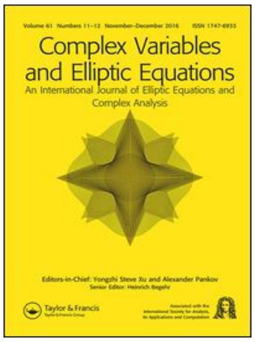

Complex Variables and Elliptic Equations

An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

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To cite this article: V. S. Guliyev, K. Koca, R. Ch. Mustafayev & T. Ünver (2017) Boundedness of operators arising from Schwarz BVP in modified local Morrey-type spaces, Complex Variables and Elliptic Equations, 62:10, 1541-1557, DOI: 10.1080/17476933.2017.1308049

To link to this article: https://doi.org/10.1080/17476933.2017.1308049

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS, 2017
VOL. 62, NO. 10, 1541–1557
https://doi.org/10.1080/17476933.2017.1308049

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Boundedness of operators arising from Schwarz BVP in modified local Morrey-type spaces

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ABSTRACT

In this paper, we prove the boundedness of a class of operators arising from Schwarz BVP in modified local Morrey-type spaces in the unit disc of the complex plane.

ARTICLE HISTORY
Received 15 April 2016
Accepted 28 January 2017

COMMUNICATED BY
T. Kalmenov

KEYWORDS
Sublinear operators; local Morrey-type spaces; Hardy inequalities

AMS SUBJECT CLASSIFICATIONS
47G10; 47B38; 30J99; 30E25

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$. The Schwarz boundary value problem (Schwarz BVP)

$$g_{\bar{z}} = f \text{ in } \mathbb{D}, \ \operatorname{Re} g = \gamma \text{ on } \partial\mathbb{D}, \ \operatorname{Im} g\left(0\right) = c, \tag{1}$$

is one of the major boundary value problems in complex analysis. It is uniquely solvable for analytic functions [1], and for polyanalytic functions [2]. The solvability of the Schwarz problem for some higher-order linear elliptic complex partial differential equations were investigated in [3,4].

The Cauchy–Riemann–Poisson–Pompeiu formula given by

$$\begin{aligned} g(z) = {} & \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + ic \\ & - \frac{1}{2\pi}\iint_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right) d\xi\, d\eta, \quad z \in \mathbb{D}, \quad \zeta = \xi + i\eta \end{aligned} \tag{2}$$

is the unique solution to the Schwarz BVP, where $f \in L^1(\mathbb{D})$, $\gamma \in C(\partial\mathbb{D}, \mathbb{R})$, $c \in \mathbb{R}$ (see [2,5,6]).

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The domain integral appearing on the right-hand side of (2), denoted by $\widetilde{T}_1$, is a modification of the Pompeiu operator

$$T_1 f(z) := -\frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{\zeta - z}, \quad z \in \mathbb{D},$$

which was studied by Vekua in [7]. The operator $\widetilde{T}_1$ is important for treating complex first-order equations (see, for instance, [7–10]). Iterating this operator with itself by the rule $\widetilde{T}_k f(z) = \widetilde{T}_1(\widetilde{T}_{k-1} f(z))$ generates the operators

$$\widetilde{T}_k f(z) := \frac{(-1)^k}{2\pi(k-1)!}\iint_{\mathbb{D}} \left(\zeta - z + \overline{\zeta - z}\right)^{k-1}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\right)\mathrm{d}\xi\,\mathrm{d}\eta$$

for $k \in \mathbb{N}$ with $\widetilde{T}_0 f(z) = f(z)$. These operators satisfy

$$\frac{\partial^l}{\partial \bar{z}^l}\widetilde{T}_k f = \widetilde{T}_{k-l} f, \quad 1 \le l \le k, \tag{3}$$

$$\operatorname{Re}\frac{\partial^l}{\partial \bar{z}^l}\widetilde{T}_k f = 0 \quad \text{on} \quad \partial\mathbb{D}, \quad 0 \le l \le k-1, \tag{4}$$

$$\operatorname{Im}\frac{\partial^l}{\partial \bar{z}^l}\widetilde{T}_k f(0) = 0, \quad 0 \le l \le k-1, \tag{5}$$

see [3,11,12]. Note that $\partial_z^l \widetilde{T}_k$ is a weakly singular integral operator for $0 \le l \le k-1$, while

$$\Pi_k f(z) := \frac{\partial^k}{\partial z^k}\widetilde{T}_k f(z) = \frac{(-1)^k k}{\pi}\iint_{\mathbb{D}}\left[\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2} + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}}\bar{\zeta} - 1\right)^{k-1}\frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2}\right]\mathrm{d}\xi\,\mathrm{d}\eta \tag{6}$$

is a strongly singular integral operator. It is known that $\|\Pi_1\|_{L^2(\mathbb{D})} = 1$ (see [7,10]). Π_k are shown to be bounded in the space L^p for $1 < p < \infty$ and in particular their L^2 norms are estimated in [13]. These operators are studied by decomposing them into two parts as $\Pi_k = T_{-k,k} + P_k$, where

$$T_{-k,k} f(z) = \frac{(-1)^k k}{\pi}\iint_{\mathbb{D}}\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{7}$$

and

$$P_k f(z) = \frac{(-1)^k k}{\pi}\iint_{\mathbb{D}}\left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}}\bar{\zeta} - 1\right)^{k-1}\frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{8}$$

which are investigated extensively in [13,14], respectively, and the boundedness of $T_{-k,k}$ and P_k in $L_p(\mathbb{D})$ are proved.

It is mentioned in [14] that the integral in (7) must be viewed as a Cauchy principal value integral,

$$T_{-k,k}f(z) = \lim_{\varepsilon \to 0} \iint_{\mathbb{D}_\varepsilon} K_{-k,k}(z-\zeta)w(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{9}$$

where $\mathbb{D}_\varepsilon$ is the domain $\mathbb{D} - \{\zeta : |\zeta - z| \le \varepsilon\}$, and the limit is taken in the norm of $L^p(\mathbb{D})$. Here

$$K_{-k,k}(z) := \frac{(-1)^k k}{\pi} z^{-k-1}\bar{z}^{k-1}.$$

These integrals can be analyzed with the well-known theory of Calderón and Zygmund [15–17] concerning singular integrals. The boundedness of P_k in $L_p(\mathbb{D})$ was proved in [13] using Schur's test (see, for instance, [18]) and Forelli–Rudin Lemma in [19].

