Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 543475, 20 pages
doi:10.1155/2012/543475

Research Article

Parabolic Fractional Maximal Operator in Modified Parabolic Morrey Spaces

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Received 30 August 2012; Revised 30 September 2012; Accepted 1 October 2012

Academic Editor: Dachun Yang

We prove that the parabolic fractional maximal operator M_α^P, $0 \leq \alpha < \gamma$, is bounded from the modified parabolic Morrey space $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to the weak modified parabolic Morrey space $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$ if and only if $\alpha/\gamma \leq 1-1/q \leq \alpha/(\gamma-\lambda)$ and from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$ if and only if $\alpha/\gamma \leq 1/p - 1/q \leq \alpha/(\gamma-\lambda)$. Here $\gamma = \mathrm{tr}P$ is the homogeneous dimension on $\mathbb{R}^n$. In the limiting case $(\gamma-\lambda)/\alpha \leq p \leq \gamma/\alpha$ we prove that the operator M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$. As an application, we prove the boundedness of M_α^P from the parabolic Besov-modified Morrey spaces $\widetilde{BM}^s_{p\theta,\lambda}(\mathbb{R}^n)$ to $\widetilde{BM}^s_{q\theta,\lambda}(\mathbb{R}^n)$. As other applications, we establish the boundedness of some Schrödinger-ype operators on modified parabolic Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, the fractional maximal operators, the fractional integral operators, and the singular integral operators, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Morrey spaces, modified Morrey spaces also play an important role (see [1, 2]).

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P (t > 0)$ and set $\gamma = \operatorname{tr} P$. Then, there exists a quasi-distance ρ associated with P such that

(a) $\rho(A_t x) = t\rho(x),\ t > 0$, for every $x \in \mathbb{R}^n$;
(b) $\rho(0) = 0,\ \rho(x-y) = \rho(y-x) \geq 0$, and $\rho(x-y) \leq k(\rho(x-z) + \rho(y-z))$;
(c) $dx = \rho^{\gamma-1} d\sigma(w) d\rho$, where $\rho = \rho(x)$, $w = A_{\rho^{-1}} x$, and $d\sigma(w)$ is a measure on the ellipsoid $\{w : \rho(w) = 1\}$.

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on ρ.

(d) For every x,

$$\begin{gathered} c_1 |x|^{\alpha_1} \leq \rho(x) \leq c_2 |x|^{\alpha_2} \quad \text{if } \rho(x) \geq 1, \\ c_3 |x|^{\alpha_3} \leq \rho(x) \leq c_4 |x|^{\alpha_4} \quad \text{if } \rho(x) \leq 1, \\ \rho(\theta x) \leq \rho(x) \quad \text{for } 0 < \theta < 1. \end{gathered} \tag{1.1}$$

Here α_i and $c_i (i = 1, \ldots, 4)$ are some positive constants. Similar properties hold for ρ^* which is associated with the matrix P^*. Here P^* is the adjoint matrix of P.

There are some important examples for the above spaces.

(1) Let $(Px, x) \geq (x, x) (x \in \mathbb{R}^n)$. In this case, $\rho(x)$ is defined by the unique solution of $|A_{t^{-1}} x| = 1$, and $k = 1$. This space is just the one studied by Calderón and Torchinsky in [3].
(2) Let P be a diagonal matrix with positive diagonal entries and let $t = \rho(x)$, $x \in \mathbb{R}^n$ be the unique solution of $|A_{t^{-1}} x| = 1$.
   (a) If all diagonal entries are greater than or equal to 1, this space was studied by Fabes and Rivière [4]. More precisely they studied the weak $(1,1)$ and L^p estimates of the singular integral operators on this space in 1966.
   (b) If there are diagonal entries smaller than 1, then ρ satisfies the above (a)–(d) with $k > 1$.

Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space [4, 5]. The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x, r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x, r)| = v_n r^\gamma$, where v_n is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x, r) = \mathbb{R}^n \setminus \mathcal{E}(x, r)$ be the complement of $\mathcal{E}(x, r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x, r) = B(x, r)$. Note that in the standard parabolic case $P_0 = \operatorname{diag}(1, \ldots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n). \tag{1.2}$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The parabolic fractional maximal function $M_\alpha^P f$ and the parabolic fractional integral $I_\alpha^P f$ are defined by

$$\begin{gathered} M_\alpha^P f(x) = \sup_{t>0} |\mathcal{E}(x, t)|^{-1+\alpha/\gamma} \int_{\mathcal{E}(x,t)} |f(y)| dy, \quad 0 \leq \alpha < \gamma. \\ I_\alpha^P f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{\gamma-\alpha}} dy, \quad 0 < \alpha < \gamma. \end{gathered} \tag{1.3}$$

If $\alpha = 0$, then $M^P \equiv M_0^P$ is the parabolic maximal operator. If $P = I$, then $M_\alpha \equiv M_\alpha^I$ is the fractional maximal operator, $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator and $I_\alpha \equiv I_\alpha^I$ is the Riesz potential.

It is well known that the fractional maximal operator, the fractional integral operator, and Calderón-Zygmund operators play an important role in harmonic analysis (see [6–9]).

Definition 1.1. Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$, $[t]_1 = \min\{1, t\}$. We denote by $M_{p,\lambda,P}(\mathbb{R}^n)$ the parabolic Morrey space, and by $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ the modified parabolic Morrey space the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms

$$\|f\|_{M_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p},$$
$$\|f\|_{\widetilde{M}_{p,\lambda,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p}, \tag{1.4}$$

respectively.

Note that

$$\widetilde{M}_{p,0,P}(\mathbb{R}^n) = M_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$
$$\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) \subset_{\!\!>} M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), \qquad \max\left\{ \|f\|_{M_{p,\lambda,P}}, \|f\|_{L_p} \right\} \le \|f\|_{\widetilde{M}_{p,\lambda,P}}, \tag{1.5}$$

and if $\lambda < 0$ or $\lambda > \gamma$, then $M_{p,\lambda,P}(\mathbb{R}^n) = \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = \Theta$, where the symbol $\subset_{\!\!>}$ means continuous embedding (let X, Y be the normed spaces, then by definition $X \subset_{\!\!>} Y$ means that there exists $C > 0$ such that $\|x\|_Y \le C\|x\|_X$ for all $x \in X$) and Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Definition 1.2 (see [10–14]). Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$. We denote by $WM_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space and by $W\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ the modified weak parabolic Morrey space the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norms:

$$\|f\|_{WM_{p,\lambda,P}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \left| \{ y \in \mathcal{E}(x,t) : |f(y)| > r \} \right| \right)^{1/p},$$
$$\|f\|_{W\widetilde{M}_{p,\lambda,P}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \left| \{ y \in \mathcal{E}(x,t) : |f(y)| > r \} \right| \right)^{1/p}, \tag{1.6}$$

respectively.

Note that

$$WL_p(\mathbb{R}^n) = WM_{p,0,P}(\mathbb{R}^n) = W\widetilde{M}_{p,0,P}(\mathbb{R}^n),$$
$$M_{p,\lambda,P}(\mathbb{R}^n) \subset WM_{p,\lambda,P}(\mathbb{R}^n), \quad \|f\|_{WM_{p,\lambda,P}} \le \|f\|_{M_{p,\lambda,P}},$$
$$\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) \subset W\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n), \quad \|f\|_{W\widetilde{M}_{p,\lambda,P}} \le \|f\|_{\widetilde{M}_{p,\lambda,P}}. \tag{1.7}$$

If $P = I$, then $M_{p,\lambda}(\mathbb{R}^n) \equiv M_{p,\lambda,I}(\mathbb{R}^n)$ is the classical Morrey spaces [15] and $\widetilde{M}_{p,\lambda}(\mathbb{R}^n) \equiv \widetilde{M}_{p,\lambda,I}(\mathbb{R}^n)$ is the modified Morrey spaces [2].

