Guliyev et al. *Boundary Value Problems* (2024) 2024:137
https://doi.org/10.1186/s13661-024-01950-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Fractional integral related to Schrödinger operator on vanishing generalized mixed Morrey spaces

Vagif S. Guliyev[1,2,3], Ali Akbulut[1*†] and Suleyman Celik[4†]

*Correspondence:
aakbulut@ahievran.edu.tr
[1]Faculty of Arts and Sciences, Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, 40100, Turkey
Full list of author information is available at the end of the article
†Equal contributors

Abstract

With b belonging to a new $BMO_\theta(\rho)$ space, $L = -\Delta + V$ is a Schrödinger operator on $\mathbb{R}^n$ with nonnegative potential V belonging to the reverse Hölder class $RH_{n/2}$. The fractional integral operator associated with L is denoted by $\mathcal{I}_\beta^L$. We investigate the boundedness of $\mathcal{I}_\beta^L$ and $[b, \mathcal{I}_\beta^L]$, which are its commutators with $b_\theta(\rho)$ on vanishing generalized mixed Morrey spaces $VM^{\alpha,V}_{\vec{p},\varphi}$ related to Schrödinger operation and generalized mixed Morrey spaces $M^{\alpha,V}_{\vec{p},\varphi}$. The boundedness of the operator $\mathcal{I}_\beta^L$ is ensured by finding sufficient conditions on the pair (φ_1, φ_2), which goes from $M^{\alpha,V}_{\vec{p},\varphi_1}$ to $M^{\alpha,V}_{\vec{q},\varphi_2}$, and from $VM^{\alpha,V}_{\vec{p},\varphi_1}$ to $VM^{\alpha,V}_{\vec{q},\varphi_2}$, $\sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i} = \beta$. When b belongs to $BMO_\theta(\rho)$ and (φ_1, φ_2) satisfies some conditions, we also show that the commutator operator $[b, \mathcal{I}_\beta^L]$ is bounded from $M^{\alpha,V}_{\vec{p},\varphi_1}$ to $M^{\alpha,V}_{\vec{q},\varphi_2}$ and from $VM^{\alpha,V}_{\vec{p},\varphi_1}$ to $VM^{\alpha,V}_{\vec{q},\varphi_2}$.

Keywords: Schrödinger operator; Fractional integral; Vanishing generalized mixed Morrey space; Commutator; BMO

1 Introduction

In this paper, we consider the second-order Schrödinger differential operator in $\mathbb{R}^n$ with $n \geq 3$, defined by

$$L = -\Delta + V,$$

where V is nonnegative and belongs to the reverse Hölder class RH_q for some exponent $q \geq n/2$. Assume that V is a nonnegative locally $L_q(\mathbb{R}^n)$ integrable function on $\mathbb{R}^n$, then we say that V belongs to RH_q $(1 < p \leq \infty)$ if there exists a positive constant C such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V^q(x) dx \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x) dx$$

holds for every $x \in \mathbb{R}^n$, where $B(x,r)$ denotes the ball centered at x with radius r. For example, the nonnegative polynomial $V \in RH_\infty$, in particular, $|x| \in RH_\infty$.

Let the potential $V \in RH_q$ with $q \geq n/2$, and the critical radius function $\rho(x)$ is defined as

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}, \; x \in \mathbb{R}^n.$$

Clearly, $0 < m_V(x) < \infty$ when $V \neq 0$, and $m_V(x) = 1$ when $V = 1$ and $m_V(x) \approx 1 + |x|$ with $V(x) = |x|^2$.

Thanks to the heat diffusion semigroup e^{-tL} for enough good function f, the negative powers $L^{-\beta/2}$ $(\beta > 0)$ related to the Schrödinger operators L can be written as

$$\mathcal{I}_\beta^L f(x) = L^{-\beta/2} f(x) = \int_0^\infty e^{-tL}(f)(x)\, t^{\beta/2-1} dt, \quad 0 < \beta < n.$$

Applying Lemma 3.3 in [1] for enough good function f holds that

$$\mathcal{I}_\beta^L f(x) = \int_{\mathbb{R}^n} K_\beta(x,y) f(y) dy, \quad 0 < \beta < n,$$

and the kernel $K_\beta(x,y)$ satisfies the following inequality:

$$|K_\beta(x,y)| \leq \frac{C}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-\beta}}. \tag{1}$$

Moreover, we have $|K_\beta(x,y)| \leq \frac{C}{|x-y|^{n-\beta}}$, $0 < \beta < n$.

The commutator of $\mathcal{I}_\beta^L$ is defined by

$$[b, \mathcal{I}_\beta^L] f(x) = b(x) \mathcal{I}_\beta^L f(x) - \mathcal{I}_\beta^L (bf)(x).$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $\mathcal{I}_\beta^L$ and $[b, \mathcal{I}_\beta^L]$ are the Riesz potential I_β and the commutator of the Riesz potential $[b, I_\beta]$, respectively, that is,

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n} dy, \quad [b, I_\beta] f(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^n} f(y) dy.$$

According to [2], the new BMO space $BMO_\theta(\rho)$ with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_B| dy \leq C \Big(1 + \frac{r}{\rho(x)}\Big)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|} \int_B b(y) dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

Throughout this paper, the letter $\vec{p}$ denotes n-tuples of the numbers in $(0, \infty]$, $n \geq 1$, $\vec{p} = (p_1, p_2, \dots, p_n)$, $1 \leq \vec{p} < \infty$ means $1 \leq p_i < \infty$ for each i. For $1 \leq \vec{p} \leq \infty$, we denote $\vec{p}\,' = (p_1', \dots, p_n')$, where p_i' satisfies $1/p_i + 1/p_i' = 1$.

In 2019, Nogayama [3] considered a new Morrey space, with the L_p norm replaced by the mixed Lebesgue norm $L_{\vec{p}}(\mathbb{R}^n)$, which is called mixed Morrey spaces.