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C.B. Morrey in 1938 [20] in relation to the study of partial differential equations, were widely investigated during the last decades, including the study of classical operators of harmonic and real analysis - maximal, singular and potential operators - in generalizations of these spaces (the so-called local Morrey-type spaces). The local Morrey-type space $LM_{p\theta,\omega}(G)$ are introduced in the doctoral thesis [21]. Some sufficient conditions for the boundedness of fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,\omega}(G)$ defined on homogeneous Lie groups G are given in [21] (also in [22,23]).

The research on local Morrey-type spaces mainly includes the study of classical operators in these spaces (cf. [24–27]). However, recently in a series of papers, authors started to study the structure of local Morrey-type spaces and relation of these spaces with other known function spaces.

The aim of this paper is to study the boundedness of integral operators (6) in modified local Morrey-type spaces $\widetilde{LM}_{p\theta,\omega}(\mathbb{D}) = LM_{p\theta,\omega}(\mathbb{D}) \cap L_p(\mathbb{D})$. By the way, we obtain norm estimates for these operators in some weighted Lebesgue spaces defined on $\mathbb{D}$.

The paper is organized as follows. Some notations and definitions are given in Section 2. Our main results are presented in Section 3. In Section 4, we prove some local estimates of sublinear operators satisfying Soria–Weiss condition (see (27)). Finally, in Section 5, we prove the boundedness of such operators in modified local Morrey-type spaces.

2. Notations and preliminaries

Now we make some conventions. Throughout the paper, we always denote by c or C a positive constant which is independent of the main parameters, but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, ($b \gtrsim a$) we mean that $a \le \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. For a measurable set E, χ_E denotes the characteristic function of E. We define the Lebesgue measure of E by $|E|$. For $0 < \rho < 1$, let $B(z,\rho) := \{\zeta \in \mathbb{D} : |z - \zeta| < \rho\}$ be the open ball centered at $z \in \mathbb{D}$ of radius ρ and ${}^{\complement}B(z,\rho) := \mathbb{D} \backslash B(z,\rho)$. Given $\lambda > 0$ and a ball B, λB denotes the ball with the same center as B and whose side is λ times that of B.

For $0 < p \le \infty$ and w a weight function on a measurable subset E of $\mathbb{C}$, that is, a locally integrable real-valued non-negative function on E, let us denote by $L_{p,w}(E)$ the weighted Lebesgue space defined as the set of all measurable functions $f : E \to \mathbb{C}$ for which the quantity

$$\|f\|_{L_{p,w}(E)} = \begin{cases} \left(\int_E |f(\zeta)|^p w(\zeta)\, d\xi\, d\eta\right)^{\frac{1}{p}} & \text{for } p < \infty, \\ \operatorname*{ess\,sup}_{\zeta \in E} |f(\zeta)| w(\zeta) & \text{for } p = \infty \end{cases} \tag{10}$$

is finite. When $w \equiv 1$, we write simply $L_p(E)$ and $\|\cdot\|_{L_p(E)}$ instead of $L_{p,w}(E)$ and $\|\cdot\|_{L_{p,w}(E)}$.

Recall the definition of weak Lebesgue space:

$$WL_p(E) := \left\{ f : E \to \mathbb{C} \text{ meas.} : \ \|f\|_{WL_p(E)} := \sup_{t \in (0,\infty)} t\, |\{z \in E : |f(z)| > t\}|^{\frac{1}{p}} < \infty \right\}.$$

Convention 2.1: We adopt the following conventions:

- Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.
- For a fixed p with $p \in [1, \infty]$, p' denotes the dual exponent of p, namely,

$$p' := \begin{cases} \infty & \text{if } \ p = 1, \\ \frac{p}{p-1} & \text{if } \ 1 < p < \infty, \\ 1 & \text{if } \ p = \infty. \end{cases} \tag{11}$$

Recall the following complete characterization of the weighted Hardy inequality on the cone of non-increasing functions (For the history of these inequalities see, for instance, [28] and [29]). We will use the notations:

$$U_*(t) := \int_t^\infty u(x)\, dx, \qquad V_*(t) := \int_t^\infty v(x)\, dx, \qquad W(t) := \int_0^t w(x)\, dx, \quad t > 0.$$

Theorem 2.2 [29, Theorem 5.2]: *Let $0 < q, p \le \infty$. Then the inequality*

$$\|H_u^* f\|_{L_{q,w}(0,\infty)} \le c \|f\|_{L_{p,v}(0,\infty)}, \quad f \in \mathfrak{M}^{\uparrow}, \tag{12}$$

where

$$H_u^* g(t) := \int_t^\infty g(s) u(s)\, ds, \qquad g \in \mathfrak{M}^+,$$

holds with the best constant c if and only if:

(i) $1 < p \le q < \infty$, *and in this case $c \approx A_0^* + A_1^*$, where*

$$A_0^* := \sup_{t \in (0,\infty)} \left(\int_t^\infty U_*^q(\tau) w(\tau)\, d\tau \right)^{\frac{1}{q}} V_*^{-\frac{1}{p}}(t),$$

$$A_1^* := \sup_{t \in (0,\infty)} W^{\frac{1}{q}}(t) \left(\int_t^\infty \left(\frac{U_*(\tau)}{V_*(\tau)} \right)^{p'} v(\tau)\, d\tau \right)^{\frac{1}{p'}};$$

(ii) $q < p < \infty$ *and* $1 < p < \infty$, *and in this case $c \approx B_0^* + B_1^*$, where*

$$B_0^* := \left(\int_0^\infty V_*^{-\frac{r}{p}}(t) \left(\int_t^\infty U_*^q(\tau) w(\tau)\, d\tau \right)^{\frac{r}{p}} U_*^q(t) w(t)\, dt \right)^{\frac{1}{r}},$$

$$B_1^* := \left(\int_0^\infty W^{\frac{r}{p}}(t) \left(\int_t^\infty \left(\frac{U_*(\tau)}{V_*(\tau)} \right)^{p'} v(\tau)\, d\tau \right)^{\frac{r}{p'}} w(t)\, dt \right)^{\frac{1}{r}};$$

(iii) *$q < p \le 1$, and in this case $c \approx B_0^* + C_1^*$, where*

$$C_1^* := \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{y \in (t,\infty)} \frac{U_*^p(y)}{V_*(y)} \right)^{\frac{r}{p}} W^{\frac{r}{p}}(t) w(t)\, dt \right)^{\frac{1}{r}};$$

