



Pointwise slant submersions from almost product Riemannian manifolds

Sezin Aykurt Sepet & Mahmut Ergut

To cite this article: Sezin Aykurt Sepet & Mahmut Ergut (2020) Pointwise slant submersions from almost product Riemannian manifolds, Journal of Interdisciplinary Mathematics, 23:3, 639-655, DOI: [10.1080/09720502.2019.1700935](https://doi.org/10.1080/09720502.2019.1700935)

To link to this article: <https://doi.org/10.1080/09720502.2019.1700935>



Published online: 02 Apr 2020.



Submit your article to this journal [↗](#)



Article views: 51



View related articles [↗](#)



View Crossmark data [↗](#)



Citing articles: 1 View citing articles [↗](#)



Pointwise slant submersions from almost product Riemannian manifolds

Sezin Aykurt Sepet *

*Department of Mathematics
Faculty of Art and Science
Kırşehir Ahi Evran University
Kırşehir
Turkey*

Mahmut Ergut

*Department of Mathematics
Faculty of Art and Science
Tekirdağ Namık Kemal University
Tekirdağ
Turkey*

Abstract

In this paper, we investigate pointwise slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We obtain some characterizations for such a submersion. Also we find curvature relations between the total manifold and the base manifold.

Subject Classification: (2010) 53C43, 53C15.

Keywords: Riemannian submersion, pointwise slant submersion, almost product Riemannian manifold.

1. Introduction

The geometry of the Riemannian submersions were firstly examined by O'Neill [17] and Gray [10]. Then in [26], Watson defined Riemannian submersions between almost Hermitian manifolds and given some

*E-mail: sezinaykurt@hotmail.com

characterizations. As a generalization of Hermitian submersions and anti-invariant submersions Sahin[23] defined slant submersions from almost Hermitian manifolds and investigated the geometry of such maps. Furthermore, Park study semi-slant submersions from almost Hermitian manifold onto Riemannian manifold. He deal with the integrability of distributions the geometry of fibers the harmonicity of these map. Later the geometric properties of submersions have been studied extensively between manifolds with different differentiable structures (see [11, 2, 4, 5, 12, 13, 14, 18, 19, 21, 24, 25]).

On the other hand, in [8], Etayo defined pointwise slant submanifolds as a generalization of slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds. Then, Chen and Gray investigate pointwise slant submanifolds and obtained several results [7].

Moreover, J. W. Lee and B. Sahin [15] as a generalization of slant submersion defined pointwise slant submersions from almost Hermitian manifolds. They also given a method for obtaining examples of such map. In addition, some functions on the riemannian manifolds are described and investigated in [20]. In this study, we define pointwise slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We obtain an example and some characterizations for such submersions.

2. Basic Properties of Riemannian submersions and almost product Riemannian manifolds

Let (\bar{M}, g) and (\bar{N}, g') are Riemannian manifolds with m and n -dimensional. A Riemannian submersion $\pi : \bar{M} \rightarrow \bar{N}$ is a surjective map that provides

1. π has the maximal rank,
2. π_* preserves the lengths of horizontal vectors.

Given a $q \in \bar{N}$, $\pi^{-1}(q)$ is an $(m - n)$ -dimensional submanifold of \bar{M} and is called fiber. If $X \in \Gamma(T\bar{M})$ is always tangent (resp. orthogonal) to fibers, then it is said to be vertical (resp. horizontal) [22]. If a vector field \bar{X} on \bar{M} is horizontal and π -related to a vector field \bar{X}_* on \bar{N} , i.e., $\pi_*\bar{X}_p = \bar{X}_{*\pi(p)}$ for all $p \in \bar{M}$, it is called basic. We will show the projection maps on the distributions $\ker\pi_*$ and $(\ker\pi_*)^\perp$ by \mathcal{V} and \mathcal{H} , respectively.