We first recall the definition of mixed Lebesgue space defined in [4].

Let $\vec{p} = (p_1, \dots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\| \cdot \|_{L_{\vec{p}}}$ or $\| \cdot \|_{L_{(p_1,\dots,p_n)}}$ is defined by

$$\begin{aligned}\|f\|_{L_{\vec{p}}} &\equiv \| \|_{L_{(p_1,\dots,p_n)}} \\ &= \bigg(\int_{\mathbb{R}} \cdots \bigg(\int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} |f(x_1, x_2, \dots, x_n)|^{p_1} dx_1 \bigg)^{\frac{p_2}{p_1}} dx_2 \bigg)^{\frac{p_3}{p_2}} \dots dx_n \bigg)^{\frac{1}{p_n}},\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. If $p_j = \infty$ for some $j = 1, \dots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L_{\vec{p}}(\mathbb{R}^n) = L_{(p_1,\dots,p_n)}(\mathbb{R}^n)$ to be the set of all $f \in L_0(\mathbb{R}^n)$ with $\|f\|_{L_{\vec{p}}} < \infty$, where $L_0(\mathbb{R}^n)$ denotes the set of measurable functions on $\mathbb{R}^n$.

The following analogue of Hölder's inequality for $L_{\vec{p}}$ is well known (see, for example, [5]).

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \le \vec{p} \le \infty$ and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$. Then, for any $f \in L_{\vec{p}}(\Omega)$ and $g \in L_{\vec{p}'}(\Omega)$, the following inequality is valid*:

$$\int_{\Omega} |f(x)g(x)| dx \le \|f\|_{L_{\vec{p}}(\Omega)} \|g\|_{L_{\vec{p}'}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1 *Let $0 < \vec{p} < \infty$ and B be a ball in $\mathbb{R}^n$. Then*

$$\|\chi_B\|_{L_{\vec{p}}} = \|\chi_B\|_{WL_{\vec{p}}} = |B|^{\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i}}.$$

By Theorem 1 and Lemma 1 we get the following estimate.

Lemma 2 *For $1 \le \vec{p} < \infty$ and for the balls $B = B(x, r)$, the following inequality is valid*:

$$\int_B |f(y)| dy \le |B|^{\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i'}} \|f\|_{L_{\vec{p}}(B)}.$$

The following lemma shows the Lebesgue differential theorem in mixed-norm Lebesgue spaces.

Lemma 3 [5, *Lemma* 2.4] *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < \vec{p} < \infty$, then*

$$\lim_{r \to 0} \|\chi_{B(x,r)}\|_{L_{\vec{p}}}^{-1} \|f\|_{L_{\vec{p}}(B(x,r))} = |f(x)| \quad a.e.\ x \in \mathbb{R}^n.$$

In [1], Morrey developed typical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ to investigate the local behavior of solutions to second order elliptic partial differential equations. We refer the readers to [1, 6–8] for information on the characteristics and uses of traditional Morrey spaces. A general nonnegative function $\varphi(x, r)$ that satisfies certain assumptions replaces r^λ in the definition of the generalized Morrey spaces (see, for example, [9–14]). Actually, a higher degree of regularity in the solutions to certain elliptic and parabolic boundary problems may be obtained thanks to the improved inclusion between the Morrey and the Hölder spaces. Generalized Morrey spaces $M_{p,\varphi}$ were separately introduced by Guliyev, Mizuhara, and Nakai [12, 13, 15] (see also [11, 16, 17]). Generally speaking, local Morrey spaces were also introduced separately by Guliyev [15] and Garcia-Cuerva and Herrero [18] (see also [19]). It should be noted that Guliyev introduced and analyzed integral operators in local Morrey-type spaces, including generalized local Morrey spaces, in [15].

We now provide the concept of potential-related generalized mixed Morrey spaces (including weak versions), which was previously presented by the first author in [20] for the case of $\vec{p} = (p,\dots,p)$.

Definition 1 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le \vec{p} < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. We denote by $M^{\alpha,V}_{\vec{p},\varphi} = M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ the generalized mixed Morrey space associated with Schrödinger operator, the space of all functions $f \in L^{\rm loc}_{\vec{p}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1}\, \|\chi_{B(x,r)}\|^{-1}_{L_{\vec{p}}(\mathbb{R}^n)}\, \|f\,\chi_{B(x,r)}\|_{L_{\vec{p}}(\mathbb{R}^n)}.$$

Remark 1 (*i*) When $\alpha = 0$ and $\varphi(x,r) = r^{(\frac{\lambda}{n}-1)\sum_{i=1}^{n}\frac{1}{p_i}}$, $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is the mixed Morrey space $L_{\vec{p},\lambda}(\mathbb{R}^n)$, see, for example, [3];

(*ii*) When $\varphi(x,r) = r^{(\frac{\lambda}{n}-1)\sum_{i=1}^{n}\frac{1}{p_i}}$, $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is the mixed Morrey space associated with Schrödinger operator $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [21] in the case of $\vec{p} = (p,\dots,p)$;

(*iii*) When $\alpha = 0$, $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is a generalized mixed Morrey space $M_{\vec{p},\varphi}(\mathbb{R}^n)$ introduced in the case of $\vec{p} = (p,\dots,p)$ by the first author, Mizuhara and Nakai in [12, 13, 15].

(*iv*) The generalized Morrey space associated with Schrödinger operator $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ was introduced by the first author in [20] in the case of $\vec{p} = (p,\dots,p)$.