(iv) *$p \le q < \infty$ and $p \le 1$, and in this case $c = D_0^*$, where*

$$D_0^* := \sup_{t \in (0,\infty)} V_*^{-\frac{1}{p}}(t) \left(\int_0^\infty U_*^q(\max\{\tau, t\}) w(\tau)\, d\tau \right)^{\frac{1}{q}};$$

(v) *$p \le 1$ and $q = \infty$, and in this case $c = E_0$, where*

$$E_0^* := \operatorname*{ess\,sup}_{t \in (0,\infty)} V_*^{-\frac{1}{p}}(t) \left(\operatorname*{ess\,sup}_{\tau \in (0,\infty)} U_*(\max\{\tau, t\}) w(\tau) \right);$$

(vi) *$1 < p < \infty$ and $q = \infty$, and in this case $c = F_0^*$, where*

$$F_0^* := \operatorname*{ess\,sup}_{t \in (0,\infty)} w(t) \left(\int_t^\infty \left(\int_t^\tau u(y) V_*^{-1}(y)\, dy \right)^{p'} v(\tau)\, d\tau \right)^{\frac{1}{p'}};$$

(vii) *$p = \infty$ and $0 < q < \infty$, and in this case $c = G_0^*$, where*

$$G_0^* := \left(\int_0^\infty \left(\int_t^\infty \frac{u(y)\, dy}{\operatorname*{ess\,sup}_{\tau \in (y,\infty)} v(\tau)} \right)^q w(t)\, dt \right)^{\frac{1}{q}};$$

(viii) *$p = q = \infty$, and in this case $c = H_0^*$, where*

$$H_0^* := \operatorname*{ess\,sup}_{t \in (0,\infty)} \left(\int_t^\infty \frac{u(y)\, dy}{\operatorname*{ess\,sup}_{\tau \in (y,\infty)} v(\tau)} \right) w(t).$$

For the sake of completeness we recall the definition of spaces we are going to use, and some properties of them.

Definition 2.3: Let $0 < p, \theta \le \infty$ and let ω be a non-negative measurable function on $(0,1)$. We denote by $LM_{p\theta,\omega}(\mathbb{D})$ the local Morrey-type space, the space of all measurable functions f on $\mathbb{D}$ with finite quasinorms

$$\|f\|_{LM_{p\theta,\omega}(\mathbb{D})} = \|\omega(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,1)}.$$

Definition 2.4: Let $0 < p, \theta \le \infty$ and let ω be a non-negative measurable function on $(0,1)$. We denote by $WLM_{p\theta,\omega}(\mathbb{D})$ the weak local Morrey-type space, the space of all measurable functions f on $\mathbb{D}$ with finite quasinorms

$$\|f\|_{WLM_{p\theta,\omega}(\mathbb{D})} = \|\omega(r)\|f\|_{WL_p(B(0,r))}\|_{L_\theta(0,1)}.$$

Remark 2.5: In view of the inequalities

$$\begin{aligned}\|\omega(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,1)} &\geq \|\omega(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(t,1)}\\ &\geq \|\omega\|_{L_\theta(t,1)}\,\|f\|_{L_p(B(0,t))}, \qquad t\in(0,1),\\ \|\omega(r)\|f\|_{WL_p(B(0,r))}\|_{L_\theta(0,1)} &\geq \|\omega(r)\|f\|_{WL_p(B(0,r))}\|_{L_\theta(t,1)}\\ &\geq \|\omega\|_{L_\theta(t,1)}\,\|f\|_{WL_p(B(0,t))}, \qquad t\in(0,1),\end{aligned}$$

it is clear that

$$LM_{p\theta,\omega}(\mathbb{D}) = WLM_{p\theta,\omega}(\mathbb{D}) = \{0\} \quad \text{when} \quad \|\omega\|_{L_\theta(t,1)} = \infty \quad \text{for all} \quad t\in(0,1).$$

Definition 2.6: We denote by Ω_θ the set of all non-negative measurable functions ω on $(0,1)$ which are non-negative, measurable on $(0,1)$, and such that

$$0 < \|\omega\|_{L_\theta(t,1)} < \infty, \quad t\in(0,1).$$

When considering $LM_{p\theta,\omega}(\mathbb{D})$ and $WLM_{p\theta,\omega}(\mathbb{D})$ we always assume that $\omega\in\Omega_\theta$.

We recall that the space $LM_{p\theta,\omega}(\mathbb{D})$ coincides with some weighted Lebesgue space.

Theorem 2.7 [21,22]: *Let $1\leq p<\infty$ and $\omega\in\Omega_p$. Then*

$$L_{p,\widetilde{\omega}(|\cdot|)}(\mathbb{D}) = LM_{pp,\omega}(\mathbb{D}),$$

and norms are equivalent, where

$$\widetilde{\omega}(\tau) := \int_\tau^1 \omega(t)^p\,\mathrm{d}t.$$

Definition 2.8: Let $0<p,\theta\leq\infty$ and let ω be a non-negative measurable function on $(0,1)$. We denote by $\widetilde{LM}_{p\theta,\omega}(\mathbb{D})$ ($\widetilde{WLM}_{p\theta,\omega}(\mathbb{D})$) the modified local Morrey-type space (the modified weak local Morrey-type space), the space of all measurable functions f on $\mathbb{D}$ with finite quasinorms

$$\|f\|_{\widetilde{LM}_{p\theta,\omega}(\mathbb{D})} = \|f\|_{LM_{p\theta,\omega}(\mathbb{D})} + \|f\|_{L_p(\mathbb{D})}$$

$$\left(\|f\|_{\widetilde{WLM}_{p\theta,\omega}(\mathbb{D})} = \|f\|_{WLM_{p\theta,\omega}(\mathbb{D})} + \|f\|_{WL_p(\mathbb{D})}\right).$$

In other words, the modified local Morrey-type spaces are nothing but the intersection of local Morrey-type spaces and the Lebesgue spaces $L_p(\mathbb{D})$, as well as, the modified weak local Morrey-type spaces are the intersection of weak local Morrey-type spaces and weak Lebesgue spaces $WL_p(\mathbb{D})$.

