ is any endomorphism of S with $S_1 \leq Im(\vartheta)$, where $\pi_\lambda : S \to S_\lambda$ for each $\lambda = 1, 2$ is the canonical projection. Since S is an $S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module, there is an $L \leq_{\oplus} S_2$ such that $S_2 = Im(\pi_2\vartheta) + L, Im(\pi_2\vartheta) \cap L \ll L$ and $Im(\pi_2\vartheta) \cap L$ is semi-simple. It is easy to see that $Im(\vartheta) \cap S_2 = Im(\pi_2\vartheta)$. Therefore $S = S_1 + S_2 = S_1 + (Im(\vartheta) \cap S_2) + L = Im(\vartheta) + L$, $Im(\vartheta) \cap L \ll L$ and $Im(\vartheta) \cap L$ is semi-simple.

Conversely, let $\vartheta \in Hom(S, S_2)$. Consider the endomorphism $\gamma = \vartheta + \pi_1 \in End_R(S)$, where $\pi_1 : S \to S_1$ is the canonical projection. Since $S_1 \leq Im(\vartheta) \oplus S_1 = Im(\gamma)$, there is an $L \leq_{\oplus} S$ such that $L \leq S_2$, $S = Im(\gamma) + L, Im(\gamma) \cap L \ll S$ and $Im(\gamma) \cap L$ is semi-simple by the assumption. Thus, $S_2 = Im(\vartheta) + L$, $Im(\vartheta) \cap L \ll L$, $Im(\vartheta) \cap L$ is semi-simple and $L \leq_{\oplus} S_2$. Hence S is an $S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Theorem 4. Let $S = S_1 \oplus S_2$ be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module. Assuming that for each $L \leq_{\oplus} S$ with $S = L + S_2$, $L \cap S_2 \leq_{\oplus} S$, then S is an $S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. Let $\vartheta = (\pi_1\vartheta, \pi_2\vartheta)$ be any endomorphism of S with $S_1 \leq Im(\vartheta)$, where $\pi_\lambda : S \to S_\lambda$ for each $\lambda = 1, 2$ is the canonical projection. We then set $\iota_2\pi_2\vartheta \in End_R(S)$, where $\iota_2 : S_2 \to S$ is the canonical injection. We denote that $Im(\iota_2\pi_2\vartheta) = Im(\vartheta) \cap S_2$. Since S is a $\mathcal{J}_{ss}^{\oplus}$ − supplemented module, there is an $L \leq_{\oplus} S$ such that $S = (Im(\vartheta) \cap S_2) + L$, $(Im(\vartheta) \cap S_2) \cap L \ll L$ and $(Im(\vartheta) \cap S_2) \cap L$ is semi-simple. Thus, $S = Im(\vartheta) + S_2$. By Lemma 1.2 [16], $S = (L \cap S_2) + Im(\vartheta)$. By hypothesis, $L \cap S_2 \leq_{\oplus} S$ as $S = L + S_2$. Therefore by using Lemma 1, S is an $S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Corollary 12. Let S be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module and $L \leq_{\oplus} S$ such that S/L is an L-projective module. Then S is an $L - \mathcal{J}_{ss}^{\oplus} -$ supplemented module.

Proof. Let $L' \leq_{\oplus} S$ with $S = L' + L$. Since $L \leq_{\oplus} S$, then $S = L \oplus K$ for some submodule K of S. Thus, K is an L-projective module. By Section 41.14 [1], there is $L'' \leq L'$ such that $S = L'' \oplus L$. Then we get $L' = L'' \oplus (L' \cap L)$, i.e. $L' \cap L \leq_{\oplus} S$. Hence S is an $L - \mathcal{J}_{ss}^{\oplus}$ −supplemented module by Theorem 4.

Corollary 13. Let $S = S_1 \oplus S_2$ be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module such that S_1 is an S_2-projective module. Then S is an $S_2 - \mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. This stems from Corollary 12.

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Corollary 14. Let $S = S_1 \oplus S_2$ be a module. If, for each $\vartheta \in End_R(S)$ with $S_1 \leq Im(\vartheta)$, there is a $K \leq_{\oplus} S$ such that $K \leq S_2$, $S = K + Im(\vartheta)$, $K \cap Im(\vartheta) \ll S$ and $K \cap Im(\vartheta)$ is semi-simple, then S_2 is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. By Lemma 1 and Theorem 1.

A module S is said to have *condition* (D_3) if when $S = S_1 + S_2$ with $S_1, S_2 \leq_{\oplus} S$, then $S_1 \cap S_2 \leq_{\oplus} S$ [17].

Corollary 15. Let S be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module with condition (D_3). Then each direct summand of S is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. By Theorem 4 and Theorem 1.

Corollary 16. Let $S = S_1 \oplus S_2$ be a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module such that S_1 is an S_2-projective module. Then S_2 is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Proof. By Corollary 13 and Theorem 1.

A module S is named *Hopfian* if each epimorphism $\vartheta \in End_R(S)$ is an isomorphism [3].

Proposition 4. Let S be a noetherian $\mathcal{J}_{ss}^{\oplus}$ −supplemented module. Suppose that for any $\vartheta \in End_R(S)$, $Im(\vartheta) \leq Soc_s(S)$ implies that $\vartheta = 0$. Then there is a decomposition $S = S_1 \oplus S_2 \oplus \dots \oplus S_n$ where S_i is an indecomposable noetherian $\mathcal{J}_{ss}^{\oplus}$ − supplemented module with $End_R(S_i)$ as a division ring.