For brevity, in the sequel, we use the notations

$$\mathfrak{A}^{\alpha,V}_{\vec{p},\varphi}(f;x,r) := \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1}\, \|\chi_{B(x,r)}\|^{-1}_{L_{\vec{p}}(\mathbb{R}^n)}\, \|f\,\chi_{B(x,r)}\|_{L_{\vec{p}}(\mathbb{R}^n)}.$$

Definition 2 The vanishing mixed generalized Morrey space associated with Schrödinger operator $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{\vec{p},\varphi}(f;x,r) = 0. \tag{2}$$

The vanishing space $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{\vec{p},\varphi}} \equiv \|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{\alpha,V}_{\vec{p},\varphi}(f;x,r).$$

In the case $\alpha = 0$ and $\varphi(x,r) = r^{(\frac{\lambda}{n}-1)\sum_{i=1}^{n}\frac{1}{p_i}}$, $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{\vec{p},\lambda}$ introduced in the case of $\vec{p} = (p,\dots,p)$ in [22], where applications to PDE were considered.

We refer to [10, 23–27] for some properties of vanishing generalized Morrey spaces.

In this paper, we consider the boundedness of the fractional integral operator $\mathcal{I}_\beta^L$ on the generalized mixed Morrey spaces $M_{\vec{p},\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized mixed Morrey spaces $VM_{\vec{p},\varphi}^{\alpha,V}(\mathbb{R}^n)$. When b belongs to the new BMO space $BMO_\theta(\rho)$, we also show that $[b,\mathcal{I}_\beta^L]$ is bounded from $M_{\vec{p},\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{\vec{q},\varphi}^{\alpha,V}(\mathbb{R}^n)$ and from $VM_{\vec{p},\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $VM_{\vec{q},\varphi}^{\alpha,V}(\mathbb{R}^n)$.

Our main results are as follows.

Theorem 2 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < \vec{p} < n/\beta$, $\sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \beta$, and $\varphi_1 \in \Omega_{\vec{p}}^{\alpha,V}$, $\varphi_2 \in \Omega_{\vec{q}}^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t} \leq c_0 \varphi_2(x,r), \tag{3}$$

where c_0 does not depend on x and r. Then the operator $\mathcal{I}_\beta^L$ is bounded on $M_{\vec{p},\varphi_1}^{\alpha,V}$ to $M_{\vec{q},\varphi_2}^{\alpha,V}$ for $p > 1$. Moreover,

$$\|\mathcal{I}_\beta^L f\|_{M_{\vec{q},\varphi_2}^{\alpha,V}} \leq C \|f\|_{M_{\vec{p},\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

Theorem 3 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < \vec{p} < n/\beta$, $\sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \beta$ and $\varphi_1 \in \Omega_{\vec{p}}^{\alpha,V}$, $\varphi_2 \in \Omega_{\vec{q}}^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t} \leq c_0 \varphi_2(x,r), \tag{4}$$

where c_0 does not depend on x and r. If $b \in BMO_\theta(\rho)$, then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $M_{\vec{p},\varphi_1}^{\alpha,V}$ to $M_{\vec{q},\varphi_2}^{\alpha,V}$ and

$$\|[b,\mathcal{I}_\beta^L] f\|_{M_{\vec{q},\varphi_2}^{\alpha,V}} \leq C [b]_\theta \|f\|_{M_{\vec{p},\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

Theorem 4 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < \vec{p} < n/\beta$, $\sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \beta$ and $\varphi_1 \in \Omega_{\vec{p},1}^{\alpha,V}$, $\varphi_2 \in \Omega_{\vec{q},1}^{\alpha,V}$ satisfy the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t) \frac{dt}{t^{1-\beta}} \leq C_0 \varphi_2(x,r), \tag{5}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $\mathcal{I}_\beta^L$ is bounded from $VM_{\vec{p},\varphi_1}^{\alpha,V}$ to $VM_{\vec{q},\varphi_2}^{\alpha,V}$.

Theorem 5 *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < \vec{p} < n/\beta$, $\sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i} = \beta$, and $\varphi_1 \in \Omega_{\vec{p},1}^{\alpha,V}$, $\varphi_2 \in \Omega_{\vec{q},1}^{\alpha,V}$ satisfy the conditions*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)\varphi_1(x,t)\frac{dt}{t^{1-\beta}} \le c_0\varphi_2(x,r), \tag{6}$$

where c_0 does not depend on x and r,

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0, \tag{7}$$

and

$$c_\delta := \int_\delta^\infty \Big(1 + |\ln t|\Big) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t^{1-\beta}} < \infty \tag{8}$$

for every $\delta > 0$. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $VM_{\vec{p},\varphi_1}^{\alpha,V}$ to $VM_{\vec{q},\varphi_2}^{\alpha,V}$.

Remark 2 Note that, in the case of $\vec{p} = (p,\dots,p)$, Theorems 2 and 3 in the case of $V \equiv 0$ were proved in [28, Corollary 5.5 and 7.5] and in the case of $\varphi(x,r) = r^{(\frac{\lambda}{n}-1)\sum_{i=1}^{n}\frac{1}{p_i}}$ in [21, Theorems 1.3 and 1.4].

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some preliminaries

We would like to recall the important properties concerning the critical function.

Lemma 4 [29] *Let $V \in RH_{n/2}$. For the associated function ρ, there exist C and $k_0 \ge 1$ such that*

$$C^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \le \rho(y) \le C\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}} \tag{9}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 5 [30] *Suppose $x \in B(x_0,r)$. Then, for $k \in N$, we have*

$$\frac{1}{\Big(1 + \frac{2^k r}{\rho(x)}\Big)^N} \lesssim \frac{1}{\Big(1 + \frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

Lemma 6 [2] *Let* $1 \le s < \infty$. *If* $b \in BMO_\theta(\rho)$, *then*

$$\left(\frac{1}{|B|}\int_B |b(y)-b_B|^s dy\right)^{1/s} \le [b]_\theta \left(1+\frac{r}{\rho(x)}\right)^{\theta'}$$

for all $B = B(x,r)$ *with* $x \in \mathbb{R}^n$ *and* $r > 0$, *where* $\theta' = (k_0+1)\theta$ *and* k_0 *is the constant appearing in* (9).