Note that the parabolic generalized Morrey spaces are defined as follows (see, e.g., [16–18], etc.)

Definition 1.3. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi,P} \equiv M_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm:

$$\|f\|_{M_{p,\varphi,P}} = \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-1} |\mathcal{E}(x,t)|^{-1/p} \|f\|_{L_p(\mathcal{E}(x,t))}. \tag{1.8}$$

Notice that if we let $\varphi(x,t) = [t]_1^{\lambda/p} |\mathcal{E}(x,t)|^{1/p}$, then we obtain the modified Morrey norm.

The anisotropic result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α^P is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \gamma(1/p - 1/q)$ and for $p = 1 < q < \infty$, I_α^P is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = \gamma(1 - 1/q)$. Spanne (see [19]) and Adams [1] studied boundedness of the Riesz potential I_α for $0 < \alpha < n$ in Morrey spaces $M_{p,\lambda}$. Later on Chiarenza and Frasca [20] reproved boundedness of the Riesz potential I_α in these spaces. By more general results of Guliyev [21] (see also [17, 18, 22, 23]) one can obtain the following generalization of the results in [1, 19, 20] to the anisotropic case.

Theorem A. *Let* $0 < \alpha < \gamma$ *and* $0 \le \lambda < \gamma - \alpha$, $1 \le p < (\gamma - \lambda)/\alpha$.

(1) *If* $1 < p < (\gamma - \lambda)/\alpha$, *then the condition* $1/p - 1/q = \alpha/(\gamma - \lambda)$ *is necessary and sufficient for the boundedness of the operator* I_α^P *from* $M_{p,\lambda,P}(\mathbb{R}^n)$ *to* $M_{q,\lambda,P}(\mathbb{R}^n)$.

(2) *If* $p = 1$, *then the condition* $1 - 1/q = \alpha/(\gamma - \lambda)$ *is necessary and sufficient for the boundedness of the operator* I_α^P *from* $M_{1,\lambda,P}(\mathbb{R}^n)$ *to* $WM_{q,\lambda,P}(\mathbb{R}^n)$.

If $\alpha = \gamma/p - \gamma/q$, then $\lambda = 0$ and the statement of Theorem A reduces to the aforementioned result by anisotropic version of Hardy-Littlewood-Sobolev.

Recall that, for $0 < \alpha < \gamma$,

$$M_\alpha^P f(x) \le v_n^{\alpha/\gamma - 1} I_\alpha^P(|f|)(x), \tag{1.9}$$

hence Theorem A also implies the boundedness of the fractional maximal operator M_α^P. It is known that the parabolic maximal operator M^P is also bounded on $M_{p,\lambda,P}$ for all $1 < p < \infty$ and $0 < \lambda < \gamma$ (see, e.g. [24]), whose isotropic counterpart was proved by Chiarenza and Frasca [20].

In this paper we study the parabolic fractional maximal integral $M_\alpha^P f$ in the modified parabolic Morrey space $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$. In the case $p = 1$ we prove that the operator M_α^P is bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$ if and only if, $\alpha/\gamma \le 1 - 1/q \le \alpha/(\gamma - \lambda)$. In the case $1 < p < (\gamma - \lambda)/\alpha$ we prove that the operator M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$ if and only if, $\alpha/\gamma \le 1/p - 1/q \le \alpha/(\gamma - \lambda)$. In the limiting case $(\gamma - \lambda)/\alpha \le p \le \gamma/\alpha$ we prove that the operator M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 1 the boundedness of the maximal operator in modified Morrey space $\widetilde{M}_{p,\lambda,P}$ is proved. The main result of the paper is the Hardy-Littlewood-Sobolev inequality in modified parabolic Morrey space for the parabolic fractional maximal operator, established in Section 2. In Section 3 by using the $(\widetilde{M}_{p,\lambda,P}, \widetilde{M}_{q,\lambda,P})$ boundedness of the parabolic fractional maximal operators we establish the boundedness of some Schrödinger type operators on modified Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Main Results

Theorem 2.1. *Let $0 < \alpha < \gamma$, $0 \leq \lambda < \gamma - \alpha$, and $1 \leq p \leq (\gamma - \lambda)/\alpha$.*

(1) *If $1 < p < (\gamma - \lambda)/\alpha$, then the condition $\alpha/\gamma \leq 1/p - 1/q \leq \alpha/(\gamma - \lambda)$ is necessary and sufficient for the boundedness of the operator M_α^P from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.*

(2) *If $p = 1 < (\gamma - \lambda)/\alpha$, then the condition $\alpha/\gamma \leq 1 - 1/q \leq \alpha/(\gamma - \lambda)$ is necessary and sufficient for the boundedness of the operator M_α^P from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.*

(3) *If $(\gamma - \lambda)/\alpha \leq p \leq \gamma/\alpha$, then the operator M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$.*

Besov-Morrey (and Triebel-Lizorkin-Morrey) spaces attracted some attention in these two decades. Kozono and Yamazaki [25] and Mazzucato [26] used these spaces in the theory of Navier-Stokes equations. Some properties of the spaces including the wavelet characterizations were described in the papers by Sawano [27, 28], Sawano and Tanaka [29, 30], Tang and Xu [31]. The most systematic and general approach can certainly be found in the very recent book [32] of Yuan et al., we also recommend this monograph for further up-to-date references on this subject.

In the following theorem we prove the boundedness of M_α^P in the parabolic Besov-modified Morrey spaces on $\mathbb{R}^n$ (see [33]), whose norm is given by

$$\widetilde{BM}^{s}_{p\theta,\lambda,P}(\mathbb{R}^n) = \left\{ f : \|f\|_{\widetilde{BM}^{s}_{p\theta,\lambda,P}} = \|f\|_{\widetilde{M}_{p,\lambda,P}} + \left(\int_{\mathbb{R}^n} \frac{\|f(x+\cdot) - f(\cdot)\|^{\theta}_{\widetilde{M}_{p,\lambda,P}}}{\rho(x)^{\gamma+s\theta}} dx \right)^{1/\theta} < \infty \right\}, \tag{2.1}$$

where $1 \leq p,\ \theta \leq \infty$, $0 < s < 1$ and $0 \leq \lambda < \gamma$.

These spaces generalize certain Besov-Morrey and Triebel-Lizorkin-Morrey spaces. As a general theory of Besov-Triebel-Lizorkin spaces, the Besov-Morrey and Triebel-Lizorkin-Morrey spaces are introduced due to the study of Navier-Stokes equations and attract some attention in recent years. Another scales of generalized Besov and Triebel-Lizorkin spaces, the Besov-type space and Triebel-Lizorkin-type space, were introduced by Yang and Yuan in [34, 35] and proved therein to be closely related to the theory of Q spaces. For further developments and applications of these spaces, we also refer to [32, 34–39].

Theorem 2.2. *Let $0 \le \alpha < \gamma$, $0 \le \lambda < \gamma - \alpha$ and $1 \le p < (\gamma - \lambda)/\alpha$. If $\alpha/Q \le 1/p - 1/q \le \alpha/(Q - \lambda)$, $1 \le \theta \le \infty$, and $0 < s < 1$, then the operator M_α^P is bounded from the space $\widetilde{BM}^s_{p\theta,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{BM}^s_{q\theta,\lambda,P}(\mathbb{R}^n)$. More precisely, there is a constant $C > 0$ such that*

$$\left\| M_\alpha^P f \right\|_{\widetilde{BM}^s_{q\theta,\lambda,P}} \le C \| f \|_{\widetilde{BM}^s_{p\theta,\lambda,P}} \tag{2.2}$$

holds for all $f \in \widetilde{BM}^s_{p\theta,\lambda,P}(\mathbb{R}^n)$.

