Set-Valued Var. Anal (2017) 25:807–828
DOI 10.1007/s11228-017-0422-y

Intrinsic Square Functions on Vanishing Generalized Orlicz-Morrey Spaces

Fatih Deringoz[1] · **Vagif S. Guliyev**[1,2,3] ·
Maria Alessandra Ragusa[4]

Received: 27 January 2017 / Accepted: 29 May 2017 / Published online: 2 June 2017

Abstract We study the boundedness of intrinsic square functions and their commutators on vanishing generalized Orlicz-Morrey spaces. In all the cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities without assuming any monotonicity property.

Keywords Vanishing generalized Orlicz-Morrey spaces · Intrinsic square functions · Commutator · BMO

Mathematics Subject Classification (2010) 42B20 · 42B35 · 46E30

Dedicated to Professor Michel Théra in honor of his 70th birthday with gratitude and friendship

Advances in Monotone Operator Theory and Optimization

✉ Maria Alessandra Ragusa
maragusa@dmi.unict.it

Fatih Deringoz
fderingoz@ahievran.edu.tr

Vagif S. Guliyev
vagif@guliyev.com

1 Department of Mathematics, Ahi Evran University, Kirsehir, Turkey

2 Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan

3 S.M. Nikolskii Institute of Mathematics at RUDN University, 6 Miklukho-Maklay St, Moscow, 117198, Russia

4 Dipartimento di Matematica e Informatica, Universitá di Catania, Catania, Italy

1 Introduction

As is well known, Morrey spaces are widely used to investigate the local behavior of solutions to second order elliptic and parabolic partial differential equations and systems. Recall that the classical Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ are defined by

$$\mathcal{M}^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\rm loc}(\mathbb{R}^n) : \|f\|_{\mathcal{M}^{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty \right\},$$

where $0 \le \lambda \le n$, $1 \le p < \infty$.

The spaces $\mathcal{M}^{p,\varphi}$, equipped with the norms

$$\|f\|_{\mathcal{M}^{p,\varphi}} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L^p(B(x,r))}, \tag{1}$$

where φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, are known as *generalized Morrey spaces*.

Orlicz spaces L^Φ (see Definition 2.2), generalizations of Lebesgue spaces L^p, introduced in [41, 42] are useful tools in harmonic analysis and its applications. For example, the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, but not on L^1, but using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$, see [4, 30, 31] and [5] for more precise statements.

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n),$$

where the "Morrey-type measuring"of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were introduced by Nakai in [37]. The boundedness of operators of harmonic analysis, in Orlicz-Morrey spaces, is studied in [8, 11, 18, 21, 23, 26, 38, 39, 46, 48].

We point out that our definition of Orlicz-Morrey spaces (see [8]) is different from the one used by Nakai [37] and Sawano et al. [48]. In words of [22], our generalized Orlicz-Morrey space is the third kind and the ones in [37] and [48] are the first kind and the second kind, respectively. According to the examples in [12], one can say that the generalized Orlicz-Morrey space of the first kind and the second kind are different. Notice that the definition of the space of the third kind relies only on the fact that L^Φ is a normed linear space, which is independent of the condition that it is generated by modulars. On the other hand, the spaces of the first and the second kind are defined via the family of modulars.

Morrey and Orlicz-Morrey spaces are not separable due to the L^∞-norm with respect to r and x. The closure of nice functions in the Morrey norm gives a subspace known under the name of vanishing Morrey space, see [45]. Such space, corresponding to the classical Morrey space, $\mathcal{M}^{p,\lambda}$, is in connection with Partial Differential Equations in [49, 50], they were also used in [44]. The vanishing generalized Morrey spaces were studied in [43] and references therein.

The intrinsic square functions were first introduced by Wilson in [55, 56]. They are defined as follows. For $0 < \alpha \le 1$, let C_α be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that the support of ϕ is contained in $\{x : |x| \le 1\}$, $\int \phi dx = 0$, and for $x,\ x' \in \mathbb{R}^n$,

$$|\phi(x) - \phi(x')| \le |x - x'|^\alpha.$$

For $(y,\ t) \in \mathbb{R}^{n+1}_{+}$ and $f \in L^1_{loc}(\mathbb{R}^n)$, set

$$A_\alpha f(t,\ y) \equiv \sup_{\phi \in C_\alpha} |f * \phi_t(y)|,$$

where $\phi_t(y) = t^{-n}\phi(\frac{y}{t})$. Then, we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x) = \left(\int\int_{\Gamma_\beta(x)} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

where $\Gamma_\beta(x) = \{(y,\ t) \in \mathbb{R}^{n+1}_{+} : |x-y| < \beta t\}$. If $\beta = 1$ we use the following notation $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as the Poisson kernel, and dominates pointwise both the classical square function (the Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ is dependent of kernels with uniform compact support, there is a pointwise relation between $G_{\alpha,\beta}(f)$ with different β:

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x) .$$

We can see details in [55].

The intrinsic Littlewood-Paley g-function and the intrinsic g^*_λ function are defined respectively by

$$g_\alpha f(x) = \left(\int_0^\infty (A_\alpha f(t,y))^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$g^*_{\lambda,\alpha} f(x) = \left(\int\int_{\mathbb{R}^{n+1}_{+}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

Let b be a locally integrable function on $\mathbb{R}^n$. Setting

$$A_{\alpha,b} f(t,y) \equiv \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} [b(x)-b(z)]\phi_t(y-z) f(z)dz \right|,$$

the commutators are defined by

$$[b, G_\alpha]f(x) = \left(\int\int_{\Gamma(x)} (A_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

$$[b, g_\alpha]f(x) = \left(\int_0^\infty (A_{\alpha,b} f(t,y))^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

and

$$[b, g^*_{\lambda,\alpha}]f(x) = \left(\int\int_{\mathbb{R}^{n+1}_{+}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} (A_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

Wilson [55] proved that G_α is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 < \alpha \le 1$. After then, Huang and Liu [27] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [52] and [53], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [51] and [40] Wang and Nakamura considered intrinsic functions and the commutators generated with *Bounded Mean Oscillation* functions (BMO) (see [29]) on weighted Morrey spaces. In [20], Guliyev and Shukurov proved the boundedness of intrinsic square functions and their commutators on generalized Morrey spaces. In [19], Guliyev and Omarova considered vector-valued intrinsic square functions and their commutators generated with BMO functions on vector-valued generalized weighted Morrey spaces. In [35] and [36], Liang, Lu et al. studied the boundedness of these operators on Musielak-Orlicz Morrey and Campaanto spaces.