Remark 2.9: It is easy to see that

$$\|f\|_{L_p(\mathbb{D})} \leq \|f\|_{\widetilde{LM}_{p\theta,\omega}(\mathbb{D})} \leq \left(1+\|\omega\|_{L_\theta(0,1)}\right)\,\|f\|_{L_p(\mathbb{D})},$$

and

$$\|f\|_{WL_p(\mathbb{D})} \leq \|f\|_{\widetilde{WLM}_{p\theta,\omega}(\mathbb{D})} \leq \left(1+\|\omega\|_{L_\theta(0,1)}\right)\,\|f\|_{WL_p(\mathbb{D})},$$

that is, $\widetilde{LM}_{p\theta,\omega}(\mathbb{D})$ and $\widetilde{WLM}_{p\theta,\omega}(\mathbb{D})$ coincide with $L_p(\mathbb{D})$ and $WL_p(\mathbb{D})$, respectively, when $\|\omega\|_{L_\theta(0,1)} < \infty$.

3. Main results

As it is mentioned in the introduction, the operators Π_k, $k \in \mathbb{N}$ are bounded on $L_p(\mathbb{D})$, $1 < p < \infty$. Our main result in this paper is to extend these results to modified local Morrey-type spaces.

Theorem 3.1: *Let $k \in \mathbb{N}$, $1 < p < \infty$, $0 < \theta_1, \theta_2 \leq \infty$ and $\omega_i \in \Omega_{\theta_i}$, $i = 1, 2$. If*

(a) $1 < \theta_1 \leq \theta_2 < \infty$, *and*

$$\sup_{t\in(0,1)} \left(\int_t^1 \left(1-\tau^{\frac{2}{p}}\right)^{\theta_2} \omega_2^{\theta_2}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\theta_2}} \left(\int_t^1 \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{\theta_1}} < \infty, \tag{13}$$

$$\sup_{t\in(0,1)} \left(\int_0^t \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{1}{\theta_2}} \times \left(\int_t^1 \left(\tau^{-\frac{2}{p}}-1\right)^{\theta_1'} \left(\int_\tau^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-\theta_1'} \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\theta_1'}} < \infty; \tag{14}$$

(b) $\theta_2 < \theta_1 < \infty$, $1 < \theta_1 < \infty$, $1/r = 1/\theta_2 - 1/\theta_1$, *and*

$$\left(\int_0^1 \left(\int_t^1 \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{-\frac{r}{\theta_1}} \times \left(\int_t^1 \left(1-\tau^{\frac{2}{p}}\right)^{\theta_2} \omega_2^{\theta_2}(\tau)\,\mathrm{d}\tau\right)^{\frac{r}{\theta_1}} \left(1-t^{\frac{2}{p}}\right)^{\theta_2} \omega_2^{\theta_2}(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty, \tag{15}$$

$$\left(\int_0^1 \left(\int_0^t \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{r}{\theta_1}} \times \left(\int_t^1 \left(\tau^{-\frac{2}{p}}-1\right)^{\theta_1'} \left(\int_\tau^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-\theta_1'} \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{\frac{r}{\theta_1'}} \omega_2^{\theta_2}(t)t^{\frac{2\theta_2}{p}}\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty; \tag{16}$$

(c) $\theta_2 < \theta_1 \leq 1$, $1/r = 1/\theta_2 - 1/\theta_1$, *and* (15) *holds and*

$$\left(\int_0^1 \left(\operatorname*{ess\,sup}_{y\in(t,1)} \left(y^{-\frac{2}{p}}-1\right)^{\theta_1} \left(\int_y^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-1}\right)^{\frac{r}{\theta_1}} \times \left(\int_0^t \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{r}{\theta_1}} \omega_2^{\theta_2}(t)t^{\frac{2\theta_2}{p}}\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty; \tag{17}$$

(d) $\theta_1 \le \theta_2 < \infty$, $\theta_1 \le 1$, *and*

$$\sup_{t\in(0,1)} \left(\int_t^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-\frac{1}{\theta_1}} \times \left(\int_0^1 \left(\min\left\{\tau^{-\frac{2}{p}}-1, t^{-\frac{2}{p}}-1\right\}\right)^{\theta_2} \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{1}{\theta_2}} < \infty; \tag{18}$$

(e) $\theta_1 = \infty$, $0 < \theta_2 < \infty$, *and*

$$\left(\int_0^1 \left(\int_t^1 \frac{y^{-\frac{2}{p}-1}\mathrm{d}y}{\operatorname*{ess\,sup}_{\tau\in(y,1)} \omega_1(\tau)}\right)^{\theta_2} \omega_2^{\theta_2}(t)\, t^{\frac{2\theta_2}{p}}\,\mathrm{d}t\right)^{\frac{1}{\theta_2}} < \infty; \tag{19}$$

(f) $\theta_1 \le 1$, $\theta_2 = \infty$, *and*

$$\operatorname*{ess\,sup}_{t\in(0,1)} \left(\int_t^1 \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{\theta_1}} \left(\operatorname*{ess\,sup}_{\tau\in(0,1)} \min\left\{\tau^{-\frac{2}{p}}-1, t^{-\frac{2}{p}}-1\right\}\omega_2(\tau)\tau^{\frac{2}{p}}\right) < \infty; \tag{20}$$

(g) $1 < \theta_1 < \infty$, $\theta_2 = \infty$, *and*

$$\operatorname*{ess\,sup}_{t\in(0,1)} \omega_2(t)t^{\frac{2}{p}} \left(\int_t^1 \left(\int_t^\tau y^{-\frac{2}{p}-1}\left(\int_y^1 \omega_1^{\theta_1}(x)\,\mathrm{d}x\right)^{-1}\mathrm{d}y\right)^{\theta_1'} \omega_1^{\theta_1}(\tau)\mathrm{d}\tau\right)^{\frac{1}{\theta_1'}} < \infty; \tag{21}$$

(h) $\theta_1 = \theta_2 = \infty$, *and*

$$\operatorname*{ess\,sup}_{t\in(0,1)} \left(\int_t^1 \frac{y^{-\frac{2}{p}-1}\mathrm{d}y}{\operatorname*{ess\,sup}_{\tau\in(y,1)} \omega_1(\tau)}\right)\omega_2(t)t^{\frac{2}{p}} < \infty, \tag{22}$$

then there exists a constant $c > 0$ *such that the inequality*

$$\|\Pi_k f\|_{\widetilde{LM}_{p\theta_2,\omega_2}(\mathbb{D})} \le c\|f\|_{\widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})}$$

holds for all $f \in \widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})$.