3. Some Several Embeddings

Lemma 3.1. *Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$. Then*

$$\begin{gathered} \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), \\ \| f \|_{\widetilde{M}_{p,\lambda,P}} = \max\left\{ \| f \|_{M_{p,\lambda,P}}, \| f \|_{L_p} \right\}. \end{gathered} \tag{3.1}$$

Proof. Let $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$. Then from (1.5) we have that $f \in M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and $\max\{\| f \|_{M_{p,\lambda,P}}, \| f \|_{L_p}\} \le \| f \|_{\widetilde{M}_{p,\lambda,P}}$.

Let now $f \in M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$. Then

$$\begin{aligned} \| f \|_{\widetilde{M}_{p,\lambda,P}} &= \sup_{x \in \mathbb{R}^n, t > 0} \left([t]_1^{-\lambda} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} \\ &= \max\left\{ \sup_{x \in \mathbb{R}^n, 0 < t \le 1} \left(t^{-\lambda} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p}, \sup_{x \in \mathbb{R}^n, t > 1} \left(\int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} \right\} \\ &\le \max\left\{ \| f \|_{M_{p,\lambda,P}}, \| f \|_{L_p} \right\}. \end{aligned} \tag{3.2}$$

Therefore, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and the embedding $M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \hookrightarrow \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ is valid.

Thus $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = M_{p,\lambda,P}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and $\max\{\| f \|_{M_{p,\lambda,P}}, \| f \|_{L_p}\} = \| f \|_{\widetilde{M}_{p,\lambda,P}}$. □

The following statement can be proved analogously.

Lemma 3.2. *Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$, then*

$$\begin{gathered} W\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n) = WM_{p,\lambda,P}(\mathbb{R}^n) \cap WL_p(\mathbb{R}^n), \\ \| f \|_{W\widetilde{M}_{p,\lambda,P}} = \max\left\{ \| f \|_{WM_{p,\lambda,P}}, \| f \|_{WL_p} \right\}. \end{gathered} \tag{3.3}$$

Lemma 3.3. *Let $0 < \alpha < \gamma$ and $0 \le \lambda \le \gamma - \alpha$. Then*

$$M_{(\gamma-\lambda)/\alpha,\lambda,P}(\mathbb{R}^n) \hookrightarrow M_{1,\gamma-\alpha,P}(\mathbb{R}^n), \qquad \| f \|_{M_{1,\gamma-\alpha,P}} \le v_n^{1-\alpha/(\gamma-\lambda)} \| f \|_{M_{(\gamma-\lambda)/\alpha,\lambda,P}}. \tag{3.4}$$

Proof. Let $0<\alpha<\gamma$, $0\le\lambda\le\gamma-\alpha$, $f\in M_{(\gamma-\lambda)/\alpha,\lambda,P}(\mathbb{R}^n)$. By the Hölder's inequality we have

$$\begin{aligned}\int_{\mathcal{E}(x,t)}|f(y)|dy &\le \left(\int_{\mathcal{E}(x,t)}|f(y)|^{(\gamma-\lambda)/\alpha}dy\right)^{\alpha/(\gamma-\lambda)}\left(\int_{\mathcal{E}(x,t)}dy\right)^{1-\alpha/(\gamma-\lambda)}\\ &\le (v_n t^{\gamma})^{1-\alpha/(\gamma-\lambda)}\left(\int_{\mathcal{E}(x,t)}|f(y)|^{(\gamma-\lambda)/\alpha}dy\right)^{\alpha/(\gamma-\lambda)}.\end{aligned} \tag{3.5}$$

Moreover,

$$\begin{aligned}t^{\alpha-\gamma}\int_{\mathcal{E}(x,t)}|f(y)|dy &\le v_n^{1-\alpha/(\gamma-\lambda)}t^{-\alpha\lambda/(\gamma-\lambda)}\left(\int_{\mathcal{E}(x,t)}|f(y)|^{(\gamma-\lambda)/\alpha}dy\right)^{\alpha/(\gamma-\lambda)}\\ &\le v_n^{1-\alpha/(\gamma-\lambda)}\left(t^{-\lambda}\int_{\mathcal{E}(x,t)}|f(y)|^{(\gamma-\lambda)/\alpha}dy\right)^{\alpha/(\gamma-\lambda)}\\ &\le v_n^{1-\alpha/(\gamma-\lambda)}\|f\|_{M_{(\gamma-\lambda)/\alpha,\lambda,P}},\end{aligned} \tag{3.6}$$

therefore$f\in M_{1,\gamma-\alpha,P}(\mathbb{R}^n)$ and

$$\|f\|_{M_{1,\gamma-\alpha,P}}\le v_n^{1-\alpha/(\gamma-\lambda)}\|f\|_{M_{(\gamma-\lambda)/\alpha,\lambda,P}}. \tag{3.7}$$

□

Lemma 3.4. *Let $0<\alpha<\gamma$, $0\le\lambda\le\gamma-\alpha$. Then for $(\gamma-\lambda)/\alpha\le p\le\gamma/\alpha$.*

$$\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)\hookrightarrow M_{1,\gamma-\alpha,P}(\mathbb{R}^n),\qquad \|f\|_{M_{1,\gamma-\alpha,P}}\le v_n^{1/p'}\|f\|_{\widetilde{M}_{p,\lambda,P}}. \tag{3.8}$$

Proof. Let $0<\alpha<\gamma$, $0\le\lambda\le\gamma-\alpha$, $f\in\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, and $(\gamma-\lambda)/\alpha\le p\le\gamma/\alpha$. By the Hölder's inequality we have

$$\begin{aligned}\|f\|_{M_{1,\gamma-\alpha,P}} &= \sup_{x\in\mathbb{R}^n,t>0}t^{\alpha-\gamma}\int_{\mathcal{E}(x,t)}|f(y)|dy\\ &\le v_n^{1/p'}\sup_{x\in\mathbb{R}^n,t>0}t^{\alpha-\gamma/p}[t]_1^{\lambda/p}\left([t]_1^{-\lambda}\int_{\mathcal{E}(x,t)}|f(y)|^p dy\right)^{1/p}\\ &\le v_n^{1/p'}\|f\|_{\widetilde{M}_{p,\lambda,P}}\sup_{t>0}t^{\alpha-\gamma/p}\ [t]_1^{\lambda/p}\\ &= v_n^{1/p'}\|f\|_{\widetilde{M}_{p,\lambda,P}}\max\left\{\sup_{0<t\le1}t^{\alpha-(\gamma-\lambda)/p},\sup_{t>1}t^{\alpha-\gamma/p}\right\}\\ &= v_n^{1/p'}\|f\|_{\widetilde{M}_{p,\lambda,P}},\end{aligned} \tag{3.9}$$

therefore $f \in M_{1,\gamma-\alpha,P}(\mathbb{R}^n)$ and

$$\|f\|_{M_{1,\gamma-\alpha,P}} \le v_n^{1/p'} \|f\|_{\widetilde{M}_{p,\lambda,P}}. \tag{3.10}$$

□

For the $0 \le \alpha < \gamma$ we define the following fractional maximal functions:

$$M_{p,\alpha}^P f(x) \equiv \left(M_\alpha^P |f|^p\right)^{1/p}(x) = \sup_{t>0}\left(|\mathcal{E}(x,t)|^{-1+\alpha/\gamma} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p}. \tag{3.11}$$

In the case $\alpha = 0$ we denote $M_{p,0}^P f$ is simply denoted by $M_p^P f$.

Lemma 3.5. *Let $1 \le p < \infty$, $0 \le \alpha < \gamma$, and $f \in M_{p,\gamma-\alpha,P}(\mathbb{R}^n)$. Then $M_{p,\alpha}^P f \in L_\infty(\mathbb{R}^n)$ and*

$$\left\|M_{p,\alpha}^P f\right\|_{L_\infty} = v_n^{(\alpha/\gamma-1)(1/p)} \|f\|_{M_{p,\gamma-\alpha,P}}. \tag{3.12}$$

Proof. We have the following.

$$\begin{aligned}\left\|M_{p,\alpha}^P f\right\|_{L_\infty} &= v_n^{(\alpha/\gamma-1)(1/p)} \sup_{x\in\mathbb{R}^n, t>0}\left(t^{\alpha-\gamma} \int_{\mathcal{E}(x,t)} |f(y)|^p dy \right)^{1/p} \\ &= v_n^{(\alpha/\gamma-1)(1/p)} \|f\|_{M_{p,\gamma-\alpha,P}}.\end{aligned} \tag{3.13}$$

□

From Lemmas 3.1 and 3.5 we get the following.