In this paper, we will consider $G_\alpha,\ g_\alpha,\ g^*_{\lambda,\alpha}$ and their commutators on vanishing generalized Orlicz-Morrey spaces. More precisely, we find sufficient conditions on general Young function Φ and functions φ_1, φ_2 which ensure the boundedness of the operators under consideration from one vanishing generalized Orlicz-Morrey spaces $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $V\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$. Our results, in the setting of Orlicz-Morrey spaces, are new, even in the case of non-vanishing spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)|$ for the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

2 Preliminaries

We recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0,+\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0}\Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty}\Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Definition 2.2 (Orlicz Space). For a Young function Φ, the set

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called a Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The space $L^\Phi_{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L^\Phi(\mathbb{R}^n)$, for all balls $B \subset \mathbb{R}^n$, where χ_B is the characteristic function of B. We refer e.g. to [32, 33, 47] for the theory of Orlicz Spaces.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leq 1\right\}.$$

We note that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi}}\Big)dx \leq 1.$$

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty. \tag{2}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

A Young function Φ is said to satisfy the Δ'-condition, denoted also by $\Phi \in \Delta'$, if

$$\Phi(rt) \le c\Phi(r)\Phi(t), \qquad r, t \ge 0,$$

for some positive constant c. Note that, each element of Δ'-class is also an element of Δ_2-class.

Definition 2.3 A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$(resp. $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \le Ct^p\Phi(s).$$

Remark 2.1 We know that if Φ is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely, if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$ (see for example [32]).

Lemma 2.1 [34] *Let Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 \le p_0 \le p_1 < \infty$. Let $\widetilde{C}$ be a positive constant. Then, there exists a positive constant C such that, for any ball B of $\mathbb{R}^n$ and $\mu \in (0, \infty)$, the inequality*

$$\int_B \Phi\left(\frac{|f(x)|}{\mu}\right) dx \le \widetilde{C}$$

implies that $\|f\|_{L^\Phi(B)} \le C\mu$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ +\infty, & r = +\infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{3}$$

Note that Young functions satisfy the properties

$$\begin{cases} \Phi(\alpha t) \le \alpha\Phi(t), & \text{if } \ 0 \le \alpha \le 1 \\ \Phi(\alpha t) \ge \alpha\Phi(t), & \text{if } \quad \alpha > 1 \end{cases} \quad \text{and} \quad \begin{cases} \Phi^{-1}(\alpha t) \ge \alpha\Phi^{-1}(t), & \text{if } \ 0 \le \alpha \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha\Phi^{-1}(t), & \text{if } \quad \alpha > 1. \end{cases}$$

The following analogue of the Hölder inequality is known, see [54].

Theorem 2.1 [54] *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\|fg\|_{L^1(\mathbb{R}^n)} \le 2\|f\|_{L^\Phi}\|g\|_{L^{\widetilde{\Phi}}}.$$

The following lemma is valid. See for example [1] for the proof.

Lemma 2.2 *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{L^\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.1, Lemma 2.2 and (3).

Lemma 2.3 *For a Young function Φ and $B = B(x,r)$, the following inequality is valid*

$$\|f\|_{L^1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L^\Phi(B)}.$$

The following theorem is an analogue of Lebesgue differentiation theorem in Orlicz spaces.

Theorem 2.2 [24] *Suppose that Φ is a Young function and let $f \in L^\Phi(\mathbb{R}^n)$ be nonnegative. Then*

$$\liminf_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \ge f(x), \quad \textit{for almost every } x \in \mathbb{R}^n. \tag{4}$$

If we moreover assume that $\Phi \in \Delta'$, then

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = f(x), \quad \textit{for almost every } x \in \mathbb{R}^n. \tag{5}$$

Definition 2.4 (generalized Orlicz-Morrey Space) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space. It is the space of all functions $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\|f\|_{L^\Phi(B(x,r))}.$$

According to this definition, we recover the generalized Morrey space $\mathcal{M}^{p,\varphi}$ under the choice $\Phi(r) = r^p$, $1 \le p < \infty$.

The following lemma was proved in [10] (see also [9]).

Lemma 2.4 *Let Φ be a Young function and φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$.*

(i) *If*

$$\sup_{t<r<\infty} \varphi(x,r)^{-1} = \infty \quad \textit{for some } t > 0 \textit{ and for all } x \in \mathbb{R}^n,$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

(ii) *If*

$$\sup_{0<r<t} \frac{1}{\varphi(x,r)\, \Phi^{-1}(|B(x,r)|^{-1})} = \infty \quad \textit{for some } t > 0 \textit{ and for almost all } x \in \mathbb{R}^n,$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.2 For the case Lemma 2.4 (ii), we impose the condition $\Phi \in \Delta'$ in [9, 10] since we use (5) in the proof. But this condition is superfluous. It is enough to use (4) to prove this fact, the details being omitted.

Remark 2.3 Let Φ be a Young function. We denote by Ω_Φ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \|\varphi(x,r)^{-1}\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{1}{\varphi(x,r)\, \Phi^{-1}(|B(x,r)|^{-1})} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.4, we always assume that $\varphi \in \Omega_\Phi$.

Extending the definition of vanishing generalized Morrey spaces to the case of Orlicz-Morrey spaces, we introduce the following definition.

Definition 2.5 (vanishing generalized Orlicz-Morrey Space) The vanishing generalized Orlicz-Morrey space $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi(x,r)} = 0. \tag{6}$$

Remark 2.4 Let Φ be a Young function. We denote by Ω_Φ^V the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that,

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \varphi(x,r) > 0, \text{ for all } \delta > 0 \tag{7}$$

and

$$\lim_{r\to 0} \frac{1}{\Phi^{-1}(r^{-n}) \inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0. \tag{8}$$

Taking into account Lemma 2.4 for the non-triviality of the space $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_\Phi^V$. For details, see [10].

The space $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ is a closed subspace of the Banach space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, by standard means.

We will also use the notation

$$\mathfrak{A}_{\Phi,\varphi}(f;x,r) := \frac{\|f\|_{L^{\Phi}(B(x,r))}}{\varphi(x,r)}$$

for brevity, so that

$$V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \left\{ f \in \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}(f;x,r) = 0 \right\}.$$

3 Intrinsic Square Functions in the Spaces $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$

The known boundedness statement for G_α on Orlicz spaces runs as follows.

Theorem 3.1 [35] *Let $\alpha \in (0,1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$. Then, G_α is bounded from $L^{\Phi}(\mathbb{R}^n)$ to itself.*

The following theorems was proved by Liang, Y., Nakai, E., Yang, D., Zhang, J. in [35].