In view of Theorem 2.7, by Theorem 3.1, we immediately get the following statement:

Corollary 3.2: *Let* $k \in \mathbb{N}$, $1 < p < \infty$ *and* $\omega_i \in \Omega_p$, $i = 1, 2$. *If*

$$\sup_{t\in(0,1)} \left(\int_t^1 \left(1-\tau^{\frac{2}{p}}\right)^p \omega_2^p(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}} \left(\int_t^1 \omega_1^p(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{p}} < \infty, \tag{23}$$

$$\sup_{t\in(0,1)} \left(\int_0^t \tau^2\omega_2^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p}} \left(\int_t^1 \left(\tau^{-\frac{2}{p}}-1\right)^{p'}\left(\int_\tau^1 \omega_1^p(s)\mathrm{d}s\right)^{-p'} \omega_1^p(\tau)\mathrm{d}\tau\right)^{\frac{1}{p'}} < \infty, \tag{24}$$

then

$$\|\Pi_k f\|_{L_{p,\bar{\omega}_2(|\cdot|)}(\mathbb{D})} \le c\|f\|_{L_{p,\bar{\omega}_1(|\cdot|)}(\mathbb{D})},$$

with constant $c > 0$ independent of f.

Here

$$\bar{\omega}_i(t) := \int_t^1 \omega_i(\tau)^p \, \mathrm{d}\tau + 1, \quad i = 1, 2. \tag{25}$$

Since the function $\widetilde{T}_1 f$ is the solution of the Schwarz BVP

$$g_{\bar{z}} = f \text{ in } \mathbb{D}, \; \mathrm{Re}\, g = 0 \text{ on } \partial\mathbb{D}, \; \mathrm{Im}\, g\left(0\right) = 0, \tag{26}$$

when $f \in L^1(\mathbb{D})$, by Theorem 3.1 and Corollary 3.2, respectively, we get the following a priori estimates for the derivative of the solution of (26).

Theorem 3.3: *Let $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$ and $\omega_i \in \Omega_{\theta_i}$, $i = 1, 2$. If conditions (a)–(h) in Theorem 3.1 hold, then for the solution of (26) the inequality*

$$\|\partial_z g\|_{\widetilde{LM}_{p\theta_2,\omega_2}(\mathbb{D})} \le c\|f\|_{\widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})}$$

holds for all $f \in \widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})$ with a constant $c > 0$ independent of f.

Corollary 3.4: *Let $1 < p < \infty$ and $\omega_i \in \Omega_p$, $i = 1, 2$. If conditions (23) and (24) hold, then for the solution of (26) the inequality*

$$\|\partial_z g\,\|_{L_{p,\bar{\omega}_2(|\cdot|)}(\mathbb{D})} \le c\|f\|_{L_{p,\bar{\omega}_1(|\cdot|)}(\mathbb{D})},$$

holds for all $f \in L_{p,\bar{\omega}_1(|\cdot|)}(\mathbb{D})$ with a constant $c > 0$ independent of f, where $\bar{\omega}_i$, $i = 1, 2$ are defined by (25).

4. Local L_p-estimates of sublinear operators

Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{D})$ and $z \notin \mathrm{supp}\, f$

$$|Tf(z)| \lesssim \iint_{\mathbb{D}} \frac{|f(\zeta|}{|z-\zeta|^2} \, \mathrm{d}\xi \, \mathrm{d}\eta, \tag{27}$$

with a constant independent of f and z.

We point out that the condition (27), when f is defined on $\mathbb{R}^n$, was introduced by Soria and Weiss in [30]. The Soria–Weiss condition is satisfied by many interesting operators in Harmonic Analysis, such as the Calderón-Zygmund singular operators, Carleson's maximal operators, Hardy-Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory integrals, the Bochner-Riesz means and so on (cf. [17,30–34]).

Theorem 4.1: *Let T be a sublinear operator satisfying condition (27).*

(i) *Let $1 < p < \infty$ and T be bounded on $L_p(\mathbb{D})$. If $f \in L_p(\mathbb{D})$ such that*

$$\int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))} \, \mathrm{d}t < \infty \quad \textit{for all} \quad \tau \in (0, 1), \tag{28}$$

then for any $\tau \in (0,1)$ *the inequality*

$$\|Tf\|_{L_p(B(0,\tau))} \leq c\tau^{\frac{2}{p}} \int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))}\, \mathrm{d}t + c\tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})} \tag{29}$$

holds with constant $c > 0$ *independent of* f *and* τ.

(ii) *Let* $1 \leq p < \infty$ *and* T *be bounded from* $L_p(\mathbb{D})$ *to* $WL_p(\mathbb{D})$. *If* $f \in L_p(\mathbb{D})$ *satisfies condition* (28), *then for any* $\tau \in (0,1)$ *the inequality*

$$\|Tf\|_{WL_p(B(0,\tau))} \leq c\tau^{\frac{2}{p}} \int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))}\, \mathrm{d}t + c\tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})} \tag{30}$$

holds with constant $c > 0$ *independent of* f *and* τ.

Proof: Let $1 \leq p < \infty$. Since

$$\tau^{\frac{2}{p}} \int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))}\, \mathrm{d}t \geq \tau^{\frac{2}{p}} \|f\|_{L_p(B(0,\tau))} \int_\tau^1 t^{-\frac{2}{p}-1}\, \mathrm{d}t$$
$$\approx \|f\|_{L_p(B(0,\tau))}(1-\tau^{\frac{2}{p}}), \quad \tau \in (0,1),$$

we get that

$$\|f\|_{L_p(B(0,\tau))} \lesssim \tau^{\frac{2}{p}} \int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))}\, \mathrm{d}t + \tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})}, \quad \tau \in (0,1). \tag{31}$$

(i) Assume that $p > 1$ and T is bounded on $L_p(\mathbb{D})$. Let $\tau \in (0, 1/2)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{B(0,2\tau)}$ and $f_2 = f\chi_{\mathbb{D}\setminus B(0,2\tau)}$.
Taking into account the sublinearity of T, we have

$$\|Tf\|_{L_p(B(0,\tau))} \leq \|Tf_1\|_{L_p(B(0,\tau))} + \|Tf_2\|_{L_p(B(0,\tau))}. \tag{32}$$