The theory of Riemannian submersions is given by O'Neill's tensors \mathcal{T} and \mathcal{A} defined by

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}_E} \mathcal{V}F + \mathcal{W}\nabla_{\mathcal{V}_E} \mathcal{H}F, \tag{2.1}$$

$$\mathcal{A}_E F = \mathcal{W}\nabla_{\mathcal{H}_E} \mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}_E} \mathcal{V}F, \tag{2.2}$$

where $E, F \in \Gamma(T\bar{M})$ and ∇ the Levi-Civita connection of (\bar{M}, g) [17].

Now we remember the following lemma from [17].

Lemma 1 : *Let π be a Riemannian submersion between Riemannian manifolds (\bar{M}, g) and (\bar{N}, g') . If \bar{X} and \bar{Y} are basic vector fields of \bar{M} , then*

- (i) $g(\bar{X}, \bar{Y}) = g'(\bar{X}_*, \bar{Y}_*) \circ \pi$,
- (ii) the horizontal part $[\bar{X}, \bar{Y}]^h$ of $[\bar{X}, \bar{Y}]$ is a basic vector field and $\pi_*([\bar{X}, \bar{Y}]^h) = [\bar{X}_*, \bar{Y}_*]$,
- (iii) $[V, \bar{X}] \in \Gamma(\ker \pi_*)$ for $V \in \ker \pi_*$,
- (iv) $(\nabla_{\bar{X}}^M \bar{Y})^h$ is the basic vector π -related to $\nabla_{\bar{X}_*}^N \bar{Y}_*$,

where ∇^M and ∇^N the Levi-Civita connections on \bar{M} and \bar{N} , respectively.

Furthermore, from (2.4) and (2.5), we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \tag{2.3}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X \tag{2.4}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{W}\nabla_X V \tag{2.5}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.6}$$

for $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and $V, W \in \Gamma(\ker \pi_*)$, where $\hat{\nabla}_V W = \mathcal{W}\nabla_V W$. Besides, if \bar{X} is a basic vector field, then $\mathcal{H}\nabla_V \bar{X} = \mathcal{A}_{\bar{X}} V$. Furthermore, for any $E \in \Gamma(TM)$, it is said that \mathcal{T} is vertical, i.e., $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}_E}$ and \mathcal{A} is horizontal, i.e., $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}_E}$. the tensor fields \mathcal{T} and \mathcal{A} that are skew-symmetric on tangent bundle of M satisfy the following equations

$$\mathcal{T}_U W = \mathcal{T}_W U, \tag{2.7}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} \mathcal{W}[X, Y] \tag{2.8}$$

for $U, W \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma((\ker \pi_*)^\perp)$. On the other hand, it is said that a Riemannian submersion $\pi: M \rightarrow N$ has totally geodesic fibers if and only if \mathcal{T} identically vanishes.

Suppose that (\bar{M}, g) and (\bar{N}, g') be Riemannian manifolds and $\psi: \bar{M} \rightarrow \bar{N}$ is a smooth mapping between them. Then the second fundamental form of ψ is defined as

$$\nabla \psi_*(X, Y) = \nabla_X^\psi \psi_*(Y) - \psi_*(\nabla_X Y) \quad (2.9)$$

for $X, Y \in \Gamma(T\bar{M})$, where ∇^ψ is the pullback connection and ∇ the Riemannian connections of the metrics g and g' . If ψ is a Riemannian submersion, then we can write

$$(\nabla \psi_*)(X, Y) = 0 \quad (2.10)$$

for $X, Y \in \Gamma((\ker \psi_*)^\perp)$.

On the other hand a smooth map $\pi: (\bar{M}, g) \rightarrow (\bar{N}, g')$ is called harmonic if

$$\text{trace} \nabla \pi_* = 0 \quad (2.11)$$

Also, from [6], π is said to be totally geodesic map if

$$(\nabla \pi_*)(X, Y) = 0. \quad (2.12)$$

for $X, Y \in \Gamma(T\bar{M})$.