Theorem 3.2 [35] *Let $\alpha \in (0,1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $x \in \mathbb{R}^n$ and $0 < r \le s < \infty$,*

$$\int_r^\infty \varphi(x,t)\frac{dt}{t} \le \widetilde{C}\,\varphi(x,r),\ \ \varphi(x,s) \le \widetilde{C}\varphi(x,r) \text{ and } \varphi(x,r)r \le \widetilde{C}\varphi(x,s)s. \tag{9}$$

Then, the operators G_α and g_α are bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Theorem 3.3 [35] *Let $\alpha \in (0,1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$ and suppose that the function $\varphi(x,r)$ satisfies the conditions (9). Then, for $\lambda > 3 + \frac{2\alpha}{n}$, the operator $g^*_{\lambda,\alpha}$ is bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

Given a weight w, will use the following statements on the boundedness of the weighted Hardy operators

$$H^*_w g(r) := \int_r^\infty g(s)w(s)ds, \quad r \in (0,\infty)$$

and

$$H^*_w g(r) := \int_r^\infty \left(1+\ln\frac{t}{r}\right) g(t)w(t)dt, \quad r \in (0,\infty).$$

The following theorem was proved in [17].

Theorem 3.4 *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{r>0} v_2(r)H^*_w g(r) \le C \sup_{r>0} v_1(r)g(r) \tag{10}$$

holds for some $C > 0$ and all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (10).

The following theorem was proved in [16].

Theorem 3.5 *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{r>0} v_2(r) H_w^* g(r) \le C \sup_{r>0} v_1(r) g(r) \tag{11}$$

holds for some $C > 0$ and all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (11).

The following lemma was generalization of the Guliyev lemma [13–15] for Orlicz spaces.

Lemma 3.1 *Let $\alpha \in (0,1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, $f \in L^{\Phi}_{loc}(\mathbb{R}^n)$, $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n$ and $r > 0$. Then, for the operator G_α, the following inequality is valid*

$$\|G_\alpha f\|_{L^\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{12}$$

Proof With the notation $2B = B(x_0, 2r)$, we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y),$$

and then

$$\|G_\alpha f\|_{L^\Phi(B)} \le \|G_\alpha f_1\|_{L^\Phi(B)} + \|G_\alpha f_2\|_{L^\Phi(B)}.$$

Since $f_1 \in L^\Phi(\mathbb{R}^n)$, by Theorem 3.1, it follows that

$$\|G_\alpha f_1\|_{L^\Phi(B)} \le \|G_\alpha f_1\|_{L^\Phi(\mathbb{R}^n)} \le C\|f_1\|_{L^\Phi(\mathbb{R}^n)} = C\|f\|_{L^\Phi(2B)}.$$

Then let us estimate $\|G_\alpha f_2\|_{L^\Phi(B)}$.

$$|f_2 * \phi_t(y)| = \left| t^{-n} \int_{|y-z|\le t} \phi\Big(\frac{y-z}{t}\Big) f_2(z)dz \right| \le t^{-n} \int_{|y-z|\le t} |f_2(z)|dz.$$

Since $x \in B(x_0, r)$, $(y,t) \in \Gamma(x)$, we have $|x-z| \le |z-y| + |x-y| \le 2t$, and

$$r \le |x_0 - z| - |x_0 - x| \le |x - z| \le |x - y| + |y - z| \le 2t.$$

So, we obtain

$$\begin{aligned} G_\alpha f_2(x) &\le \left(\int\int_{\Gamma(x)} \left| t^{-n}\int_{|y-z|\le t} |f_2(z)|dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\ &\le \left(\int_{t>r/2}\int_{|x-y|<t}\left(\int_{|x-z|\le 2t}|f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{t>r/2}\left(\int_{|x-z|\le 2t}|f_2(z)|dz\right)^2 \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}. \end{aligned}$$

By Minkowski's inequality and the relations $|x-z| \ge |x_0-z| - |x_0-x| \ge \frac{1}{2}|x_0-z|$, we have

$$\begin{aligned} G_\alpha f_2(x) &\lesssim \int_{\mathbb{R}^n}\left(\int_{t>\frac{|x-z|}{2}} \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} |f_2(z)|dz \\ &\lesssim \int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^n}dz \lesssim \int_{|x_0-z|>2r} \frac{|f(z)|}{|x_0-z|^n}dz \\ &= \int_{|x_0-z|>2r} |f(z)| \int_{|x_0-z|}^{+\infty} \frac{dt}{t^{n+1}}dz \\ &= \int_{2r}^{\infty}\int_{2r<|x_0-z|<t} |f(z)|dz\, \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}. \end{aligned} \tag{13}$$

The last inequality follows from Lemma 2.3. Moreover,

$$\|G_\alpha f_2\|_{L^\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

Thus

$$\|G_\alpha f\|_{L^\Phi(B)} \lesssim \|f\|_{L^\Phi(2B)} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

On the other hand, by (3) we get

$$\begin{aligned} \Phi^{-1}\big(r^{-n}\big) &\approx \Phi^{-1}\big(r^{-n}\big)r^n \int_{2r}^{\infty}\frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \end{aligned}$$

and then

$$\|f\|_{L^\Phi(2B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}. \tag{14}$$

Thus

$$\|G_\alpha f\|_{L^\Phi(B)} \lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

□

Theorem 3.6 *Let $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, $\varphi_1, \varphi_2 \in \Omega_\Phi$ and suppose that the functions (φ_1, φ_2) and Φ satisfy the condition*

$$\int_r^\infty \Big(\operatorname*{ess\ inf}_{t<s<\infty} \varphi_1(x, s) \Big) \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C \, \Phi^{-1}\big(r^{-n}\big) \varphi_2(x, r), \tag{15}$$

where C does not depend on x and r. Then G_α is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof By Lemma 3.1 and Theorem 3.4 we have

$$\begin{aligned} \|G_\alpha f\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)} \int_r^\infty \Phi^{-1}\big(t^{-n}\big) \|f\|_{L^\Phi(B(x,t))} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x, r)^{-1} \|f\|_{L^\Phi(B(x,r))} \\ &= \|f\|_{\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)}. \end{aligned}$$

□

In [55], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 3.6, we have the following result.