Lemma 7 [2] *Let* $1 \le s < \infty$, $b \in BMO_\theta(\rho)$, *and* $B = B(x,r)$. *Then*

$$\left(\frac{1}{|2^k B|}\int_{2^k B} |b(y)-b_B|^s dy\right)^{1/s} \le [b]_\theta k \left(1+\frac{2^k r}{\rho(x)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$ *with* θ' *as in Lemma* 6.

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 8 [31] *Let* f *be a real-valued nonnegative function and measurable on* E. *Then*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

Lemma 9 *Let* $\varphi(x,r)$ *be a positive measurable function on* $\mathbb{R}^n \times (0,\infty)$, $1 \le \vec{p} < \infty$, $\alpha \ge 0$, *and* $V \in RH_q$, $q \ge 1$.

(i) *If*

$$\sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)} = \infty \quad \textit{for some } t>0 \textit{ and for all } x \in \mathbb{R}^n, \tag{10}$$

then $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau>0 \textit{ and for all } x \in \mathbb{R}^n, \tag{11}$$

then $M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n) = \Theta$.

Proof (i) Let (10) be satisfied and f be not equivalent to zero. Then $\sup_{x\in\mathbb{R}^n} \|f\|_{L_{\vec{p}}(B(x,t))} > 0$, hence

$$\begin{aligned}\|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} &\ge \sup_{x\in\mathbb{R}^n}\sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-\sum_{i=1}^{n}\frac{1}{p_i}} \|f\|_{L_{\vec{p}}(B(x,r))} \\ &\ge \sup_{x\in\mathbb{R}^n} \|f\|_{L_{\vec{p}}(B(x,t))} \sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-\sum_{i=1}^{n}\frac{1}{p_i}}.\end{aligned}$$

Therefore $\|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} = \infty$.

(ii) Let $f \in M^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$ and (11) be satisfied. Then there are two possibilities:

Case 1: $\sup_{0<r<t} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} = \infty$ for all $t > 0$.

Case 2: $\sup_{0<r<t} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} < \infty$ for some $t \in (0,\tau)$.

For Case 1, by Lebesgue differentiation theorem (see, Lemma 3), for almost all $x \in \mathbb{R}^n$,

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L_{\vec{p}}}}{\|\chi_{B(x,r)}\|_{L_{\vec{p}}}} = |f(x)|. \tag{12}$$

We claim that $f(x) = 0$ for all those x. Indeed, fix x and assume $|f(x)| > 0$. Then by (12) there exists $t_0 > 0$ such that

$$r^{-\sum_{i=1}^{n}\frac{1}{p_i}} \|f\|_{L_{\vec{p}}(B(x,r))} \geq 2^{-1}\, v_n^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}} |f(x)|$$

for all $0 < r \leq t_0$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Consequently,

$$\begin{aligned} \|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} &\geq \sup_{0<r<t_0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-\sum_{i=1}^{n}\frac{1}{p_i}} \|f\|_{L_{\vec{p}}(B(x,r))} \\ &\geq 2^{-1} v_n^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}} |f(x)| \sup_{0<r<t_0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1}. \end{aligned}$$

Hence $\|f\|_{M^{\alpha,V}_{\vec{p},\varphi}} = \infty$, so $f \notin M_{\vec{p},\varphi}(\mathbb{R}^n)$, and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{t<r<\tau} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} = \infty$, hence

$$\begin{aligned} \sup_{s<r<\infty} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-\sum_{i=1}^{n}\frac{1}{p_i}} &\geq \sup_{t<r<\tau} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-\sum_{i=1}^{n}\frac{1}{p_i}} \\ &\geq \tau^{-\frac{n}{p}} \sup_{t<r<\tau} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} = \infty, \end{aligned}$$

which is the case in (i). □

Remark 3 We denote by $\Omega^{\alpha,V}_{\vec{p}}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that, for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \left\| \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 9, we always assume that $\varphi \in \Omega^{\alpha,V}_{\vec{p}}$.

Remark 4 We denote by $\Omega^{\alpha,V}_{\vec{p},1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \left(1 + \frac{r}{\rho(x)}\right)^{-\alpha} \varphi(x,r) > 0 \text{ for some } \delta > 0 \tag{13}$$

and

$$\lim_{r\to 0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{\sum_{i=1}^{n}\frac{1}{p_i}}}{\varphi(x,r)} = 0.$$

For the nontriviality of the space $VM^{\alpha,V}_{\vec{p},\varphi}(\mathbb{R}^n)$, we always assume that $\varphi \in \Omega^{\alpha,V}_{\vec{p},1}$.

3 Proof of Theorem 2

The following Guliyev-type local estimates play an essential role in the proof of our results, see for example [11, 15, 20].

Theorem 6 *Let $V \in RH_{n/2}$. If $1 < \vec{p} < n/\beta$, $\sum_{i=1}^{n} \frac{1}{q_i} = \sum_{i=1}^{n} \frac{1}{p_i} - \beta$, then the inequality*

$$\|\mathcal{I}_\beta^L(f)\|_{L_{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t}$$

holds for any $f \in L_{\vec{p}}^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ and $\lambda B = B(x_0, \lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0, 2r)$. Then

$$\|\mathcal{I}_\beta^L(f)\|_{L_{\vec{q}}(B(x_0,r))} \le \|\mathcal{I}_\beta^L(f_1)\|_{L_{\vec{q}}(B(x_0,r))} + \|\mathcal{I}_\beta^L(f_2)\|_{L_{\vec{q}}(B(x_0,r))}.$$