We give the following curvature relations for a Riemannian submersion.

Theorem 1 : *Let (\bar{M}, g) and (\bar{N}, g) be two Riemannian manifolds with the corresponding curvature tensors \mathcal{R} and \mathcal{R}' , respectively. Let $\pi: (\bar{M}, g) \rightarrow (\bar{N}, g')$ be a Riemannian submersion and $\hat{\mathcal{R}}$ the curvature tensor of fibers of π . If X, Y, Z, H are horizontal and U, V, W, F vertical vectors, then*

$$\mathcal{R}(U, V, W, F) = \hat{\mathcal{R}}(U, V, W, F) + g(\mathcal{T}_U F, \mathcal{T}_U W) - g(\mathcal{T}_U F, \mathcal{T}_V W) \quad (2.13)$$

$$\begin{aligned} \mathcal{R}(X, Y, Z, H) &= \mathcal{R}'(\pi_* X, \pi_* Y, \pi_* Z, \pi_* H) + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ &\quad - g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H) \end{aligned} \quad (2.14)$$

$$\begin{aligned} \mathcal{R}(X, V, Y, W) &= -g((\nabla_X \mathcal{T})_V W, Y) + g((\nabla_V \mathcal{A})_X Y, W) \\ &\quad - g(\mathcal{T}_V X, \mathcal{T}_W Y) + g(\mathcal{A}_X V, \mathcal{A}_Y W) \end{aligned} \quad (2.15)$$

(see [9]).

An m -dimensional manifold \bar{M} is called an almost product manifold with almost product structure if a tensor F of type $(1, 1)$ is given in \bar{M} and satisfies

$$F^2 = I, \tag{2.16}$$

where I denotes the identity map. Then it can be written

$$P = \frac{1}{2}(I + F), Q = \frac{1}{2}(I - F).$$

Furthermore the following equations are given

$$P + Q = I, P^2 = P, Q^2 = Q, PQ = QP = 0, F = P - Q.$$

Thus, we say that the eigenvalues of F are $+1$ or -1 .

Given a Riemannian metric g in almost product manifold \bar{M} defined as follows

$$g(FX, FY) = g(X, Y) \tag{2.17}$$

for $X, Y \in \Gamma(T\bar{M})$, then (\bar{M}, F, g) is called an almost product Riemannian manifold [11]. We denote the Levi-Civita connection on \bar{M} with respect to g by ∇ . If F is parallel with respect to ∇ , i.e.

$$\nabla_X F = 0, X \in \Gamma(T\bar{M}) \tag{2.18}$$

we say that \bar{M} is a locally product Riemannian manifold [3, 27].

3. Pointwise slant submersions

Definition 1 : Let (\bar{M}, g, F) be an almost product Riemannian manifold and (\bar{N}, g') a Riemannian manifold. A Riemannian submersion $\pi: (\bar{M}, g, F) \rightarrow (\bar{N}, g')$ is called a pointwise slant submersion if for any point $p \in \bar{M}$, the angle $\theta(X)$ between FX and the space $(ker \pi_*)_p$ is independent of the choice of the nonzero vector $X \in \Gamma(ker \pi_*)$. The angle θ is called the slant function of the pointwise slant submersion.

Now, we consider an example for pointwise slant submersions.

Example 1 : Let R^4 denote the standard Euclidean space with the standard metric g . Suppose that F_1 and F_2 are almost product Riemannian structure on R^4 such that

$$F_1(x_1, x_2, x_3, x_4) = (-x_3, x_4, -x_1, x_2), F_2(x_1, x_2, x_3, x_4) = (x_2, x_1, x_4, x_3).$$