Lemma 3.6. *Let $1 \le p < \infty$, $0 \le \alpha < \gamma$ and $f \in \widetilde{M}_{p,\gamma-\alpha,P}(\mathbb{R}^n)$. Then $M_{p,\alpha}^P f \in L_\infty(\mathbb{R}^n)$ and*

$$\left\|M_{p,\alpha}^P f\right\|_{L_\infty} \le v_n^{(\alpha/\gamma-1)(1/p)} \|f\|_{\widetilde{M}_{p,\gamma-\alpha,P}}. \tag{3.14}$$

In the case $\alpha = 0$ from Lemmas 3.5 and 3.6 one gets that for the $M_p^P f$ the following property is valid.

Corollary 3.7. *Let $1 \le p < \infty$ and $f \in L_\infty(\mathbb{R}^n)$. Then $M_p^P f \in L_\infty(\mathbb{R}^n)$ and*

$$\left\|M_p^P f\right\|_{L_\infty} = v_n^{-1/p} \|f\|_{L_\infty}. \tag{3.15}$$

In the case $p = 1$ from Lemmas 3.3 and 3.5 we get for the $M_\alpha^P f$ the following property is valid.

Corollary 3.8. *Let $0 \le \alpha < \gamma$, $0 \le \lambda \le \gamma - \alpha$ and $f \in M_{(\gamma-\lambda)/\alpha,\lambda,P}(\mathbb{R}^n)$. Then $M_\alpha^P f \in L_\infty(\mathbb{R}^n)$ and*

$$\left\|M_\alpha^P f\right\|_{L_\infty} = v_n^{\alpha/\gamma-1} \|f\|_{M_{1,\gamma-\alpha,P}} \le v_n^{\alpha/\gamma-\alpha/(n-\lambda)} \|f\|_{M_{(\gamma-\lambda)/\alpha,\lambda,P}}. \tag{3.16}$$

From Lemmas 3.4 and 3.6 one gets that for $M_\alpha^P f$ the following property is valid.

Corollary 3.9. *Let $0 \le \alpha < \gamma$, $0 \le \lambda \le \gamma - \alpha$ and $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$. Then $M_\alpha^P f \in L_\infty(\mathbb{R}^n)$ for $(\gamma-\lambda)/\alpha \le p \le \gamma/\alpha$ and*

$$\left\| M_\alpha^P f \right\|_{L_\infty} = v_n^{\alpha/\gamma - 1} \| f \|_{M_{1,\gamma-\alpha,P}} \le v_n^{\alpha/\gamma - 1/p} \| f \|_{\widetilde{M}_{p,\lambda,P}}. \tag{3.17}$$

4. $\widetilde{M}_{p,\lambda,P}$ Boundedness of the Parabolic Maximal Operator

In this section we study the $\widetilde{M}_{p,\lambda,P}$ boundedness of the maximal operator M^P.

Theorem 4.1 (see [24]). (1) *If $f \in M_{1,\lambda,P}(\mathbb{R}^n)$, $0 \le \lambda < \gamma$, then $M^P f \in WM_{1,\lambda,P}(\mathbb{R}^n)$ and*

$$\left\| M^P f \right\|_{WM_{1,\lambda,P}} \le C_\lambda \| f \|_{M_{1,\lambda,P}}, \tag{4.1}$$

where C_λ depends only on n and λ.

(2) *If $f \in M_{p,\lambda,P}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \lambda < \gamma$, then $M^P f \in M_{p,\lambda,P}(\mathbb{R}^n)$ and*

$$\left\| M^P f \right\|_{M_{p,\lambda,P}} \le C_{p,\lambda,P} \| f \|_{M_{p,\lambda,P}}, \tag{4.2}$$

where $C_{p,\lambda,P}$ depends only on n, p, λ, and P.

Applying Theorem 4.1, one obtains the following result.

Theorem 4.2. (1) *If $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$, $0 \le \lambda < \gamma$, then $M^P f \in W\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ and*

$$\left\| M^P f \right\|_{W\widetilde{M}_{1,\lambda,P}} \le \overline{C}_{1,\lambda,P} \| f \|_{\widetilde{M}_{1,\lambda,P}}, \tag{4.3}$$

where $\overline{C}_{1,\lambda,P}$ depends only on λ and n.

(2) *If $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \lambda < \gamma$, then $M^P f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and*

$$\left\| M^P f \right\|_{\widetilde{M}_{p,\lambda,P}} \le \overline{C}_{p,\lambda,P} \| f \|_{\widetilde{M}_{p,\lambda,P}}, \tag{4.4}$$

where $\overline{C}_{p,\lambda,P}$ depends only on p, λ, P, and n.

Proof. It is obvious that (see Lemmas 3.1 and 3.2)

$$\left\| M^P f \right\|_{\widetilde{M}_{p,\lambda,P}} = \max\left\{ \left\| M^P f \right\|_{M_{p,\lambda,P}}, \left\| M^P f \right\|_{L_p} \right\} \tag{4.5}$$

for $1 < p < \infty$ and

$$\left\|M^P f\right\|_{W\widetilde{L}_{1,\lambda,P}} = \max\left\{\left\|M^P f\right\|_{WM_{1,\lambda,P}}, \left\|M^P f\right\|_{WL_1}\right\} \tag{4.6}$$

for $p = 1$.

Let $1 < p < \infty$. By the boundedness of M^P on $L_p(\mathbb{R}^n)$ (see, e.g. [3]) and from Theorem 4.1 we get

$$\left\|M^P f\right\|_{\widetilde{M}_{p,\lambda,P}} \leq \max\{C_p, C_{p,\lambda,P}\}\|f\|_{\widetilde{M}_{p,\lambda,P}}. \tag{4.7}$$

□

Let $p = 1$. By the boundedness of M^P from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ (see, e.g., [3]) and from Theorem 4.1 we have

$$\left\|M^P f\right\|_{W\widetilde{M}_{1,\lambda,P}} \leq \max\{C_1, C_{1,\lambda,P}\}\|f\|_{\widetilde{M}_{1,\lambda,P}}. \tag{4.8}$$

5. Proof of Main Results

Proof of Theorem 2.1. (1) *Sufficiency.* Let $r > 0$, $0 < \alpha < \gamma$, $0 < \lambda < \gamma - \alpha$, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$, and $1 < p < (\gamma - \lambda)/\alpha$. Then

$$\begin{aligned} M_\alpha^P f(x) &\leq \sup_{t\leq r}|\mathcal{E}(x,t)|^{-1+\alpha/\gamma}\int_{\mathcal{E}(x,t)}|f(y)|dy \\ &\quad + \sup_{t>r}|\mathcal{E}(x,t)|^{-1+\alpha/\gamma}\int_{\mathcal{E}(x,t)}|f(y)|dy \equiv A(x,r) + C(x,r). \end{aligned} \tag{5.1}$$

□

For $A(x,r)$ we get

$$\begin{aligned} |A(x,r)| &\lesssim \sup_{t\leq r} t^\alpha|\mathcal{E}(x,t)|^{-1}\int_{\mathcal{E}(x,t)}|f(y)|dy \\ &\leq r^\alpha M^P f(x). \end{aligned} \tag{5.2}$$