Since $f_1 \in L_{\vec{p}}(\mathbb{R}^n)$ and from the boundedness of $\mathcal{I}_\beta^L$ from $L_{\vec{p}}(\mathbb{R}^n)$ to $L_{\vec{q}}(\mathbb{R}^n)$ (see [4]), it follows that

$$\begin{aligned}
\|\mathcal{I}_\beta^L(f_1)\|_{L_{\vec{q}}(B(x_0,r))} &\lesssim \|f\|_{L_{\vec{p}}(B(x_0,2r))} \\
&\lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \|f\|_{L_{\vec{p}}(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\sum_{i=1}^{n}\frac{1}{q_i}+1}} \\
&\lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t}. \qquad (14)
\end{aligned}$$

To estimate $\|\mathcal{I}_\beta^L(f_2)\|_{L_{\vec{p}}(B(x_0,r))}$, observe that $x \in B$, $y \in (2B)^c$ implies $|x - y| \approx |x_0 - y|$. Then by (1) we have

$$\begin{aligned}
\sup_{x\in B} |\mathcal{I}_\beta^L(f_2)(x)| &\le \sup_{x\in B} \int_{(2B)^c} |K_\beta(x,y)f(y)|dy \\
&\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\beta}} dy \\
&\lesssim \sum_{k=1}^{\infty} (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)|dy.
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
\sup_{x\in B} |\mathcal{I}_\beta^L(f_2)(x)| &\lesssim \sum_{k=1}^{\infty} \|f\|_{L_{\vec{p}}(2^{k+1}B)} (2^{k+1}r)^{-1-\sum_{i=1}^{n}\frac{1}{p_i}+\beta} \int_{2^k r}^{2^{k+1}r} dt \\
&\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t}
\end{aligned}$$

$$\lesssim \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t}. \tag{15}$$

Then

$$\|\mathcal{I}_\beta^L(f_2)\|_{L_{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t} \tag{16}$$

holds for $1 < \vec{p} < n/\beta$. Therefore, by (14) and (16) we get

$$\|\mathcal{I}_\beta^L(f)\|_{L_{\vec{q}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^{n}\frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t} \tag{17}$$

holds for $1 < \vec{p} < n/\beta$. □

Proof of Theorem 2 From Lemma 8, we have

$$\frac{1}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}}.$$

Note the fact that $\|f\|_{L_{\vec{p}}(B(x_0,t))}$ is a nondecreasing function of t and $f \in M^{\alpha,V}_{\vec{p},\varphi_1}$, then

$$\begin{aligned}
&\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_{\vec{p}}(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \\
&\lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_{\vec{p}}(B(x_0,t))}}{\varphi_1(x_0,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \\
&\lesssim \sup_{0<s<\infty} \frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_{\vec{p}}(B(x_0,s))}}{\varphi_1(x_0,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \\
&\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}}.
\end{aligned}$$

Since $\alpha \geq 0$ and (φ_1,φ_2) satisfies condition (3), then

$$\begin{aligned}
&\int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t} \\
&= \int_{2r}^{\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_{\vec{p}}(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\sum_{i=1}^{n}\frac{1}{p_i}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\sum_{i=1}^{n}\frac{1}{q_i}}} \frac{dt}{t}
\end{aligned}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}} \int_{2r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \tag{18}$$

Then by Theorem 6 we get

$$\begin{aligned}
&\|\mathcal{I}^L_\beta(f)\|_{M^{\alpha,V}_{\vec{q},\varphi_2}} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{q_i}} \|\mathcal{I}^L_\beta(f)\|_{L_{\vec{p}}(B(x_0,r))} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}}.
\end{aligned}$$

□

4 Proof of Theorem 3

As the proof of Theorem 2, it suffices to prove the following result.

Theorem 7 *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$. If $1 < \vec{p} < n/\beta$, $\sum_{i=1}^n \frac{1}{q_i} = \sum_{i=1}^n \frac{1}{p_i} - \beta$, then the inequality*

$$\|[b,\mathcal{I}^L_\beta(f)]\|_{L_{\vec{q}}(B(x_0,r))} \lesssim [b]_\theta r^{\sum_{i=1}^n \frac{1}{q_i}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t} \tag{19}$$

holds for any $f \in L^{\mathrm{loc}}_{\vec{p}}(\mathbb{R}^n)$.

Proof We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)(y)}$. Then

$$\|[b,\mathcal{I}^L_\beta](f)\|_{L_{\vec{q}}(B(x_0,r))} \le \|[b,\mathcal{I}^L_\beta](f_1)\|_{L_{\vec{q}}(B(x_0,r))} + \|[b,\mathcal{I}^L_\beta](f_2)\|_{L_{\vec{q}}(B(x_0,r))}.$$

By the boundedness of $[b,\mathcal{I}^L_\beta]$ on $L_{\vec{p}}(\mathbb{R}^n)$ to $L_{\vec{q}}(\mathbb{R}^n)$ (see [21]) and (14) we get

$$\begin{aligned}
\|[b,\mathcal{I}^L_\beta](f_1)\|_{L_{\vec{q}}(B(x_0,r))} &\lesssim [b]_\theta \|f\|_{L_{\vec{p}}(B(x_0,2r))} \\
&\lesssim [b]_\theta r^{\sum_{i=1}^n \frac{1}{q_i}} \int_{2r}^{\infty} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t} \\
&\lesssim [b]_\theta r^{\sum_{i=1}^n \frac{1}{q_i}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t}.
\end{aligned} \tag{20}$$

We now turn to dealing with the term $\|[b,\mathcal{I}_\beta^L](f_2)\|_{L_{\vec{q}}(B(x_0,r))}$. For any given $x\in B(x_0,r)$, we have

$$|[b,\mathcal{I}_\beta^L]f_2(x)| \le |b(x)-b_{2B}|\,|\mathcal{I}_\beta^L(f_2)(x)| + |\mathcal{I}_\beta^L((b-b_{2B})f_2)(x)|.$$