Corollary 3.1 *Let $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, $\varphi_1, \varphi_2 \in \Omega_\Phi$ and suppose that the functions (φ_1, φ_2) and Φ satisfy the condition (15), then g_α is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.*

Theorem 3.7 *Let $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, $\varphi_1, \varphi_2 \in \Omega_\Phi^V$, suppose that the functions (φ_1, φ_2) and Φ satisfy the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x, t) \frac{\Phi^{-1}\big(t^{-n}\big)}{t} dt < \infty \tag{16}$$

for every $\delta > 0$, and

$$\frac{1}{\varphi_2(x, r)\Phi^{-1}\big(r^{-n}\big)} \int_r^\infty \varphi_1(x, t) \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C_0, \tag{17}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then G_α is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof Since the norm inequality is already provided by Theorem 3.6, so we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi_1}(f; x, r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi_2}(G_\alpha f; x, r) = 0, \tag{18}$$

To check (18), i.e. to show that

$$\sup_{x\in\mathbb{R}^n} \frac{\|G_\alpha f\|_{L^\Phi(B(x,r))}}{\varphi_2(x, r)} < \varepsilon \quad \text{for small } r,$$

we use the estimate (12) where we split the right-hand side:

$$\frac{\|G_\alpha f\|_{L^\Phi(B(x,r))}}{\varphi_2(x, r)} \le C[I_{\delta_0}(x, r) + J_{\delta_0}(x, r)], \tag{19}$$

with $\delta_0 > 0$ and $r < \delta_0$, where

$$I_{\delta_0}(x,r) := \frac{1}{\Phi^{-1}(r^{-n})\varphi_2(x,r)} \int_r^{\delta_0} \frac{\Phi^{-1}\big(t^{-n}\big)}{t} \|f\|_{L^\Phi(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\Phi^{-1}(r^{-n})\varphi_2(x,r)} \int_{\delta_0}^{\infty} \frac{\Phi^{-1}\big(t^{-n}\big)}{t} \|f\|_{L^\Phi(B(x,t))} dt.$$

We use the fact that $f \in V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \frac{\|f\|_{L^\Phi(B(x,t))}}{\varphi_1(x,t)} < \frac{\varepsilon}{2CC_0}, \quad t \le \delta_0,$$

where C and C_0 are constants from (19) and (17). This yields a uniform estimate of the first term with respect to $r \in (0, \delta_0)$:

$$\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \;\; 0 < r < \delta_0.$$

For the second term, we have $J_{\delta_0}(x,r) \le c_{\delta_0} \frac{\|f\|_{\mathcal{M}^{\Phi,\varphi_1}}}{\Phi^{-1}(r^{-n})\varphi_2(x,r)}$, where c_{δ_0} is the constant from (16) with $\delta = \delta_0$. Then, by (8) we choose a small r such that $\sup_{x\in\mathbb{R}^n} \frac{1}{\Phi^{-1}(r^{-n})\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{\mathcal{M}^{\Phi,\varphi_1}}}$. The proof of the estimate of the second term is complete. □

Corollary 3.2 *Let $\alpha \in (0,1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, $\varphi_1, \varphi_2 \in \Omega_\Phi^V$. Moreover, suppose that the functions (φ_1, φ_2) and Φ satisfy the conditions (16) and (17), then g_α is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.*

The following lemma is an easy consequence of the inequality

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x)$$

which was proved in [55] and of the monotonicity of the norm $\|\cdot\|_{L^\Phi}$.

Lemma 3.2 *For $j \in \mathrm{Z}^+$, denote*

$$G_{\alpha,2^j}(f)(x) = \left(\int_0^\infty \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

Let Φ be a Young function and $0 < \alpha \le 1$, then we have

$$\|G_{\alpha,2^j}(f)\|_{L^\Phi(\mathbb{R}^n)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \|G_\alpha(f)\|_{L^\Phi(\mathbb{R}^n)}.$$

Theorem 3.8 *Let $\alpha \in (0,1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, $\varphi_1, \varphi_2 \in \Omega_\Phi$ and the functions (φ_1, φ_2) and Φ satisfy the condition (15). Then for $\lambda > 3 + \dfrac{2\alpha}{n}$, $g^*_{\lambda,\alpha}$ is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.*

Proof

$$\begin{aligned}[g^*_{\lambda,\alpha}(f)(x)]^2 &= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \\ &\quad + \int_0^\infty \int_{|x-y|\geq t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \\ &:= I + II.\end{aligned}$$

First, let us estimate I.

$$I \leq \int_0^{+\infty} \int_{|x-y|<t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \leq (G_\alpha f(x))^2$$

Now, let us estimate II.

$$\begin{aligned} II &\leq \sum_{j=1}^{\infty} \int_0^\infty \int_{2^{j-1}t\leq|x-y|\leq 2^j t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \\ &\lesssim \sum_{j=1}^{\infty} \int_0^\infty \int_{2^{j-1}t\leq|x-y|\leq 2^j t} 2^{-jn\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \\ &\lesssim \sum_{j=1}^{\infty} 2^{-jn\lambda} \int_0^\infty \int_{|x-y|\leq 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \\ &:= \sum_{j=1}^{\infty} 2^{-jn\lambda} (G_{\alpha,2^j}(f)(x))^2 \end{aligned}$$

Thus,

$$\|g^*_{\lambda,\alpha}(f)\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)} \leq \|G_\alpha f\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(f)\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)}. \tag{20}$$

By Theorem 3.6, we have

$$\|G_\alpha f\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)}. \tag{21}$$

In the following, we will estimate $\|G_{\alpha,2^j}(f)\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)}$. We divide $\|G_{\alpha,2^j}(f)\|_{L^\Phi(B)}$ into two parts.

$$\|G_{\alpha,2^j}(f)\|_{L^\Phi(B)} \leq \|G_{\alpha,2^j}(f_1)\|_{L^\Phi(B)} + \|G_{\alpha,2^j}(f_2)\|_{L^\Phi(B)}, \tag{22}$$

where $f_1(y) = f(y)\chi_{2B}(y)$, $f_2(y) = f(y) - f_1(y)$. For the first part, by Lemma 3.2 and (14),

$$\|G_{\alpha,2^j}(f_1)\|_{L^\Phi(B)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \|G_\alpha(f_1)\|_{L^\Phi(\mathbb{R}^n)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{L^\Phi(2B)}$$

$$\lesssim 2^{j(\frac{3n}{2}+\alpha)} \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{23}$$

For the second part.

$$
\begin{aligned}
G_{\alpha,2^j}(f_2)(x) &= \left(\int_0^\infty \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\sup_{\phi\in C_\alpha} |f*\phi_t(y)|\right)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\int_{|z-y|\le t} |f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $|x-z| \le |z-y| + |x-y| \le 2^{j+1}t$, we get

$$
\begin{aligned}
G_{\alpha,2^j}(f_2)(x) &\le \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\int_{|x-z|\le 2^{j+1}t} |f_2(z)|dz\right)^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\infty \left(\int_{|x-z|\le 2^{j+1}t} |f_2(z)|dz\right)^2 \frac{2^{jn}dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\
&\le 2^{\frac{jn}{2}} \int_{\mathbb{R}^n} \left(\int_{t\ge \frac{|x-z|}{2^{j+1}}} |f_2(z)|^2 \frac{1}{t^{2n+1}} dt\right)^{\frac{1}{2}} dz \\
&\le 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^n} dz.
\end{aligned}
$$