Since $f_1 \in L_p(\mathbb{D})$, the boundedness of T in $L_p(\mathbb{D})$ implies that

$$\|Tf_1\|_{L_p(B(0,\tau))} \leq \|Tf_1\|_{L_p(\mathbb{D})} \lesssim \|f_1\|_{L_p(\mathbb{D})} \approx \|f\|_{L_p(B(0,2\tau))}, \tag{33}$$

where the constant is independent of f and τ.
In view of (31) we get that

$$\|Tf_1\|_{L_p(B(0,\tau))} \lesssim \tau^{\frac{2}{p}} \int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))}\, \mathrm{d}t + \tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})}. \tag{34}$$

By (27), we have that

$$|Tf_2(t)| \lesssim \iint_{\mathbb{D}\setminus B(0,2\tau)} \frac{|f(\zeta)|}{|t-\zeta|^2}\, \mathrm{d}\xi\, \mathrm{d}\eta, \qquad t \in B(0,\tau).$$

It is clear that $t \in B(0,\tau)$, $\zeta \in \mathbb{D}\backslash B(0,2\tau)$ implies $(1/2)|\zeta| \le |t-\zeta| < (3/2)|\zeta|$. Therefore we obtain that

$$\|Tf_2\|_{L_p(B(0,\tau))} \lesssim \tau^{\frac{2}{p}} \iint_{\mathbb{D}\backslash B(0,2\tau)} \frac{|f(\zeta)|}{|\zeta|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

By Fubini's theorem, we get that

$$\begin{aligned}
&\iint_{\mathbb{D}\backslash B(0,2\tau)} \frac{|f(\zeta)|}{|\zeta|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\approx \iint_{\mathbb{D}\backslash B(0,2\tau)} |f(\zeta)| \left(1+\int_{|\zeta|}^1 \frac{\mathrm{d}s}{s^3}\right)\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \iint_{\mathbb{D}\backslash B(0,2\tau)} |f(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta + \iint_{D\backslash B(0,2\tau)} |f(\zeta)| \left(\int_{|\zeta|}^1 \frac{\mathrm{d}s}{s^3}\right)\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \iint_{\mathbb{D}\backslash B(0,2\tau)} |f(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{2\tau}^1 \left(\iint_{2\tau\le|\zeta|\le s} |f(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta\right)\frac{\mathrm{d}s}{s^3}\\
&\le \iint_{\mathbb{D}} |f(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{2\tau}^1 \left(\iint_{B(0,s)} |f(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta\right)\frac{\mathrm{d}s}{s^3}.
\end{aligned}$$

Applying Hölder's inequality, we arrive at

$$\iint_{\mathbb{D}\backslash B(0,2\tau)} \frac{|f(\zeta)|}{|\zeta|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \lesssim \|f\|_{L_p(\mathbb{D})} + \int_{2\tau}^1 s^{-\frac{2}{p}-1}\|f\|_{L_p(B(0,s))}\,\mathrm{d}s.$$

Thus the inequality

$$\|Tf_2\|_{L_p(B(0,\tau))} \lesssim \tau^{\frac{2}{p}} \int_{\tau}^1 s^{-\frac{2}{p}-1}\|f\|_{L_p(B(0,s))}\,\mathrm{d}s + \tau^{\frac{2}{p}}\|f\|_{L_p(\mathbb{D})} \tag{35}$$

holds for all $\tau \in (0,1/2)$.
Finally, combining (32), (34) and (35), we obtain that

$$\|Tf\|_{L_p(B(0,\tau))} \lesssim \tau^{\frac{2}{p}} \int_{\tau}^1 s^{-\frac{2}{p}-1}\|f\|_{L_p(B(0,s))}\,\mathrm{d}s + \tau^{\frac{2}{p}}\|f\|_{L_p(\mathbb{D})}$$

holds for all $\tau \in (0,1/2)$ with a constant independent of f and τ.
Let now $\tau \in [1/2,1)$. Then, using $L_p(\mathbb{D})$-boundedness of T, we obtain

$$\|Tf\|_{L_p(B(0,\tau))} \le \|Tf\|_{L_p(\mathbb{D})} \lesssim \|f\|_{L_p(\mathbb{D})} \approx \tau^{\frac{2}{p}}\|f\|_{L_p(\mathbb{D})},$$

and, inequality (29) holds.

(ii) Assume that $1 \le p < \infty$ and T is bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$. Let again $\tau \in (0,1/2)$, and write $f = f_1 + f_2$ with $f_1 = f\chi_{B(0,2\tau)}$ and $f_2 = f\chi_{\mathbb{D}\backslash B(0,2\tau)}$. Taking into account the sublinearity of T, we have

$$\|Tf\|_{WL_p(B(0,\tau))} \le \|Tf_1\|_{WL_p(B(0,\tau))} + \|Tf_2\|_{WL_p(B(0,\tau))}. \tag{36}$$

Since $f_1 \in L_p(\mathbb{D})$, in view of (31), the boundedness of T from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$ implies that

$$\begin{aligned} \|Tf_1\|_{WL_p(B(0,\tau))} &\le \|Tf_1\|_{WL_p(\mathbb{D})} \lesssim \|f_1\|_{L_p(\mathbb{D})} \approx \|f\|_{L_p(B(0,2\tau))} \\ &\lesssim \tau^{\frac{2}{p}} \int_\tau^1 t^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,t))}\, \mathrm{d}t + \tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})}, \end{aligned} \tag{37}$$

where the constant is independent of f and τ.

On the other hand, since

$$\|Tf_2\|_{WL_p(B(0,\tau))} \le \|Tf_2\|_{L_p(B(0,\tau))},$$

using (35), we get that

$$\|Tf_2\|_{WL_p(B(0,\tau))} \lesssim \tau^{\frac{2}{p}} \int_\tau^1 s^{-\frac{2}{p}-1} \|f\|_{L_p(B(0,s))}\, \mathrm{d}s + \tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})} \tag{38}$$

holds true for all $\tau \in (0, 1/2)$.

Combining (36), (37) and (38), we see that inequality (30) holds true for all $\tau \in (0, 1/2)$ with a constant independent of f and τ.

If $\tau \in [1/2, 1)$, then, using the boundedness of T from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$, we obtain that

$$\|Tf\|_{WL_p(B(0,\tau))} \le \|Tf\|_{WL_p(\mathbb{D})} \lesssim \|f\|_{L_p(\mathbb{D})} \approx \tau^{\frac{2}{p}} \|f\|_{L_p(\mathbb{D})},$$

and, inequality (30) holds.