Then, by (15), Lemma 6, and taking $N\ge (k_0+1)\theta$, we get

$$\begin{aligned}
&\|(b(x)-b_{2B})\mathcal{I}_\beta^L(f_2)\|_{L_{\vec{q}}(B(x_0,r))}\\
&\lesssim [b]_\theta r^{\sum_{i=1}^n \frac{1}{q_i}}\Big(1+\frac{2r}{\rho(x_0)}\Big)^{\theta-N/(k_0+1)}\int_{2r}^\infty \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\
&\lesssim [b]_\theta r^{\sum_{i=1}^n \frac{1}{q_i}}\int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}.
\end{aligned}\tag{21}$$

Finally, let us estimate $\|\mathcal{I}_\beta^L((b-b_{2B})f_2)\|_{L_{\vec{q}}(B(x_0,r))}$. By (1), Lemma 5, and (15) we have

$$\begin{aligned}
&\sup_{x\in B}|\mathcal{I}_\beta^L((b-b_{2B})f_2)(x)|\\
&\lesssim \sup_{x\in B}\int_{(2B)^c}\frac{1}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^N}\frac{|b(y)-b_{2B}||f(y)|}{|x_0-y|^{n-\beta}}dy\\
&\lesssim \sup_{x\in B}\sum_{k=1}^\infty \frac{1}{(2^k r)^{n-\beta}\Big(1+\frac{2^k r}{\rho(x)}\Big)^N}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy\\
&\lesssim \sum_{k=1}^\infty \frac{1}{(2^k r)^{n-\beta}\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy &\lesssim \|b(\cdot)-b_{2B}\|_{L_{\vec{p}'}(2^{k+1}B)}\,\|f\|_{L_{\vec{p}}(2^{k+1}B)}\\
&\lesssim [b]_\theta k\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{\theta'}(2^k r)^{\sum_{i=1}^n \frac{1}{p_i'}}\,\|f\|_{L_{\vec{p}}(B(x_0,2^{k+1}r))}.
\end{aligned}$$

Then

$$\begin{aligned}
\sup_{x\in B}|\mathcal{I}_\beta^L((b-b_{2B})f_2)(x)| &\lesssim [b]_\theta\sum_{k=1}^\infty \frac{k(2^k r)^{-\sum_{i=1}^n \frac{1}{p_i}+\beta}}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)-\theta'}}\|f\|_{L_{\vec{p}}(B(x_0,2^{k+1}r))}\\
&\lesssim [b]_\theta\sum_{k=1}^\infty k(2^k r)^{-\sum_{i=1}^n \frac{1}{q_i}}\,\|f\|_{L_{\vec{p}}(B(x_0,2^{k+1}r))}\\
&\lesssim [b]_\theta\sum_{k=1}^\infty k\int_{2^k r}^{2^{k+1}r}\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}.
\end{aligned}$$

Since $2^k r \le t \le 2^{k+1}r$, then $k \approx \ln \frac{t}{r}$. Thus

$$\begin{aligned}\sup_{x\in B}|\mathcal{I}_\beta^L((b-b_{2B})f_2)(x)| &\lesssim [b]_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\ &\lesssim [b]_\theta \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \ln\frac{t}{r} \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\ &\lesssim [b]_\theta \int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}.\end{aligned}$$

Then

$$\|\mathcal{I}_\beta^L((b-b_{2B})f_2)\|_{L_{\vec{q}}(B(x_0,r))} \lesssim [b]_\theta r^{\sum_{i=1}^n \frac{1}{q_i}} \int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}. \tag{22}$$

Combining (20), (21), and (22), the proof of Theorem 7 is completed.

Proof of Theorem 3. Since $f \in M^{\alpha,V}_{\vec{p},\varphi_1}$ and (φ_1,φ_2) satisfies condition (4), by (18) we have

$$\begin{aligned}&\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\ &= \int_{2r}^{\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_{\vec{p}}(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^n \frac{1}{p_i}}}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^n \frac{1}{p_i}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\ &\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^n \frac{1}{p_i}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\ &\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\sum_{i=1}^n \frac{1}{p_i}}}{t^{\sum_{i=1}^n \frac{1}{q_i}}}\frac{dt}{t}\\ &\lesssim \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r).\end{aligned} \tag{23}$$

Then from Theorem 7 and by (23) we get

$$\begin{aligned}&\|[b,\mathcal{I}_\beta^L](f)\|_{M^{\alpha,V}_{\vec{q},\varphi_2}}\\ &\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-\sum_{i=1}^n \frac{1}{q_i}}\|[b,\mathcal{I}_\beta^L](f)\|_{L_{\vec{q}}(B(x_0,r))}\end{aligned}$$

$$\lesssim [b]_\theta \sup_{x_0\in\mathbb{R}^n, r>0} \Big(1+\frac{r}{\rho(x_0)}\Big)^\alpha \varphi_2(x_0,r)^{-1} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^n \frac{1}{q_i}}} \frac{dt}{t}$$

$$\lesssim [b]_\theta \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}}. \qquad \square$$

5 Proof of Theorem 4

The statement is derived from estimate (17). The estimation of the norm of the operator, that is, the boundedness in the nonvanishing space, immediately follows from Theorem 2. So we only have to prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{\vec{p},\varphi_1}(f;x,r)=0 \;\;\Rightarrow\;\; \lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{\vec{q},\varphi_2}(\mathcal{I}^L_\beta(f);x,r)=0. \tag{24}$$