By the Hölder inequality

$$\begin{aligned} C(x,r) &\approx \sup_{t>r} t^{\alpha-\gamma}\int_{\mathcal{E}(x,t)}|f(y)|dy \\ &\leq \min\left\{r^{\alpha-\gamma/p}\|f\|_{L_p}, r^{\alpha-(\gamma-\lambda)/p}\|f\|_{M_{p,\lambda,P}}\right\}. \end{aligned} \tag{5.3}$$

Thus for all $r > 0$

$$\left|M_\alpha^P f(x)\right| \lesssim \min\left\{r^\alpha M^P f(x) + r^{\alpha-\gamma/p}\|f\|_{\widetilde{M}_{p,\lambda,P}}, r^\alpha M^P f(x) + t^{\alpha-(\gamma-\lambda)/p}\|f\|_{\widetilde{M}_{p,\lambda,P}}\right\}. \tag{5.4}$$

Minimizing with respect to r, at

$$r = \left[\left(M^P f(x)\right)^{-1} \|f\|_{\widetilde{M}_{p,\lambda,P}}\right]^{p/(\gamma-\lambda)},$$
$$r = \left[\left(M^P f(x)\right)^{-1} \|f\|_{\widetilde{M}_{p,\lambda,P}}\right]^{p/n}, \tag{5.5}$$

we have

$$\left|M_\alpha^P f(x)\right| \lesssim \min\left\{\left(\frac{M^P f(x)}{\|f\|_{\widetilde{M}_{p,\lambda,P}}}\right)^{1-p\alpha/(\gamma-\lambda)}, \left(\frac{M^P f(x)}{\|f\|_{\widetilde{M}_{p,\lambda,P}}}\right)^{1-p\alpha/\gamma}\right\} \|f\|_{\widetilde{M}_{p,\lambda,P}}. \tag{5.6}$$

Then

$$\left|M_\alpha^P f(x)\right| \lesssim \left(M^P f(x)\right)^{p/q} \|f\|_{\widetilde{M}_{p,\lambda,P}}^{1-p/q}. \tag{5.7}$$

Hence, by Theorem 4.2, we have

$$\int_{\mathcal{E}(x,t)} \left|M_\alpha^P f(y)\right|^q dy \lesssim \|f\|_{\widetilde{M}_{p,\lambda,P}}^{q-p} \int_{\mathcal{E}(x,t)} \left(M^P f(y)\right)^p dy$$
$$\lesssim [t]_1^\lambda \|f\|_{\widetilde{M}_{p,\lambda,P}}^q, \tag{5.8}$$

which implies that M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

Necessity. Let $1 < p < \gamma - \lambda/\alpha$, $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ and assume that M_α^P is bounded from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1,t\}$. Then

$$\begin{aligned}
\|f_t\|_{\widetilde{M}_{p,\lambda,P}} &= \sup_{r>0,\, x\in\mathbb{R}^n} \left([r]_1^{-\lambda} \int_{\mathcal{E}(x,r)} |f_t(y)|^p dy\right)^{1/p} \\
&= t^{-\gamma/p} \sup_{x\in\mathbb{R}^n,\, r>0} \left([r]_1^{-\lambda} \int_{\mathcal{E}(x,tr)} |f(y)|^p dy\right)^{1/p} \\
&= t^{-\gamma/p} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/p} \sup_{r>0,\, x\in\mathbb{R}^n} \left([tr]_1^{-\lambda} \int_{\mathcal{E}(x,tr)} |f(y)|^p dy\right)^{1/p} \\
&= t^{-\gamma/p} [t]_{1,+}^{\lambda/p} \|f\|_{\widetilde{M}_{p,\lambda,P}},
\end{aligned}$$
$$M_\alpha^P f_t(x) = t^{-\alpha} M_\alpha^P f(tx),$$

$$\begin{aligned}\left\|M_\alpha^P f_t\right\|_{\widetilde{M}_{q,\lambda,P}} &= t^{-\alpha}\sup_{x\in\mathbb{R}^n,\, r>0}\left([r]_1^{-\lambda}\int_{\mathcal{E}(x,r)}\left|M_\alpha^P f(ty)\right|^q dy\right)^{1/q}\\ &= t^{-\alpha-\gamma/q}\sup_{r>0}\left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/q}\sup_{r>0,\, x\in\mathbb{R}^n}\left([tr]_1^{-\lambda}\int_{\mathcal{E}(tx,tr)}\left|M_\alpha^P f(y)\right|^q dy\right)^{1/q}\\ &= t^{-\alpha-\gamma/q}[t]_{1,+}^{\lambda/q}\left\|M_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}}.\end{aligned}\tag{5.9}$$

By the boundedness of M_α^P from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$

$$\begin{aligned}\left\|M_\alpha^P f\right\|_{\widetilde{L}_{q,\lambda,P}} &= t^{\alpha+\gamma/q}[t]_{1,+}^{-\lambda/q}\left\|M_\alpha^P f_t\right\|_{\widetilde{L}_{q,\lambda,P}}\\ &\le t^{\alpha+\gamma/q}[t]_{1,+}^{-\lambda/q}\|f_t\|_{\widetilde{L}_{p,\lambda,P}}\\ &\lesssim t^{\alpha+\gamma/q-\gamma/p}[t]_{1,+}^{\lambda/p-\lambda/q}\|f\|_{\widetilde{M}_{p,\lambda,P}}.\end{aligned}\tag{5.10}$$

If $1/p < 1/q + \alpha/\gamma$, then by letting $t \to 0$ we have $\|M_\alpha^P f\|_{\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$.

As well as if $1/p > 1/q + \alpha/(\gamma-\lambda)$, then at $t \to \infty$ we obtain $\|M_\alpha^P f\|_{\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$.

Therefore $\alpha/\gamma \le 1/p - 1/q \le \alpha/(\gamma-\lambda)$.

(2) *Sufficiency.* Let $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$. From (5.1) we have

$$\left|\left\{y\in\mathcal{E}(x,t):\left|M_\alpha^P f(y)\right|>2\beta\right\}\right|\le\left|\left\{y\in\mathcal{E}(x,t):\left|A(y,t)\right|>\beta\right\}\right|+\left|\left\{y\in\mathcal{E}(x,t):\left|C(y,t)\right|>\beta\right\}\right|.\tag{5.11}$$

Then

$$\begin{aligned}C(y,t) &\approx \sup_{\tau>t} t^{\alpha-\gamma}\int_{\mathcal{E}(y,\tau)}|f(y)|dy\\ &\le \min\left\{t^{\alpha-\gamma}\|f\|_{L_1}, t^{\alpha-(\gamma-\lambda)}\|f\|_{M_{1,\lambda,P}}\right\}.\\ &\lesssim [t]_1^{\lambda}t^{\alpha-\gamma}\|f\|_{\widetilde{M}_{1,\lambda,P}}.\end{aligned}\tag{5.12}$$

Taking into account inequality (5.2) and Theorem 4.2, we have

$$\begin{aligned}&\left|\{y\in\mathcal{E}(x,t):|A(y,t)|>\beta\}\right|\\ &\quad\le\left|\left\{y\in\mathcal{E}(x,t):M^P f(y)>\frac{\beta}{C_1 t^{\alpha}}\right\}\right|\\ &\quad\le\frac{C_2 t^{\alpha}}{\beta}\cdot[t]_1^{\lambda}\|f\|_{\widetilde{M}_{1,\lambda,P}},\end{aligned}\tag{5.13}$$

where $C_2 = C_1 \cdot C_{1,\lambda,P}$ and thus if $C_2[2t]_1^{\lambda} t^{\alpha-\gamma}\|f\|_{\widetilde{M}_{1,\lambda,P}} = \beta$, then $|C(y,t)| \le \beta$ and consequently, $|\{y \in \mathcal{E}(x,t) : |C(y,t)| > \beta\}| = 0$.