For $|x-z| \ge |x_0-z| - |x_0-x| \ge |x_0-z| - \frac{1}{2}|x_0-z| = \frac{1}{2}|x_0-z|$, so by Fubini's theorem and Lemma 2.3, we obtain

$$
\begin{aligned}
G_{\alpha,2^j}(f_2)(x) &\le 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \frac{|f(z)|}{|x_0-z|^n} dz \\
&= 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} |f(z)| \int_{|x_0-z|}^\infty \frac{1}{t^{n+1}} dtdz \\
&\le 2^{\frac{3jn}{2}} \int_{2r}^\infty \int_{|x_0-z|<t} |f(z)| \frac{1}{t^{n+1}} dzdt \\
&\le 2^{\frac{3jn}{2}} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}.
\end{aligned}
$$

So,

$$
\|G_{\alpha,2^j}(f_2)\|_{L^\Phi(B)} \lesssim 2^{\frac{3jn}{2}} \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{24}
$$

Combining (22), (23) and (24), we have

$$
\|G_{\alpha,2^j}(f)\|_{L^\Phi(B)} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}.
$$

Thus by Theorem 3.4 we have

$$\begin{aligned}\|G_{\alpha,2^j} f\|_{\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)} &\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)\Phi^{-1}(r^{-n})} \int_r^\infty \Phi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x,t))} \frac{dt}{t} \\ &\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} \|f\|_{L^\Phi(B(x,r))} \\ &= 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)}. \end{aligned} \tag{25}$$

Since $\lambda > 3 + \frac{2\alpha}{n}$, by (20), (21) and (25), we have the desired theorem. □

The proof of the following theorem is similar to the Theorem 3.7. So we omit the details here.

Theorem 3.9 *Let* $\alpha \in (0, 1]$, Φ *be a Young function which is of lower type* p_0 *and of upper type* p_1 *with* $1 < p_0 \le p_1 < \infty$, $\varphi_1, \varphi_2 \in \Omega_\Phi^V$ *and the functions* (φ_1, φ_2) *and* Φ *satisfy the conditions (16) and (17). Then for* $\lambda > 3 + \frac{2\alpha}{n}$, $g^*_{\lambda,\alpha}$ *is bounded from* $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $V\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Remark 3.1 Note that from Theorems 3.6, 3.8 and Corollary 3.1 in particular we get Theorems 3.2 and 3.3 which proved in [35].

4 Commutators of the Intrinsic Square Functions in the Spaces $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$

It is well known that the commutator is an important integral operator and plays a key role in harmonic analysis. In 1965, Calderón [2, 3] studied a kind of commutators appearing in Cauchy integral problems of Lip-line. Let K be a Calderón–Zygmund singular integral operator and b be a $BMO(\mathbb{R}^n)$ function. A well known result of Coifman et al. [6] states that the commutator operator $[b, K]f = K(bf) - bKf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón–Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order. The boundedness result was generalized to other contexts and important applications to some non-linear Partial Differential Equations were given by Coifman et al. [7].

A function $f \in L^1_{loc}(\mathbb{R}^n)$ is said to be a $BMO(\mathbb{R}^n)$ function if, for

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy,$$

the quantity

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy < \infty,$$

is finite.

Before proving the main theorems, we need the following lemmas.

Lemma 4.1 [29] (1) *The John–Nirenberg inequality: there are constants C_1, $C_2 > 0$, such that for all b is a $BMO(\mathbb{R}^n)$ function and $\beta > 0$*

$$|\{x \in B : |b(x) - b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) *The John–Nirenberg inequality implies that*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{26}$$

for $1 < p < \infty$.

(3) *Let b is a $BMO(\mathbb{R}^n)$ function. Then, there is a constant $C > 0$ such that*

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \le C\|b\|_* \ln \frac{t}{r} \quad \textit{for} \quad 0 < 2r < t, \tag{27}$$

where C is independent of b, x, r and t.

Lemma 4.2 *Let b be a BMO function and Φ be a Young function. Let Φ is of lower type p_0 and of upper type p_1 with $1 \le p_0 \le p_1 < \infty$, then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-n}\big) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L^\Phi(B(x,r))}.$$

Proof By Hölder's inequality, we have

$$\|b\|_* \lesssim \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-n}\big) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L^\Phi(B(x,r))}.$$

Now we show that

$$\sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-n}\big) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L^\Phi(B(x,r))} \lesssim \|b\|_*.$$

Without loss of generality, we may assume that $\|b\|_* = 1$; otherwise, we replace b by $b/\|b\|_*$. Using the fact that Φ is of lower type p_0 and of upper type p_1 and (2) it follows that

$$\begin{aligned} &\int_{B(x,r)} \Phi\left(\frac{|b(y) - b_{B(x,r)}|\, \Phi^{-1}\big(|B(x,r)|^{-1}\big)}{\|b\|_*} \right) dy \\ &= \int_{B(x,r)} \Phi\Big(|b(y) - b_{B(x,r)}| \Phi^{-1}\big(|B(x,r)|^{-1}\big)\Big) dy \\ &\lesssim \frac{1}{|B(x,r)|} \int_{B(x,r)} \left[|b(y) - b_{B(x,r)}|^{p_0} + |b(y) - b_{B(x,r)}|^{p_1} \right] dy \lesssim 1. \end{aligned}$$

By Lemma 2.1 we get the desired result. □

Remark 4.1 Note that statements of type of Lemma 4.2 are known in a more general case of rearrangement invariant spaces and also variable exponent Lebesgue spaces $L^{p(\cdot)}$, see, for instance, [25] and [28]. Hovewer, we gave a short proof of Lemma 4.2 for completeness of presentation.

The known boundedness statement for $[b, G_\alpha]$ on Orlicz spaces runs as follows.

Theorem 4.1 [35] *Let b a BMO function, $\alpha \in (0, 1]$, Φ to be a Young function of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$. Then, $[b, G_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to itself.*

The following theorems was proved by Liang et al. in [35].