To show that $\sup_{x\in\mathbb{R}^n}\big(1+\frac{r}{\rho(x)}\big)^\alpha \varphi_2(x,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{p_i}} \|\mathcal{I}^L_\beta(f)\|_{L_{\vec{q}}(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (17):

$$\Big(1+\frac{r}{\rho(x)}\Big)^\alpha \varphi_2(x,r)^{-1} r^{-\sum_{i=1}^n \frac{1}{p_i}} \|\mathcal{I}^L_\beta(f)\|_{L_{\vec{q}}(B(x,r))} \le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{25}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(x,r) := \frac{\big(1+\frac{r}{\rho(x)}\big)^\alpha}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\sum_{i=1}^n \frac{1}{q_i}-1} \|f\|_{L_{\vec{p}}(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{\big(1+\frac{r}{\rho(x)}\big)^\alpha}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} t^{-\sum_{i=1}^n \frac{1}{q_i}-1} \|f\|_{L_{\vec{p}}(B(x,t))} dt,$$

and it is supposed that $r<\delta_0$. We use the fact that $f\in VM^{\alpha,V}_{\vec{p},\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\Big(1+\frac{t}{\rho(x)}\Big)^\alpha \varphi_1(x,t)^{-1} t^{-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L_{\vec{p}}(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (5) and (25). This allows to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0<r<\delta_0.$$

The estimation of the second term now may be made already by the choice of r sufficiently small. Indeed, thanks to condition (13), we have

$$J_{\delta_0}(x,r) \le c_{\sigma_0} \frac{\big(1+\frac{r}{\rho(x)}\big)^\alpha}{\varphi_1(x,r)} \|f\|_{VM^{\alpha,V}_{\vec{p},\varphi_1}},$$

where c_{σ_0} is the constant from (2). Then, by (13), it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{\vec{p},\varphi_1}}},$$

which completes the proof of (24).

6 Proof of Theorem 5

The norm inequality having already been provided by Theorem 3, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_1(x,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|f\|_{L_{\vec{p}}(B(x,r))}=0$$
$$\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|[b,\mathcal{I}^L_{\beta}(f)]\|_{L_{\vec{q}}(B(x,r))}=0.$$

To check that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|[b,\mathcal{I}^L_{\beta}(f)]\|_{L_{\vec{q}}(B(x,r))}<\varepsilon \quad \text{for small } r,$$

we use estimate (19):

$$\varphi_2(x,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|[b,\mathcal{I}^L_{\beta}(f)]\|_{L_{\vec{q}}(B(x,r))}\lesssim \frac{[b]_\theta}{\varphi_2(x,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}.$$

We take $r<\delta_0$, where δ_0 will be chosen small enough, and split the integration:

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|[b,\mathcal{I}^L_{\beta}(f)]\|_{L_{\vec{q}}(B(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{26}$$

where

$$I_{\delta_0}(x,r):=\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r):=\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)}\int_{\delta_0}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{\vec{p}}(B(x_0,t))}}{t^{\sum_{i=1}^{n}\frac{1}{q_i}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_1(x,r)^{-1}r^{-\sum_{i=1}^{n}\frac{1}{p_i}}\|f\|_{L_{\vec{p}}(B(x,r))}<\frac{\varepsilon}{2CC_0}, \quad r\le\delta_0,$$

where C and C_0 are constants from (26) and (6), which yields the estimate of the first term uniform in $r \in (0,\delta_0)$: $\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, $0 < r < \delta_0$.

For the second term, writing $1 + \ln\frac{t}{r} \le 1 + |\ln t| + \ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \le \frac{c_{\delta_0} + \widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M^{\alpha,V}_{\vec{p},\varphi_1}},$$

where c_{δ_0} is the constant from (8) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (7), we can choose small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, which completes the proof.

7 Conclusion

In this paper, we study the boundedness of the fractional integral operator $\mathcal{I}^L_\beta$ associated with the Schrödinger operator and its commutators $[b,\mathcal{I}^L_\beta]$ with $b \in BMO_\theta(\rho)$ on generalized mixed Morrey spaces $M^{\alpha,V}_{\vec{p},\varphi}$ associated with the Schrödinger operator and vanishing generalized mixed Morrey spaces $VM^{\alpha,V}_{\vec{p},\varphi}$ associated with the Schrödinger operator. We find the sufficient conditions on the pair (φ_1,φ_2), which ensures the boundedness of the operator $\mathcal{I}^L_\beta$ from $M^{\alpha,V}_{\vec{p},\varphi_1}$ to $M^{\alpha,V}_{\vec{q},\varphi_2}$ and from $VM^{\alpha,V}_{\vec{p},\varphi_1}$ to $VM^{\alpha,V}_{\vec{q},\varphi_2}$, $\sum_{i=1}^n \frac{1}{q_i} = \sum_{i=1}^n \frac{1}{p_i} - \beta$. When b belongs to $BMO_\theta(\rho)$ and (φ_1,φ_2) satisfies some conditions, we also show that the commutator operator $[b,\mathcal{I}^L_\beta]$ is bounded from $M^{\alpha,V}_{\vec{p},\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and from $VM^{\alpha,V}_{\vec{p},\varphi_1}$ to $VM^{\alpha,V}_{\vec{q},\varphi_2}$, $\sum_{i=1}^n \frac{1}{q_i} = \sum_{i=1}^n \frac{1}{p_i} - \beta$.

Acknowledgements
The author thanks the referee(s) for carefully reading the paper and useful comments. The research of V.S. Guliyev was supported by the RUDN University Strategic Academic Leadership Program.

Author contributions
The three authors worked together to complete this project. These intriguing issues were brought up by V.S.G. in the study. The essay was written, the findings analyzed, and the theorems proven by V.S.G., A.A., and S.C., V.S.G., A.A., and S.C. oversaw the study's analysis. The final draft was read and approved by all three writers.

Funding
No external funding was received for this work.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Faculty of Arts and Sciences, Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, 40100, Turkey. [2]Institute of Applied Mathematics, Baku State University, Z. Khalilov Str., 23, Baku, AZ1148, Azerbaijan. [3]Department of Mathematics, Peoples Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya Str., Moscow, 117198, Russian Federation. [4]Social Sciences Vocational School, Department of Banking, Finance and Insurance, Firat University, Elazig, 23119, Turkey.