Then we can define new almost product Riemannian structure F_θ on R^4 by

$$F_\theta = (\cos \theta)F_1 + (\sin \theta)F_2,$$

where $\theta : R^4 \rightarrow R$ is a real-valued function. Thus, $R_\theta^4 = (R^4, F_\theta, g)$ is an almost product Riemannian manifold. Let $\pi : R^4 \rightarrow R^2$ be also a map defined by

$$\pi(x_1, x_2, x_3, x_4) = \left(\frac{x_1 - x_3}{\sqrt{2}}, \frac{x_2 - x_4}{\sqrt{2}} \right).$$

Then we obtain

$$\begin{aligned} \ker \pi_* &= \text{Span} \left\{ U = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, V = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} \right\} \\ (\ker \pi_*)^\perp &= \text{Span} \left\{ X = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}, Y = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4} \right\}. \end{aligned}$$

π is a Riemannian submersion. Moreover, π is a pointwise slant submersion with slant function θ such that $g(F_\theta U, V) = 2 \cos \theta$.

Let π be a pointwise slant submersion from an almost product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') . Then for $U, V \in \Gamma(\ker \pi_*)$, we have

$$FU = \varphi U + \omega U, \quad (3.1)$$

where φU and ωU are vertical and horizontal components of FU , respectively. Also, for $V \in \Gamma((\ker \pi_*)^\perp)$, we have

$$FX = \mathcal{B}X + \mathcal{C}X, \quad (3.2)$$

where $\mathcal{B}X$ and $\mathcal{C}X$ are vertical and horizontal parts of FX , respectively. Also, using (2.6), (2.7), (3.1) and (3.2), we can write the following equations

$$(\nabla_U \omega)V = \mathcal{C}T_U V - T_U \varphi V, \quad (3.3)$$

$$(\nabla_U \varphi)V = \mathcal{B}T_U V - T_U \omega V, \quad (3.4)$$

where

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega \hat{\nabla}_U V, \quad (3.5)$$

$$(\nabla_U \varphi)V = \hat{\nabla}_U \varphi V - \varphi \hat{\nabla}_U V \quad (3.6)$$

for $U, V \in \Gamma(\ker \pi_*)$, where ∇ is the Levi-Civita connection on \bar{M} . We say that ω is parallel if

$$(\nabla_U \omega)V = 0$$

for $U, V \in \Gamma(\ker \pi_*)$.

Let (\bar{M}, g) and (\bar{N}, g') are two Riemannian manifolds and $\pi : (\bar{M}, g) \rightarrow (\bar{N}, g')$ is a map between them, then the adjoint map ${}^* \pi_*$ of π_* is defined by

$$g(x, {}^* \pi_* y) = g'(\pi_* x, y) \tag{3.7}$$

for $x \in T_p \bar{M}, y \in T_{\pi(p)} \bar{N}$ and $p \in \bar{M}$. Suppose that π_*^h is a linear transformation

$$\pi_*^h : \left((\ker \pi_*)^\perp(p), g_{p((\ker \pi_*)^\perp(p))} \right) \rightarrow (range \pi_*(q), g_{q(range \pi_*(q))})$$

for each $p \in \bar{M}$. Denote the adjoint of π_*^h by

$$({}^* \pi_{*p})^h : range \pi_*(q) \rightarrow (\ker \pi_*)^\perp(p)$$

defined by

$$({}^* \pi_{*p})^h y = {}^* \pi_{*p} y, \tag{3.8}$$

where $y \in \Gamma(range \pi_{*p}), q = \pi(p)$. Then this map is an isomorphism and

$$(\pi_{*p}^h)^{-1} = ({}^* \pi_{*p})^h = ({}^* \pi_{*p}^h) \tag{3.9}$$

[15].

Theorem 2 : Let (\bar{M}, g, F) be an almost product Riemannian manifold and (\bar{N}, g') a Riemannian manifold. A Riemannian submersion $\pi : (\bar{M}, g, F) \rightarrow (\bar{N}, g')$ is a pointwise slant submersion if and only if there exist a slant function θ such that

$$\varphi^2 = (\cos^2 \theta)U$$

for $U \in \Gamma(\ker \pi_*)$.