In the case $2t < 1$

$$C_2[2t]_1^{\lambda} t^{\alpha-\gamma}\|f\|_{\widetilde{M}_{1,\lambda,P}} = \beta \Longleftrightarrow \frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta} \approx t^{\gamma-\alpha-\lambda}$$
$$\Longleftrightarrow t^{\alpha} \approx \left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\alpha/(\gamma-\lambda-\alpha)} \tag{5.14}$$
$$\Longleftrightarrow t^{\alpha}\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta} \approx \left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{(\gamma-\lambda)/(\gamma-\lambda-\alpha)},$$

then

$$\left|\left\{y \in \mathcal{E}(x,t) : \left|M_{\alpha}^{P}f(y)\right| > 2\beta\right\}\right| \lesssim \frac{1}{\beta}\ [t]_1^{\lambda}\ t^{\alpha}\ \|f\|_{\widetilde{M}_{1,\lambda,P}}$$
$$\approx [t]_1^{\lambda}\left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{(\gamma-\lambda)/(\gamma-\lambda-\alpha)}. \tag{5.15}$$

In the case $2t \ge 1$

$$C_2[2t]_1^{\lambda} t^{\alpha-\gamma}\|f\|_{\widetilde{M}_{1,\lambda,P}} = \beta \Longleftrightarrow \frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta} \approx t^{\gamma-\alpha}$$
$$\Longleftrightarrow t^{\alpha} \approx \left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\alpha/(\gamma-\alpha)} \tag{5.16}$$
$$\Longleftrightarrow t^{\alpha}\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta} \approx \left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\alpha/(\gamma-\alpha)},$$

then

$$\left|\left\{y \in \mathcal{E}(x,t) : \left|M_{\alpha}^{P}f(y)\right| > 2\beta\right\}\right| \lesssim \frac{1}{\beta}\ [t]_1^{\lambda}\ t^{\alpha}\ \|f\|_{\widetilde{M}_{1,\lambda,P}}$$
$$\approx [t]_1^{\lambda}\left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\gamma/(\gamma-\alpha)}. \tag{5.17}$$

Finally we have

$$\left|\left\{y \in \mathcal{E}(x,t) : \left|M_{\alpha}^{P}f(y)\right| > 2\beta\right\}\right|$$
$$\lesssim [t]_1^{\lambda}\min\left\{\left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{(\gamma-\lambda)/(\gamma-\lambda-\alpha)}, \left(\frac{\|f\|_{\widetilde{M}_{1,\lambda,P}}}{\beta}\right)^{\gamma/(\gamma-\alpha)}\right\} \tag{5.18}$$
$$\lesssim [t]_1^{\lambda}\left(\frac{1}{\beta}\|f\|_{\widetilde{M}_{1,\lambda,P}}\right)^{q}.$$

Necessity. Let M_α^P be bounded from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$. We have

$$\begin{aligned}\left\|M_\alpha^P f_t\right\|_{W\widetilde{L}_{q,\lambda,P}} &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\tau>0}\left([\tau]_1^{-\lambda}\int_{\{y\in\mathcal{E}(x,\tau):|M_\alpha^P f_t(y)|>r\}} dy\right)^{1/q} \\ &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\tau>0}\left([\tau]_1^{-\lambda}\int_{\{y\in\mathcal{E}(tx,\tau):|M_\alpha^P f(ty)|>rt^\alpha\}} dy\right)^{1/q} \\ &= t^{-\alpha-\gamma/q}\sup_{\tau>0}\left(\frac{[t\tau]_1}{[\tau]_1}\right)^{\lambda/q}\sup_{r>0} rt^\alpha \\ &\quad\times \sup_{x\in\mathbb{R}^n,\tau>0}\left([t\tau]_1^{-\lambda}\int_{\{y\in\mathcal{E}(x,t\tau):|M_\alpha^P f(y)|>rt^\alpha\}} dy\right)^{1/q} \\ &= t^{-\alpha-\gamma/q}[t]_{1,+}^{\lambda/q}\left\|M_\alpha^P f\right\|_{W\widetilde{M}_{q,\lambda,P}}.\end{aligned} \tag{5.19}$$

By the boundedness of M_α^P from $\widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$ to $W\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$

$$\left\|M_\alpha^P f\right\|_{W\widetilde{L}_{q,\lambda,P}} \le C_{1,q,\lambda} t^{\alpha+\gamma/q-n}[t]_{1,+}^{\lambda-\lambda/q}\|f\|_{\widetilde{M}_{1,\lambda,P}}, \tag{5.20}$$

where $C_{1,q,\lambda}$ depends only on q, λ, and n.

If $1 < 1/q + \alpha/\gamma$, then by letting $t \to 0$ we have $\|M_\alpha^P f\|_{W\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$, which is impossible.

Similarly, if $1 > 1/q + \alpha/(\gamma-\lambda)$, then for $t \to \infty$ we obtain $\|M_\alpha^P f\|_{W\widetilde{M}_{q,\lambda,P}} = 0$ for all $f \in \widetilde{M}_{1,\lambda,P}(\mathbb{R}^n)$, which is impossible.

Therefore $\alpha/\gamma \le 1 - 1/q \le \alpha/(\gamma-\lambda)$.

Thus Theorem 2.1 is proved.

Proof of Theorem 2.2. By the definition of the parabolic Besov-modified Morrey spaces on $\mathbb{R}^n$ it suffices to show that

$$\left\|\tau_y M_\alpha^P f - M_\alpha^P f\right\|_{\widetilde{M}_{q,\lambda,P}} \lesssim \|\tau_y f - f\|_{\widetilde{M}_{p,\lambda,P}}, \tag{5.21}$$

where $\tau_y f(x) = f(x+y)$.

It is easy to see that $\tau_y f$ commutes with M_α^P, that is, $\tau_y M_\alpha^P f = M_\alpha^P(\tau_y f)$. Hence we obtain

$$\left|\tau_y M_\alpha^P f - M_\alpha^P f\right| = \left|M_\alpha^P(\tau_y f) - M_\alpha^P f\right| \le M_\alpha^P\left(\left|\tau_y f - f\right|\right). \tag{5.22}$$

Taking $\widetilde{M}_{p,\lambda,P}$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of M_α^P from $\widetilde{M}_{p,\lambda,P}(\mathbb{R}^n)$ to $\widetilde{M}_{q,\lambda,P}(\mathbb{R}^n)$.

Thus the proof of the Theorem 2.2 is completed. □

6. Parabolic Schrödinger-Type Operators $V^{\mu}(\partial/\partial t - \Delta + V)^{-\beta}$ and $V^{\mu}\nabla^2(\partial/\partial t - \Delta + V)^{-\beta}$

In this section we consider the parabolic Schrödinger operator

$$\frac{\partial}{\partial t} - \Delta + V \quad \text{on } \mathbb{R}^{n+1}, \tag{6.1}$$

where $V = V(x,t)$ is a nonnegative potential which belongs to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$. Examples of such potentials are all positive polynomials but also singular functions like $\max\{|x|, t^{1/2}\}^{\alpha}$ for $\alpha > -(n+2)/q$. We prove the modified parabolic Morrey space $\widetilde{M}_{p,\lambda,P_0}(\mathbb{R}^{n+1}) \to \widetilde{M}_{q,\lambda,P_0}(\mathbb{R}^{n+1})$ estimates for the operators $V^{\mu}(\partial/\partial t - \Delta + V)^{-\beta}$ and $V^{\mu}\nabla^2(\partial/\partial t - \Delta + V)^{-\beta}$, where $P_0 = \mathrm{diag}(1,\dots,1,2)$.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [40–42]). Shen [41] studied the Schrödinger operator $-\Delta + V$, assuming that the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the L_p boundedness of the operators $(-\Delta + V)^{i\mu}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$, and $\nabla(-\Delta + V)^{-1}$. Kurata and Sugano generalized Shen's results to uniformly elliptic operators in [43]. Sugano [44] also extended some results of Shen to the operator $V^{\mu}(-\Delta + V)^{-\beta}$, $0 \le \mu \le \beta \le 1$ and $V^{\mu}\nabla(-\Delta + V)^{-\beta}$, $0 \le \mu \le 1/2 \le \beta \le 1$ and $\beta - \mu \ge 1/2$. Following Shen's approach, Gao and Jiang extend the results to the parabolic case. In [45], they consider the parabolic operator $\partial/\partial t - \Delta + V$ where $V \in B_q(\mathbb{R}^{n+1})$ is a nonnegative potential depending only on the space variables and, under the assumptions $n \ge 3$ and $p > (n+2)/2$, they proved the boundedness of $V(\partial/\partial t - \Delta + V)^{-1}$ in $L_p(\mathbb{R}^{n+1})$.