Received: 19 July 2024 Accepted: 16 October 2024 Published online: 21 October 2024

References

1. Morrey, C.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**, 126–166 (1938)
2. Bongioanni, B., Harboure, E., Salinas, O.: Commutators of Riesz transforms related to Schrodinger operators. J. Fourier Anal. Appl. **17**(11), 115–134 (2011)

3. Nogayama, T.: Mixed Morrey spaces. Positivity **23**(4), 961–1000 (2019)
4. Benedek, A., Panzone, R.: The space L^p, with mixed norm. Duke Math. J. **28**(3), 301–324 (1961)
5. Zhang, H., Zhou, J.: The boundedness of fractional integral operators in local and global mixed Morrey-type spaces. Positivity **26**, 22 (2022)
6. Chiarenza, F., Frasca, M.: Morrey spaces and Hardy-Littlewood maximal function. Rend. Mat. **7**, 273–279 (1987)
7. Di Fazio, G., Ragusa, M.A.: Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. J. Funct. Anal. **112**, 241–256 (1993)
8. Fan, D., Lu, S., Yang, D.: Boundedness of operators in Morrey spaces on homogeneous spaces and its applications. J. Fourier Anal. Appl. **14**, 625–634 (1998)
9. Akbulut, A., Dziri, M., Ekincioglu, I.: Maximal function and fractional integral associated with Laplace-Bessel differential operators on generalized Morrey spaces. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. **42**(1), 8–25 (2022)
10. Akbulut, A., Celik, S., Omarova, M.N.: Fractional maximal operator associated with Schrodinger operator and its commutators on vanishing generalized Morrey spaces. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. **44**(1), 3–19 (2024). https://doi.org/10.30546/2617-7900.44.1.2024.3
11. Guliyev, V.S.: Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal. Appl., **20**, 503948 (2009)
12. Mizuhara, T.: Boundedness of some classical operators on generalized Morrey spaces, ICM90 satellite proceedings. In: Igari, S. (ed.) Harmonic Analysis, pp. 183–189. Springer, Tokyo (1992)
13. Nakai, E.: Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. Math. Nachr. **166**, 95–103 (1994)
14. Softova, L.: Singular integrals and commutators in generalized Morrey spaces. Acta Math. Sin. Engl. Ser. **22**(3), 757–766 (2006)
15. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. In: Russian Doctor's Degree Dissertation, pp. 329. Mat. Inst. Steklov, Moscow (1994)
16. Akbulut, A., Gadjiev, T.S., Serbetci, A., Rustamov, Y.I.: Regularity of solutions to nonlinear elliptic equations in generalized Morrey spaces. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. **43**(4), 14–31 (2023). https://doi.org/10.30546/2617-7900.43.4.2023.14
17. Sawano, Y.: A thought on generalized Morrey spaces. J. Indones. Math. Soc. **25**(3), 210–281 (2019)
18. Garcia-Cuerva, Y., Herrero, M.J.L.: A theory of Hardy spaces associated to Herz spaces. Proc. Lond. Math. Soc. **69**(3), 605–628 (1994)
19. Alvarez, J., Lakey, J., Guzman-Partida, M.: Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures. Collect. Math. **51**(1), 1–47 (2000)
20. Guliyev, V.S.: Function spaces and integral operators associated with Schrodinger operators: an overview. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40**, 178–202 (2014)
21. Tang, L., Dong, J.: Boundedness for some Schrodinger type operator on Morrey spaces related to certain nonnegative potentials. J. Math. Anal. Appl. **355**, 101–109 (2009)
22. Vitanza, C.: Functions with vanishing Morrey norm and elliptic partial differential equations. In: Proceedings of Methods of Real Analysis and Partial Differential Equations, pp. 147–150. Springer, Capri (1990)
23. Akbulut, A., Kuzu, O.: Marcinkiewicz integrals with rough kernel associated with Schrodinger operator on vanishing generalized Morrey spaces. Azerb. J. Math. **4**(1), 40–54 (2014)
24. Cao, X., Chen, D.: The boundedness of Toeplitz-type operators on vanishing-Morrey spaces. Anal. Theory Appl. **27**(4), 309–319 (2011)
25. Ragusa, M.A.: Commutators of fractional integral operators on vanishing-Morrey spaces. J. Glob. Optim. **40**(1–3), 361–368 (2008)
26. Samko, N.: Maximal, potential and singular operators in vanishing generalized Morrey spaces. J. Glob. Optim. **57**(4), 1385–1399 (2013)
27. Guliyev, V.S., Guliyev, R.V., Omarova, M.A., Ragusa, M.A.: Schrodinger type operators on local generalized Morrey spaces related to certain nonnegative potentials. Discrete Contin. Dyn. Syst., Ser. B **25**(2), 671–690 (2020)
28. Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.: Boundedness of sublinear operators and commutators on generalized Morrey spaces. Integral Equ. Oper. Theory **71**(3), 327–355 (2011)
29. Shen, Z.: L_p estimates for Schrodinger operators with certain potentials. Ann. Inst. Fourier (Grenoble) **45**(2), 513–546 (1995)
30. Akbulut, A., Guliyev, V.S., Omarova, M.N.: Marcinkiewicz integrals associated with Schrodinger operators and their commutators on vanishing generalized Morrey spaces. Bound. Value Probl. **2017**(121), 121 (2017). https://doi.org/10.1186/s13661-017-0851-4
31. Wheeden, R., Zygmund, A.: Measure and Integral: An Introduction to Real Analysis. Pure and Applied Mathematics, pp. 43. Dekker, New York-Basel (1977)

Publisher's Note