Theorem 4.2 [35] *Let $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$, b be a BMO function. Suppose, also, that the mapping $\varphi(x, r)$ satisfy the conditions (9). Then, the operators $[b, G_\alpha]$ and $[b, g_\alpha]$ are bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

Theorem 4.3 [35] *Let $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$. Suppose, also, that the mapping b be a BMO function and that the function $\varphi(x, r)$ satisfy the conditions (9). Then, for $\lambda > 3 + \frac{2\alpha}{n}$, the operator $[b, g^*_{\lambda,\alpha}]$ is bounded on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.*

Lemma 4.3 *Let b be a BMO function, $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$. Then, the inequality*

$$\|[b, G_\alpha]f\|_{L^\Phi(B(x_0,r))} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big)\|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \tag{28}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^\Phi_{loc}(\mathbb{R}^n)$.

Proof For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence

$$\|[b, G_\alpha]f\|_{L^\Phi(B)} \le \|[b, G_\alpha]f_1\|_{L^\Phi(B)} + \|[b, G_\alpha]f_2\|_{L^\Phi(B)}.$$

From the boundedness of $[b, G_\alpha]$ in $L^\Phi(\mathbb{R}^n)$ it follows that

$$\begin{aligned}\|[b, G_\alpha]f_1\|_{L^\Phi(B)} &\le \|[b, G_\alpha]f_1\|_{L^\Phi(\mathbb{R}^n)}\\ &\lesssim \|b\|_*\|f_1\|_{L^\Phi(\mathbb{R}^n)} = \|b\|_*\|f\|_{L^\Phi(2B)}.\end{aligned}$$

We split the second part into two parts.

$$\begin{aligned}[b, G_\alpha]f_2(x) &= \left(\int\int_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(x) - b(z)]\phi_t(y - z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &\le \left(\int\int_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b(x) - b_B]\phi_t(y - z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &\quad + \left(\int\int_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}[b_B - b(z)]\phi_t(y - z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &:= A + B.\end{aligned}$$

First, for A, we find that

$$A = |b(x) - b_B|\left(\iint_{\Gamma(x)} \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}\phi_t(y - z)f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} = |b(x) - b_B|G_\alpha f_2(x).$$

By (13) and Lemma 4.2, we derive

$$\begin{aligned}\|A\|_{L^\Phi(B)} &= \||b(\cdot)-b_B|G_\alpha f_2(\cdot)\|_{L^\Phi(B)} \\ &\le \|b(\cdot)-b_B\|_{L^\Phi(B)} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \\ &\le \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.\end{aligned}$$

For B, since $|x-y|<t$, we get $|x-z|<2t$. Thus, by Minkowski's inequality,

$$\begin{aligned}B &\le \left(\int\int_{\Gamma(x)}\left|\int_{|x-z|<2t}|b_B-b(z)||f_2(z)|dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^\infty\left|\int_{|x-z|<2t}|b_B-b(z)||f_2(z)|dz\right|^2\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\ &\le \int_{|x_0-z|>2r}|b_B-b(z)||f(z)|\frac{dz}{|x-z|^n}.\end{aligned}$$

Since $|x-z|\ge\frac{1}{2}|x_0-z|$, we have

$$\begin{aligned}\|B\|_{L^\Phi(B)} &\lesssim \left\|\int_{\complement_{(2B)}}\frac{|b(z)-b_B|}{|x_0-z|^n}|f(z)|dz\right\|_{L^\Phi(B)} \\ &\approx \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{\complement_{(2B)}}\frac{|b(z)-b_B|}{|x_0-z|^n}|f(z)|dz \\ &\approx \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{\complement_{(2B)}}|b(z)-b_B||f(z)|\int_{|x_0-z|}^\infty\frac{dt}{t^{n+1}}dz \\ &\approx \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty\int_{2r\le|x_0-z|\le t}|b(z)-b_B||f(z)|dz\frac{dt}{t^{n+1}} \\ &\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty\int_{B(x_0,t)}|b(z)-b_B||f(z)|dz\frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality, by Lemma 4.2 and (27) we get

$$\begin{aligned}\|B\|_{L^\Phi(B)} &\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty\int_{B(x_0,t)}|b(z)-b_{B(x_0,t)}||f(z)|dz\frac{dt}{t^{n+1}} \\ &\quad+\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty|b_B-b_{B(x_0,t)}|\int_{B(x_0,t)}|f(z)|dz\frac{dt}{t^{n+1}} \\ &\lesssim \frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty\left\|b(\cdot)-b_{B(x_0,t)}\right\|_{L^{\widetilde{\Phi}}(B)}\|f\|_{L^\Phi(B(x_0,t))}\frac{dt}{t^{n+1}} \\ &\quad+\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty|b_B-b_{B(x_0,t)}|\|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \\ &\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)\|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.\end{aligned}$$

Summing up $\|A\|_{L^\Phi(B)}$ and $\|B\|_{L^\Phi(B)}$, we have

$$\|[b, G_\alpha]f_2\|_{L^\Phi(B)} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}.$$

Finally, we obtain

$$\|[b, G_\alpha]f\|_{L^\Phi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi(2B)} + \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t},$$

and the statement of Lemma 4.3 follows by (14). □

Theorem 4.4 *Let b be a BMO function, $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$ and $\varphi_1, \varphi_2 \in \Omega_\Phi$. Suppose, also, that the mappings (φ_1, φ_2) and Φ satisfy the condition*

$$\int_r^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \Big(\underset{t<s<\infty}{ess\ \ inf} \varphi_1(x, s)\Big) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \Phi^{-1}\big(r^{-n}\big) \varphi_2(x, r), \tag{29}$$

where C does not depend on x and r. Then, the operator $[b, G_\alpha]$ is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof The statement of Theorem 4.4 follows in a similar way as for Lemma 4.3 and Theorem 3.5 in the same manner as in the proof of Theorem 3.6. □

In [55], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 4.4, we have the following result.

Corollary 4.1 *Let b be a BMO function, $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$ and $\varphi_1, \varphi_2 \in \Omega_\Phi$. Suppose, also, that the mappings (φ_1, φ_2) and Φ satisfy the condition (29). Then, $[b, g_\alpha]$ is bounded from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.*

Theorem 4.5 *Let b be a BMO function, $\alpha \in (0, 1]$, Φ be a Young function which is of lower type p_0 and of upper type p_1 with $1 < p_0 \le p_1 < \infty$ and $\varphi_1, \varphi_2 \in \Omega_\Phi^V$. Suppose, also, that the mappings (φ_1, φ_2) and Φ satisfy*

$$\int_r^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \varphi_1(x, t)\, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C_0 \varphi_2(x, r) \Phi^{-1}\big(r^{-n}\big), \tag{30}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$ and the conditions

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\Phi^{-1}(r^{-n}) \inf_{x\in\mathbb{R}^n} \varphi_2(x, r)} = 0 \tag{31}$$

and

$$c_\delta := \int_\delta^{\infty} (1 + |\ln t|) \sup_{x\in\mathbb{R}^n} \varphi_1(x, t) \frac{\Phi^{-1}\big(t^{-n}\big)}{t} dt < \infty \tag{32}$$

for every $\delta > 0$. Then, the operator $[b, G_\alpha]$ is bounded from $V\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $V\mathcal{M}^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Proof The proof follows more or less the same lines as for Theorem 3.7, but now the arguments are different due to the necessity to introduce the logarithmic factor into the assumptions.