The main purpose of this section is to investigate the modified parabolic Morrey space $\widetilde{M}_{p,\lambda,P_0}(\mathbb{R}^{n+1}) \to \widetilde{M}_{q,\lambda,P_0}(\mathbb{R}^{n+1})$ boundedness of the operators

$$\begin{aligned} \mathcal{T}_1 &= V^{\mu}\left(\frac{\partial}{\partial t} - \Delta + V\right)^{-\beta}, \quad 0 \le \mu \le \beta \le 1, \\ \mathcal{T}_2 &= V^{\mu}\nabla^2\left(\frac{\partial}{\partial t} - \Delta + V\right)^{-\beta}, \quad 0 \le \mu \le \frac{1}{2} \le \beta \le 1,\ \beta - \mu \ge \frac{1}{2}. \end{aligned} \tag{6.2}$$

Note that the operator $\nabla^2(\partial/\partial t - \Delta + V)^{-1}$ in [45] is the special case of $\mathcal{T}_2$.

It is worth pointing out that we need to establish pointwise estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^{n+1}$ in [45]. And we prove the modified parabolic Morrey space $\widetilde{M}_{p,\lambda,P_0}(\mathbb{R}^{n+1}) \to \widetilde{M}_{q,\lambda,P_0}(\mathbb{R}^{n+1})$ boundedness of the parabolic fractional maximal operators.

Definition 6.1. (1) A nonnegative locally L_q integrable function V on $\mathbb{R}^{n+1}$ is said to belong to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$ $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|K|}\int_K V(y,\tau)^q dy\, d\tau\right)^{1/q} \le \frac{C}{|K|}\int_K V(y,\tau)\, dy\, d\tau \tag{6.3}$$

holds for every parabolic cylinder

$$K = K((x,t),r) = \left\{ (y,\tau) \in \mathbb{R}^{n+1} : |x_i - y_i| < r, |t-\tau| < r^2, i = 1,\dots,n \right\} \tag{6.4}$$

of center (x,t) and radius r in $\mathbb{R}^{n+1}$.

(2) Let $V = V(x,t) \geq 0$. We say $V \in B_\infty(\mathbb{R}^{n+1})$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_\infty(K)} \leq \frac{C}{|K|} \int_K V(y,\tau)\,dy\,d\tau \tag{6.5}$$

holds for every parabolic cylinder $K = K((x,t),r)$ in $\mathbb{R}^{n+1}$.

Clearly, $B_\infty(\mathbb{R}^{n+1}) \subset B_q(\mathbb{R}^{n+1})$ for $1 < q < \infty$. But it is important that the $B_q(\mathbb{R}^{n+1})$ class has a property of "self-improvement", that is, if $V \in B_q(\mathbb{R}^{n+1})$, then $V \in B_{q+\varepsilon}(\mathbb{R}^{n+1})$ for some $\varepsilon > 0$ (see [46]).

By the functional calculus, we may write, for all $0 < \beta < 1$,

$$\left(\frac{\partial}{\partial t} - \Delta + V\right)^{-\beta} = \frac{1}{\pi}\int_0^\infty \lambda^{-\beta}\left(\frac{\partial}{\partial t} - \Delta + V + \lambda\right)^{-1} d\lambda. \tag{6.6}$$

Let $f \in C_0^\infty(\mathbb{R}^{n+1})$. From

$$\left(\frac{\partial}{\partial t} - \Delta + V + \lambda\right)^{-1} f(x,t) = \int_{\mathbb{R}^{n+1}} \Gamma(x,t;y,\tau;\lambda) f(y,\tau)\,dy\,d\tau, \tag{6.7}$$

it follows that

$$\mathcal{T}_1 f(x,t) = \int_{\mathbb{R}^{n+1}} K_1(x,t;y,\tau) V(y,\tau)^\mu f(y,\tau)\,dy\,d\tau, \tag{6.8}$$

where

$$K_1(x,t;y,\tau) = \begin{cases} \dfrac{1}{\pi}\int_0^\infty \lambda^{-\beta}\Gamma(x,t;y,\tau;\lambda)\,d\lambda & \text{for } 0 < \beta < 1 \\ \Gamma(x,t;y,\tau;0) & \text{for } \beta = 1. \end{cases} \tag{6.9}$$

The following two pointwise estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ which were proved in [42], Lemma 3.2 with the potential $V \in B_\infty(\mathbb{R}^{n+1})$.

Theorem A. *Suppose $V \in B_\infty(\mathbb{R}^{n+1})$ and $0 \leq \mu \leq \beta \leq 1$. Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$*

$$|\mathcal{T}_1 f(x,t)| \lesssim M_\alpha^{P_0} f(x,t), \tag{6.10}$$

where $\alpha = 2(\beta - \mu)$.

Theorem B. *Suppose* $V \in B_\infty(\mathbb{R}^{n+1})$, $0 \le \mu \le 1/2 \le \beta \le 1$ *and* $\beta - \mu \ge 1/2$. *Then, for any* $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\left|\mathcal{T}_2 f(x,t)\right| \lesssim M_\alpha^{P_0} f(x,t), \tag{6.11}$$

where $\alpha = 2(\beta - \mu) - 1$.

Note that the similar estimates for the adjoint operators T_1^* and T_2^* with the potential $V \in B_{q_1}$ for some $q_1 > (n+2)/2$ are also valid (see [47]).

Theorem C. *Suppose* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for some* $q_1 > (n+2)/2$, $0 \le \mu \le \beta \le 1$ *and let* $1/q_2 = 1 - \mu/q_1$. *Then there exists a constant* $C > 0$ *such that*

$$\left|T_1^* f(x,t)\right| \le C\left(M_{\alpha q_2}^{P_0}\left(|f|^{q_2}\right)(x,t)\right)^{1/q_2}, \quad f \in C_0^\infty\left(\mathbb{R}^{n+1}\right), \tag{6.12}$$

where $\alpha = 2(\beta - \mu)$.

Theorem D. *Suppose* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for some* $q_1 > (n + 2)/2$, $0 \le \mu \le 1/2 < \beta \le 1$, *and* $\beta - \mu \ge 1/2$. *And let*

$$\frac{1}{q_2} = \begin{cases} 1 - \dfrac{\mu}{q_1}, & if \ q_1 > n+2, \\ 1 - \dfrac{\mu+1}{q_1} + \dfrac{1}{n+2}, & if \ \dfrac{n+2}{2} < q_1 < n+2. \end{cases} \tag{6.13}$$

Then there exists a constant $C > 0$ such that

$$\left|T_2^* f(x,t)\right| \le C\left(M_{\alpha q_2}^{P_0}\left(|f|^{q_2}\right)(x,t)\right)^{1/q_2}, \quad f \in C_0^\infty\left(\mathbb{R}^{n+1}\right), \tag{6.14}$$

where $\alpha = 2(\beta - \mu) - 1$.

The above theorems will yield the modified parabolic Morrey estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$.