□

5. Boundedness of sublinear operators in modified local Morrey-type spaces

In this section we prove the boundedness of sublinear operators satisfying condition (27) in modified local Morrey-type spaces $\widetilde{LM}_{p\theta,\omega}(\mathbb{D})$.

The following statement is true.

Theorem 5.1: *Let $0 < \theta_1, \theta_2 \le \infty$ and $\omega_i \in \Omega_{\theta_i}$, $i = 1, 2$. Assume that T is a sublinear operator satisfying condition (27).*

(i) *Let $1 < p < \infty$ and T be bounded on $L_p(\mathbb{D})$. If conditions (a)–(h) in Theorem 3.1 hold, then*

$$\|Tf\|_{\widetilde{LM}_{p\theta_2,\omega_2}(\mathbb{D})} \le c\|f\|_{\widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})}, \tag{39}$$

where constant $c > 0$ is independent of f.

(ii) *Let $1 \le p < \infty$ and T be bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$. If conditions (a)–(h) in Theorem 3.1 hold, then*

$$\|Tf\|_{\widetilde{WLM}_{p\theta_2,\omega_2}(\mathbb{D})} \le c\|f\|_{\widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})}. \tag{40}$$

where constant $c > 0$ is independent of f.

***Proof*:** At first we show that

$$\left\| s^{2/p}\omega_2(s)\right\|_{L_{\theta_2}(0,1)} < \infty, \tag{41}$$

when conditions (a)–(h) hold.

Since $\omega_2 \in \Omega_{\theta_2}$, then $\|\omega_2\|_{L_{\theta_2}(t,1)} < \infty$ for any $t \in (0,1)$. Thus

$$\left\| s^{2/p}\omega_2(s)\right\|_{L_{\theta_2}(t,1)} \le \|\omega_2\|_{L_{\theta_2}(t,1)} < \infty. \tag{42}$$

(a) Condition (14), in particular, implies that

$$\left(\int_0^t \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{1}{\theta_2}}\left(\int_t^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-1}\left(\int_t^1 \left(\tau^{-\frac{2}{p}}-1\right)^{\theta_1'}\omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\theta_1'}} < \infty$$

for all $t \in (0,1)$. Since $\omega_1 \in \Omega_{\theta_1}$, we get that inequality

$$\left\| s^{2/p}\omega_2(s)\right\|_{L_{\theta_2}(0,t)} < \infty \qquad t \in (0,1) \tag{43}$$

holds.

(b) Condition (16), in particular, implies that

$$\left(\int_0^t \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{1}{\theta_1}+\frac{1}{r}}\left(\int_t^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-1}\left(\int_t^1 \left(\tau^{-\frac{2}{p}}-1\right)^{\theta_1'}\omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\theta_1'}} < \infty$$

for all $t \in (0,1)$. Since $\omega_1 \in \Omega_{\theta_1}$, then (43) holds.

(c) In view of (17), we have that

$$\operatorname*{ess\,sup}_{y\in(t,1)} \left(y^{-\frac{2}{p}}-1\right)\left(\int_y^1 \omega_1^{\theta_1}(s)\,\mathrm{d}s\right)^{-\frac{1}{\theta_1}}\left(\int_0^t \omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{1}{\theta_1}+\frac{1}{r}} < \infty$$

for all $t \in (0,1)$. Since $\omega_1 \in \Omega_{\theta_1}$, then (43) holds.

(d) In view of $\omega_1 \in \Omega_{\theta_1}$, condition (18) yields that

$$\left(\int_0^t \left(\min\left\{\tau^{-\frac{2}{p}}-1, t^{-\frac{2}{p}}-1\right\}\right)^{\theta_2}\omega_2^{\theta_2}(\tau)\tau^{\frac{2\theta_2}{p}}\,\mathrm{d}\tau\right)^{\frac{1}{\theta_2}} < \infty$$

for all $t \in (0,1)$, which implies (43).

(e) Using (19), we obtain that

$$\left(\int_0^s \left(\int_t^1 \frac{y^{-\frac{2}{p}-1}\mathrm{d}y}{\operatorname*{ess\,sup}_{\tau\in(y,1)}\omega_1(\tau)}\right)^{\theta_2}\omega_2^{\theta_2}(t)\,t^{\frac{2\theta_2}{p}}\,\mathrm{d}t\right)^{\frac{1}{\theta_2}} < \infty, \qquad s \in (0,1).$$

Therefore,

$$\left(\operatorname*{ess\,sup}_{\tau\in(s,1)}\omega_1(\tau)\right)^{-1}\left(s^{-\frac{2}{p}}-1\right)\left(\int_0^s \omega_2^{\theta_2}(t)\,t^{\frac{2\theta_2}{p}}\,\mathrm{d}t\right)^{\frac{1}{\theta_2}} < \infty, \qquad s \in (0,1)$$

and, in view of $\omega_1 \in \Omega_\infty$, (43) follows.

(f) Finiteness of (20) implies that

$$\operatorname*{ess\,sup}_{t\in(s,1)}\left(\int_t^1 \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{\theta_1}}\left(\operatorname*{ess\,sup}_{\tau\in(0,t)}\min\left\{\tau^{-\frac{2}{p}}-1,t^{-\frac{2}{p}}-1\right\}\omega_2(\tau)\tau^{\frac{2}{p}}\right)<\infty$$

for all $s \in (0,1)$. Consequently,

$$\left(\int_s^1 \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{\theta_1}}\operatorname*{ess\,sup}_{t\in(s,1)}\left(t^{-\frac{2}{p}}-1\right)\left(\operatorname*{ess\,sup}_{\tau\in(0,t)}\omega_2(\tau)\tau^{\frac{2}{p}}\right)<\infty,$$

and whence

$$\left(\int_s^1 \omega_1^{\theta_1}(\tau)\,\mathrm{d}\tau\right)^{-\frac{1}{\theta_1}}\left(s^{-\frac{2}{p}}-1\right)\left(\operatorname*{ess\,sup}_{\tau\in(0,s)}\omega_2(\tau)\tau^{\frac{2}{p}}\right)<\infty,$$

for all $s \in (0,1)$. In view of $\omega_1 \in \Omega_{\theta_1}$, (43) holds.