The norm inequality having already been provided by Theorem 4.4, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{\|f\|_{L^\Phi(B(x,r))}}{\varphi_1(x,r)}=0 \implies \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{\|[b,G_\alpha]f\|_{L^\Phi(B(x,r))}}{\varphi_2(x,r)}=0. \tag{33}$$

To check that

$$\sup_{x\in\mathbb{R}^n}\frac{\|[b,G_\alpha]f\|_{L^\Phi(B(x,r))}}{\varphi_2(x,r)}<\varepsilon \quad \text{for small } r,$$

we use the estimate (28):

$$\frac{\|[b,G_\alpha]f\|_{L^\Phi(B(x,r))}}{\varphi_2(x,r)}\lesssim\frac{\|b\|_*}{\varphi_2(x,r)\Phi^{-1}\big(r^{-n}\big)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

We take $r<\delta_0$ where δ_0 will be chosen small enough and we split the integration:

$$\frac{\|[b,G_\alpha]f\|_{L^\Phi(B(x,r))}}{\varphi_2(x,r)}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{34}$$

where

$$I_{\delta_0}(x,r):=\frac{1}{\Phi^{-1}(r^{-n})\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\frac{\Phi^{-1}\big(t^{-n}\big)}{t}\|f\|_{L^\Phi(B(x,t))}dt$$

and

$$J_{\delta_0}(x,r):=\frac{1}{\Phi^{-1}(r^{-n})\varphi_2(x,r)}\int_{\delta_0}^\infty\left(1+\ln\frac{t}{r}\right)\frac{\Phi^{-1}\big(t^{-n}\big)}{t}\|f\|_{L^\Phi(B(x,t))}dt.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\frac{\|f\|_{L^\Phi(B(x,t))}}{\varphi_1(x,t)}<\frac{\varepsilon}{2CC_0},\quad t\le\delta_0,$$

where C and C_0 are constants from (34) and (30), which yields the estimate of the first term uniformly in $r\in(0,\delta_0)$: $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, $0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r)\le\frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\Phi^{-1}(r^{-n})\varphi_2(x,r)}\|f\|_{\mathcal{M}^{\Phi,\varphi}},$$

where c_{δ_0} is the constant from (32) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with the omitted logarithmic factor in the integrand. Then, by (31) we can choose a small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, and the proof is established. □

Remark 4.2 By using the argument similarly as the above proofs and also in the proof of Theorem 3.8, we can also show the boundedness of $[b,g^*_{\lambda,\alpha}]$ on $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ and $V\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$.

Remark 4.3 Let us notice that from Theorem 4.4, Corollary 4.1 and Remark 4.2 we recapture Theorems 4.2 and 4.3 proved in [35].

Acknowledgements The research of V.S. Guliyev was partially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number: 02.a03.21.0008).

The research of F. Deringoz was partially supported by the grant of Ahi Evran University Scientific Research Project (FEF.A3.16.011).

The research of M.A. Ragusa was partially supported by the grant of the Research Project FIR 2014.

References

1. Bennett, C., Sharpley, R.: Interpolation of operators. Academic Press, Boston (1988)
2. Calderon, A.P.: Commutators of singular integral operators. Proc. Nat. Acad. Sci. U.S.A. **53**, 1092–1099 (1965)
3. Calderon, A.P.: Cauchy integrals on Lipschitz curves and related operators. Proc. Nat. Acad. Sci. U.S.A. **74**(4), 1324–1327 (1977)
4. Carozza, M., Passarelli di Napoli, A.: Composition of maximal operators. Publicacions Matematiques **40**(2), 397–409 (1996)
5. Cianchi, A.: Strong and weak type inequalities for some classical operators in Orlicz spaces. J. London Math. Soc. **60**(2), 187–202 (1999). no. 1
6. Coifman, R.R., Rochberg, R., Weiss, G.: Factorization theorems for Hardy spaces in several variables. Ann. Math.(2) **103**(839), 611–635 (1976)
7. Coifman, R., Lions, P., Meyer, Y., Semmes, S.: Compensated compactness and Hardy spaces. J. Math. Pures Appl. **72**, 247–286 (1993)
8. Deringoz, F., Guliyev, V.S., Samko, S.: Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces, Operator Theory, Operator Algebras and Applications. Series: Operator Theory: Adv. Appl. **242**, 139–158 (2014)
9. Deringoz, F., Guliyev, V.S., Hasanov, S.: A characterization for Adams type boundedness of the fractional maximal operator on generalized Orlicz-Morrey spaces. Integral Transforms Special Funct. **28**(4), 284–299 (2017)
10. Deringoz, F., Guliyev, V.S., Samko, S.: Vanishing generalized Orlicz-Morrey spaces and fractional maximal operator. Publ. Math. Debrecen **90**(1–2), 125–147 (2017)
11. Gala, S., Ragusa, M.A., Sawano, Y., Tanaka, H.: Uniqueness criterion of weak solutions for the dissipative quasi-geostrophic equations in Orlicz-Morrey spaces. Appl. Anal. **93**(2), 356–368 (2014)
12. Gala, S., Sawano, Y., Tanaka, H.: A remark on two generalized Orlicz-Morrey spaces. J. Approx. Theory **98**, 1–9 (2015)
13. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$, Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 329 pp. (in Russian) (1994)
14. Guliyev, V.S.: Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications, Casioglu, Baku, 332 pp. (1999)
15. Guliyev, V.S.: Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl. Art ID 503948 (2009)
16. Guliyev, V.S.: Generalized weighted Morrey spaces and higher order commutators of sublinear operators. J. Eurasian Math. **3**(3), 33–61 (2012)
17. Guliyev, V.S.: Generalized local Morrey spaces and fractional integral operators with rough kernel. J. Math. Sci. (N. Y.) **193**(2), 211–227 (2013)
18. Guliyev, V.S., Deringoz, F.: On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces, J. Funct. Spaces, Volume 2014, Article ID 617414, 11 pages. doi:10.1155/2014/617414
19. Guliyev, V.S., Omarova, M.N.: Higher order commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces. Azer. J. Math. **4**(2), 64–85 (2014). http://azjm.org/index.php/azjm/article/view/221
20. Guliyev, V.S., Shukurov, P.S.: Commutators of intrinsic square functions on generalized Morrey spaces. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **39**, 33–46 (2013)
21. Guliyev, V.S., Deringoz, F., Hasanov, J.J.: Φ-admissible singular operators and their commutators on vanishing generalized Orlicz-Morrey spaces. J. Inequal. Appl. **2014**, 143 (2014). http://www.journalofinequalitiesandapplications.com/content/2014/1/143
22. Guliyev, V.S., Hasanov, S.G., Sawano, Y., Noi, T.: Non-smooth atomic decompositions for generalized Orlicz-Morrey spaces of the third kind. Acta Appl. Math. **145**, 133–174 (2016)
23. Hasanov, J.J.: Φ-admissible sublinear singular operators and generalized Orlicz-Morrey spaces, J. Funct. Spaces, Volume 2014, Article ID 505237 7 pages. doi:10.1155/2014/505237 (2014)