Proof. As S is a noetherian module, S possesses a finite decomposition such that direct summands are indecomposable noetherian. Thus, according to Theorem 1, each direct summand is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module. Since each noetherian module is Hopfian, then each direct summand is an indecomposable Hopfian d-Rickart module by Corollary 5, and so their endomorphism rings are division ring by Corollary 4.8 [2].

Theorem 5. Consider the assertions below in relation to a ring *R*:

1) R is a semi-simple artinian ring.
2) Each left R −module is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.
3) Each free left R −module is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

Then $(1) \Longrightarrow (2) \Longrightarrow (3)$. Moreover, if $Soc_s({}_RR) = 0$, then $(3) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$: Let S be an R −module and ϑ be an arbitrary endomorphism of S. Since R is a semi-simple artinian ring, then $Im(\vartheta) \leq_{\oplus} S$ by Proposition 3.7 [18]. Hence S is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.

$(2) \Longrightarrow (3)$: This is explicit.

$(3) \Longrightarrow (1)$: Let I be a left ideal of R. Then there is a free R −module F and an epimorphism $\vartheta : F \to I$. By the assumption, there is a $K \leq_{\oplus} F$ such that $F = K + I, K \cap I \ll K$ and $K \cap I$ is semi-simple. Then $K \cap I \leq Soc_s({}_RR)$. By the assumption, since $Soc_s({}_RR) = 0$, then $I \leq_{\oplus} F$, and similarly for ${}_RR$. Hence R is a semi-simple artinian ring.

It is recalled from Section 19.4 [1] that a projective module *S*, together with an epimorphism $f : S \to N$ such that $Ker(f) \ll S$, is named a $projective\ cover$ of N.

Proposition 5. Let S be a projective module with $Rad(S) \leq Soc(S)$. Then the statements below are equivalent:

1) S is a $\mathcal{J}_{ss}^{\oplus}$ −supplemented module.
2) $S/Im(\vartheta)$ possesses a projective cover for each $\vartheta \in End_R(S)$.

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Proof. (1) $\Longrightarrow$ (2): Let ϑ be the arbitrary endomorphism of S. Then by hypothesis, there is an $L \leq_{\oplus} S$ with $S = Im(\vartheta) + L, Im(\vartheta) \cap L \ll L$ and $Im(\vartheta) \cap L$ is semi-simple. Since L is a projective module by Section 18.1 [1], we have the epimorphism $f : L \to S \to S/Im(\vartheta)$. Thus, $Ker(f) = Im(\vartheta) \cap L \ll L$. Hence $S/Im(\vartheta)$ possesses a projective cover.

(2) $\Longrightarrow$ (1): Let $\vartheta \in End_R(S)$ and $f : P \to S/Im(\vartheta)$ be a projective cover. Then by projectivity of S, there is a homomorphism $g : S \to P$ such that $fg = \pi$, where $\pi: S \to S/Im(\vartheta)$ is the canonical projection. It is explicit that g is surjective. Thus, g splits, that is there is a homomorphism $h : P \to S$ such that $gh = 1_P$. Then $f = fgh = \pi h$. Therefore $S = Im(\vartheta) + h(P)$ and $Im(\vartheta) \cap h(P) \ll h(P)$. Note that $Im(\vartheta) \cap h(P) \leq Rad(S)$. By the assumption, $Im(\vartheta) \cap h(P)$ is semi-simple. Hence S is a $\mathcal{I}_{ss}^{\oplus}-$supplemented module.

Corollary 17. Let R be a ring with $Soc_s({}_RR) = Rad(R)$. Then the statements below are equivalent:

1) ${}_RR$ is a $\mathcal{I}_{ss}^{\oplus}-$supplemented module.
2) ${}_RR/Im(\vartheta)$ possesses a projective cover for each $\vartheta \in End_R({}_RR)$.

Proof. By Lemma 2 [10], $Soc_s({}_RR) = Rad(R) \cap Soc({}_RR)$. Then by the assumption, $Rad(R) \leq Soc({}_RR)$. Hence the result stems from Proposition 5.

Finally, in the next theorem we characterise projective $R-$modules via $\mathcal{I}_{ss}^{\oplus}-$supplemented $R-$modules over a left perfect ring R with $Rad(R) \leq Soc({}_RR)$.

Theorem 6. Let R be a ring. Consider the conditions below.

1) R is a left perfect ring with $Rad(R) \leq Soc({}_RR)$.
2) Each projective left $R-$module is a $\mathcal{I}_{ss}^{\oplus}-$supplemented module.
3) Each free left $R-$module is a $\mathcal{I}_{ss}^{\oplus}-$supplemented module.

Then (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2): Let S be a projective $R-$module. Then since S is a projective module, $Rad(S) = Rad(R)S \leq Soc({}_RR)S = Soc(S)$ by the assumption. Since R is a left perfect ring, each left $R-$module possesses a projective cover by Section 43.9 [1], and so $S/Im(\vartheta)$ possesses a projective cover for each $\vartheta \in End_R(S)$. Hence S is a $\mathcal{I}_{ss}^{\oplus}-$supplemented module by Proposition 5.

(2) $\Longrightarrow$ (3): This is explicit.

(3) $\Longrightarrow$ (2): This stems from Corollary 7.

CONCLUSIONS

In this paper the concept of '$\mathcal{I}_{ss}^{\oplus}-$supplemented module' is defined based on the known concepts which are in the literature. Generalising *ss* $-$ supplements in this paper to $\delta_{ss}-$supplements, the notion of '$\mathcal{I}_{\delta_{ss}}^{\oplus}-$supplemented module' can be described and basic algebraic properties can be examined analogously.

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