(g) Condition (21) implies that

$$\left(\int_s^1 \omega_1^{\theta_1}(x)\,\mathrm{d}x\right)^{-1}\left(\int_s^1\left(\int_s^t y^{-\frac{2}{p}-1}\,\mathrm{d}y\right)^{\theta_1'}\omega_1^{\theta_1}(t)\,\mathrm{d}t\right)^{\frac{1}{\theta_1'}}\operatorname*{ess\,sup}_{\tau\in(0,s)}\omega_2(\tau)\tau^{\frac{2}{p}}<\infty$$

for all $s \in (0,1)$. In view of $\omega_1 \in \Omega_{\theta_1}$, (43) holds.

(h) By (22), we have that

$$\left(\operatorname*{ess\,sup}_{\tau\in(s,1)}\omega_1(\tau)\right)^{-1}\left(s^{-\frac{2}{p}}-1\right)\operatorname*{ess\,sup}_{t\in(0,s)}\omega_2(t)t^{\frac{2}{p}}<\infty \quad s\in(0,1).$$

Again, in view of $\omega_1 \in \Omega_\infty$, we have that (43) holds.
Combining (42) and (43), we find that (41) holds.

(i) Let $1 < p < \infty$, T be bounded on $L_p(\mathbb{D})$ and conditions (a)–(h) in Theorem 3.1 hold. Assume that $f \in \widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})$. In view of Theorem 4.1 (i), using $L_p(\mathbb{D})$ boundedness of T, Theorem 2.2, and inequality (41), we obtain that

$$\begin{aligned}
\|Tf\|_{\widetilde{LM}_{p\theta_2,\omega_2}(\mathbb{D})} &= \left\|\omega_2(\tau)\|Tf\|_{L_p(B(0,\tau))}\right\|_{L_{\theta_2}(0,1)}+\|Tf\|_{L_p(\mathbb{D})}\\
&\lesssim c\left\|\omega_2(\tau)\tau^{2/p}\int_\tau^1 t^{-2/p-1}\|f\|_{L_p(B(0,t))}\,\mathrm{d}t\right\|_{L_{\theta_2}(0,1)}\\
&\quad + c\left\|\tau^{2/p}\omega_2(\tau)\right\|_{L_{\theta_2}(0,1)}\|f\|_{L_p(\mathbb{D})}+c\,\|f\|_{L_p(\mathbb{D})}\\
&\le c\left\|\omega_1(\tau)\|f\|_{L_p(B(0,\tau))}\right\|_{L_{\theta_1}(0,1)}+c\left\|\tau^{2/p}\omega_2(\tau)\right\|_{L_{\theta_2}(0,1)}\|f\|_{L_p(\mathbb{D})}\\
&\quad + c\,\|f\|_{L_p(\mathbb{D})}\\
&= c\left(\|f\|_{LM_{p\theta_1,\omega_1}(\mathbb{D})}+\|f\|_{L_p(\mathbb{D})}\right)=c\,\|f\|_{\widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})}.
\end{aligned}$$

(ii) Let $1 \leq p < \infty$, T be bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$, and conditions (a)–(h) in Theorem 3.1 hold. Assume that $f \in \widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})$. In view of Theorem 4.1 (ii), using the boundedness of T from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$, Theorem 2.2, and inequality (41), we arrive at

$$\begin{aligned}\|Tf\|_{\widetilde{WLM}_{p\theta_2,\omega_2}(\mathbb{D})} &= \left\|\omega_2(\tau)\|Tf\|_{WL_p(B(0,\tau))}\right\|_{L_{\theta_2}(0,1)} + \|Tf\|_{WL_p(\mathbb{D})} \\ &\lesssim c\left\|\omega_2(\tau)\tau^{2/p}\int_\tau^1 t^{-2/p-1}\|f\|_{L_p(B(0,t))}\,\mathrm{d}t\right\|_{L_{\theta_2}(0,1)} \\ &\quad + c\left\|\tau^{2/p}\omega_2(\tau)\right\|_{L_{\theta_2}(0,1)}\|f\|_{L_p(\mathbb{D})} + c\,\|f\|_{L_p(\mathbb{D})} \\ &\leq c\left(\|f\|_{LM_{p\theta_1,\omega_1}(\mathbb{D})} + \|f\|_{L_p(\mathbb{D})}\right) = c\,\|f\|_{\widetilde{LM}_{p\theta_1,\omega_1}(\mathbb{D})}.\end{aligned}$$

□

Corollary 5.2: *Let $1 < p < \infty$ and $\omega_i \in \Omega_p$, $i = 1, 2$. Assume that T is a sublinear operator satisfying condition (27), and being bounded on $L_p(\mathbb{D})$.*

If conditions (23) and (24) hold, then

$$\|Tf\|_{L_{p,\bar{\omega}_2(|\cdot|)}(\mathbb{D})} \leq c\|f\|_{L_{p,\bar{\omega}_1(|\cdot|)}(\mathbb{D})}, \tag{44}$$

with constant $c > 0$ independent of f, where $\bar{\omega}_i$, $i = 1, 2$ are defined by (25).

***Proof*:** The statement follows from Theorem 5.1 (i) and Theorem 2.7, when $\theta_1 = \theta_2 = p$. □

Proof of Theorem 3.1: Note that operators $T_{-k,k}$ and P_k, and consequently, the operator Π_k, satisfy condition (27). It is obvious for the operator $T_{-k,k}$, and easily follows for the operator P_k by the inequality

$$\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right| < 1 \quad \text{if} \quad |z| < 1 \quad \text{and} \quad |\zeta| < 1. \tag{45}$$

Indeed,

$$\begin{aligned}|P_k f(z)| &\leq \frac{k}{\pi}\iint_{\mathbb{D}}\left|\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}}\bar{\zeta} - 1\right|^{k-1}\frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq \frac{k}{\pi}\iint_{\mathbb{D}}\left(2\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right||\zeta| + 1\right)^{k-1}\frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq \frac{k3^{k-1}}{\pi}\iint_{\mathbb{D}}\frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq \frac{k3^{k-1}}{\pi}\iint_{\mathbb{D}}\frac{|f(\zeta)|}{|\zeta - z|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Since the operator Π_k is bounded in $L_p(\mathbb{D})$, the statement of Theorem 3.1 follows from Theorem 5.1 (i).

Acknowledgements

We thank the anonymous referee for his remarks, which have improved the final version of this paper. The research of V.S. Guliyev was partially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No: 02.a03.21.0008).

Disclosure statement

No potential conflict of interest was reported by the authors.

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