Corollary 6.2. *Assume that* $V \in B_\infty(\mathbb{R}^{n+1})$, *and* $0 \le \mu \le \beta \le 1$. *Let* $1 \le p \le q < \infty$, $0 \le \lambda < n + 2 - 2(\beta - \mu)$, *and*

$$\frac{2(\beta-\mu)}{n+2} \le \frac{1}{p} - \frac{1}{q} \le \frac{2(\beta-\mu)}{n+2-\lambda}. \tag{6.15}$$

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\begin{aligned} \left\|\mathcal{T}_1 f\right\|_{\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{p,\lambda,P_0}} \quad \text{for } p > 1, \\ \left\|\mathcal{T}_1 f\right\|_{W\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{1,\lambda,P_0}} \quad \text{for } p = 1. \end{aligned} \tag{6.16}$$

Corollary 6.3. *Assume that* $V \in B_\infty(\mathbb{R}^{n+1})$, $0 \le \mu \le 1/2 \le \beta \le 1$ *and* $\beta - \mu \ge 1/2$. *Let* $1 \le p \le q < \infty$, $0 \le \lambda < n + 1 - 2(\beta - \mu)$, *and*

$$\frac{2(\beta-\mu)-1}{n+2} \le \frac{1}{p} - \frac{1}{q} \le \frac{2(\beta-\mu)-1}{n+2-\lambda}. \tag{6.17}$$

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\begin{aligned} \|\mathcal{T}_2 f\|_{\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{p,\lambda,P_0}} \quad \text{for } p > 1, \\ \|\mathcal{T}_2 f\|_{W\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{1,\lambda,P_0}} \quad \text{for } p = 1. \end{aligned} \tag{6.18}$$

Corollary 6.4. *Assume that* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for* $q_1 > (n+2)/2$ *and* $0 \le \mu \le \beta \le 1$. *Let* $\alpha = 2(\beta - \mu)$, $1 \le p < 1/(\mu/q_1 + \alpha/(n+2))$, $\alpha/(n+2) \le 1/p - 1/q \le \alpha/(n+2-\lambda)$, $1/q_2 = 1 - \mu/q_1$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\begin{aligned} \|\mathcal{T}_1 f\|_{\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{p,\lambda,P_0}} \quad \textit{for } p > 1, \\ \|\mathcal{T}_1 f\|_{W\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{1,\lambda,P_0}} \quad \textit{for } p = 1. \end{aligned} \tag{6.19}$$

Corollary 6.5. *Assume that* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for* $q_1 > (n+2)/2$, *and*

$$\begin{aligned} 0 \le \mu \le \frac{1}{2} \le \beta \le 1 \quad &\textit{if } q_1 > n+2, \\ 0 \le \mu \le \frac{1}{2} < \beta \le 1 \quad &\textit{if } \frac{n+2}{2} < q_1 < n+2. \end{aligned} \tag{6.20}$$

Let $\alpha = 2(\beta - \mu) - 1$, $\beta - \mu \ge 1/2$, $1 \le p < 1/(\mu/q_1 + \alpha/(n+2))$, $\alpha/(n+2) \le 1/p - 1/q \le \alpha/(n+2-\lambda)$, $1/q_2 = 1 - \mu/q_1$, where

$$\frac{1}{p_1} = \begin{cases} \dfrac{\mu}{q_1}, & \text{if } q_1 > n+2, \\ \dfrac{\mu+1}{q_1} - \dfrac{1}{n+2}, & \text{if } \dfrac{n+2}{2} < q_1 < n+2. \end{cases} \tag{6.21}$$

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\begin{aligned} \|\mathcal{T}_2 f\|_{\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{p,\lambda,P_0}} \quad \text{for } p > 1, \\ \|\mathcal{T}_2 f\|_{W\widetilde{M}_{q,\lambda,P_0}} &\lesssim \|f\|_{\widetilde{M}_{1,\lambda,P_0}} \quad \text{for } p = 1. \end{aligned} \tag{6.22}$$

Acknowledgments

The authors would like to express their gratitude to the referees for their very valuable comments and suggestions. The research of V. Guliyev was partially supported by the Grant of

Science Development Foundation under the President of the Republic of Azerbaijan Project EIF-2010-1(1)-40/06-1 and by the Grant of 2011-Ahi Evran University Scientific Research Projects (BAP FBA-11-13) and by the Scientific and Technological Research Council of Turkey (TUBITAK Project no. 110T695).

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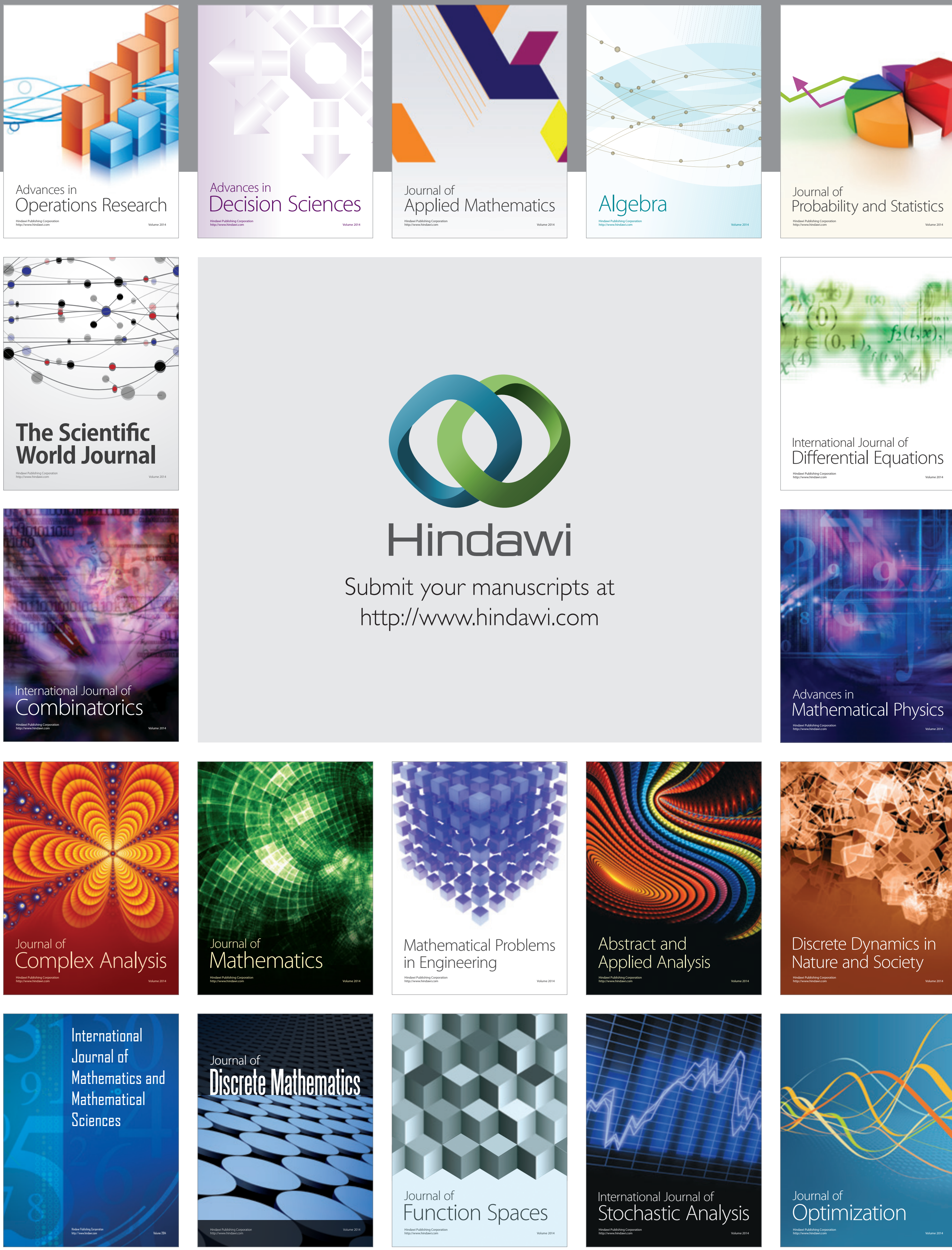
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