24. Hencl, S., Kleprlik, J.: Composition of q-quasiconformal mappings and functions in Orlicz-Sobolev spaces. Illinois J. Math. **56**(3), 931–955 (2012)
25. Ho, K.-P.: Characterization of BMO in terms of rearrangement-invariant Banach function spaces. Expo. Math. **27**, 363–372 (2009)
26. Ho, K.-P.: The fractional integral operators on Morrey spaces with variable exponent domains. Math. Ineq. and Appl. **16**(2), 363–373 (2013)
27. Huang, J.Z., Liu, Y.: Some characterizations of weighted Hardy spaces. J. Math. Anal. Appl. **363**, 121–127 (2010)
28. Izuki, M., Sawano, Y.: Variable Lebesgue norm estimates for BMO functions. Czechoslovak Math. J. **62**(137), 717–727 (2012). (3)
29. John, F., Nirenberg, L.: On functions of bounded mean oscillation. Comm. Pure Appl. Math. **14**, 415–426 (1961)
30. Kita, H.: On maximal functions in Orlicz spaces. Proc. Amer. Math. Soc. **124**, 3019–3025 (1996)
31. Kita, H.: On Hardy-Littlewood maximal functions in Orlicz spaces. Math. Nachr. **183**, 135–155 (1997)
32. Kokilashvili, V., Krbec, M.M.: Weighted Inequalities in Lorentz and Orlicz Spaces. World Scientific, Singapore (1991)
33. Krasnoselskii, M.A., Rutickii, Y.B.: Convex Functions and Orlicz Spaces, English Translation P. Noordhoff Ltd., Groningen (1961)
34. Ky, L.D.: New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integr. Equ. Oper. Theory **78**(1), 115–150 (2014)
35. Liang, Y., Nakai, E., Yang, D., Zhang, J.: Boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz morrey and campanato spaces, banach. J. Math. Anal. **8**(1), 221–268 (2014)
36. Lu, S.S., Shi, S.: A characterization of Campanato space via commutator of fractional integral. J. Math. Anal. Appl. **419**(1), 123–137 (2014)
37. Nakai, E.: Generalized Fractional Integrals on Orlicz-Morrey Spaces. In: Banach and Function Spaces. (Kitakyushu, 2003), Yokohama Publishers, Yokohama, 323–333 (2004)
38. Nakai, E.: Orlicz-morrey spaces and the Hardy-Littlewood maximal function. Studia Math. **188**(3), 193–221 (2008)
39. Nakai, E.: Calderón-zygmund operators on Orlicz-Morrey spaces and modular inequalities, Banach and function spaces II, Yokohama Publ, Yokohama, 393–410 (2008)
40. Nakamura, S.: Generalized weighted Morrey spaces and classical operators. Mathematische Nachrichten **289**(17–18), 2235–2262 (2016)
41. Orlicz, W.: ÜBer eine gewisse Klasse von räumen vom Typus B, Bull. Acad. Polon. A, 207–220 (1932), reprinted in: Collected Papers, PWN, Warszawa 217–230 (1988)
42. Orlicz, W.: ÜBer räume (l^M), Bull. Acad. Polon. A, 93–107 (1936), reprinted in: Collected Papers, PWN, Warszawa 345–359 (1988)
43. Persson, L.E., Ragusa, M.A., Samko, N., Wall, P.: Commutators of Hardy operators in vanishing Morrey spaces. AIP Conf. Proc. **1493**, 859–866 (2012). doi:10.1063/1.4765588
44. Ragusa, M.A.: Commutators of fractional integral operators on vanishing-Morrey spaces. J. Global Optim. **40**(1-3), 361–368 (2008)
45. Ragusa, M.A.: Necessary and sufficient condition for a VMO function. Appl. Math. Comp. **218**, 11952–11958 (2012)
46. Ragusa, M.A.: Embeddings for Morrey-Lorentz spaces. J. Optim. Theory Appl. **154**(2), 491–499 (2012)
47. Rao, M.M., Ren, Z.D.: Theory of orlicz spaces. M. Dekker, Inc., New York (1991)
48. Sawano, Y., Sugano, S., Tanaka, H.: Orlicz-morrey spaces and fractional operators. Potential Anal. **36**(4), 517–556 (2012)
49. Vitanza, C.: Functions with Vanishing Morrey Norm and Elliptic Partial Differential Equations. In: Proceedings of Methods of Real Analysis and Partial Differential Equations, Capri, pp. 147–150 (1990)
50. Vitanza, C.: Regularity results for a class of elliptic equations with coefficients in Morrey spaces. Ricerche Mat. **42**(2), 265–281 (1993)
51. Wang, H.: Intrinsic square functions on the weighted Morrey spaces. J. Math. Anal. Appl. **396**, 302–314 (2012)
52. Wang, H.: Boundedness of intrinsic square functions on the weighted weak Hardy spaces. Integr. Equ. Oper. Theory **75**, 135–149 (2013)
53. Wang, H., Liu, H.P.: Weak type estimates of intrinsic square functions on the weighted Hardy spaces. Arch. Math. **97**, 49–59 (2011)
54. Weiss, G.: A note on Orlicz spaces. Portugal Math. **15**, 35–47 (1956)
55. Wilson, M.: The intrinsic square function. Rev. Mat. Iberoam. **23**, 771–791 (2007)
56. Wilson, M.: Weighted Littlewood-Paley theory and Exponential-square integrability, Lecture Notes in Math, vol. 1924, p. 227. Springer, Berlin (2007)