Proof : Suppose that π is pointwise slant submersion. Then for any nonzero $U \in \Gamma(\ker \pi_*)$, we get

$$\cos \theta = \frac{\|\varphi U\|}{\|FU\|}, \tag{3.10}$$

where θ is the slant angle. By using (2.2) and (3.10), we obtain

$$\begin{aligned} g(\varphi^2 U, U) &= g(\varphi U, \varphi U) \\ &= (\cos^2 \theta)g(U, U) \end{aligned} \tag{3.11}$$

for all $U \in \Gamma(\ker \pi_*)$. From (3.11), we have

$$\varphi^2 U = (\cos^2 \theta)U, U \in \Gamma(\ker \pi_*).$$

Thus, the proof is completed. □

Lemma 2 : Let π be a pointwise slant submersion from locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') . Then

$$\begin{aligned}g(\varphi X, \varphi Y) &= \cos^2 \theta g(X, Y) \\g(\omega X, \omega Y) &= \sin^2 \theta g(X, Y)\end{aligned}$$

for any $X, Y \in (\ker \pi_*)$

Proof : From the equations (2.2) and Theorem 2, we arrive the proof. \square

Theorem 3 : Let π be a pointwise slant submersion from locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') . Then π is harmonic if and only if

$$\begin{aligned}\text{trace}^* \pi_*((\nabla \pi_*)(U), \omega \varphi(U)) - \text{trace} \omega \mathcal{T}_U \omega(U) \\+ \text{trace} \mathcal{C}^* \pi_*((\nabla \pi_*)(U), \omega(U)) = 0.\end{aligned}$$

for $U \in \Gamma(\ker \pi_*)$.

Proof : From the equations (2.2), (2.3), (2.6) and (3.1) we get

$$g(\mathcal{T}_U U, X) = g(\nabla_U \varphi U, FX) + g(\nabla_U \omega U, FX)$$

for $U \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^\perp)$. Then we have

$$g(\mathcal{T}_U U, X) = g(\nabla_U F \varphi U, X) + g(\nabla_U \omega U, FX).$$

Using (2.3), (3.2) and Theorem 2 we deduce

$$\sin^2 \theta g(\mathcal{T}_U U, X) = g(\nabla_U \omega \varphi U, X) + g(\mathcal{T}_U \omega U, BX) + g(\nabla_U \omega U, CX).$$

Using (2.12) we obtain

$$\begin{aligned}\sin^2 \theta g(\mathcal{T}_U U, X) &= -g'((\nabla \pi_*)(U, \omega \varphi U), \pi_*(X)) + g(\mathcal{T}_U \omega U, X) \\&\quad - g'((\nabla \pi_*)(U, \omega U), \pi_*(CX))\end{aligned}$$

From the last equation, we derive

$$\begin{aligned}\sin^2 \theta g(\mathcal{T}_U U, X) &= -g^*(\pi_*((\nabla \pi_*)(U, \omega \varphi U)), X) + g(\omega \mathcal{T}_U \omega U, X) \\&\quad - g(\mathcal{C}^* \pi_*((\nabla \pi_*)(U, \omega U)), X).\end{aligned}$$

Thus we get the desired equality.

Conversely, a direct computation gives the proof. \square

Theorem 4 : *Let π be a pointwise slant submersion from a locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') . Then the fibers are totally geodesic submanifolds in \bar{M} if and only if*

$$g'(\nabla_{X_*}^N \pi_*(\omega U), \pi_*(\omega V)) = -\sin^2 \theta g([U, X], V) + \sin 2\theta X[\theta]g(U, V) - g(\mathcal{A}_X \omega \phi U, V) - g(\mathcal{A}_X \omega U, \phi V),$$

for $U, V \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^\perp)$, where X and X_* are π -related vector fields and ∇^N is the Levi-Civita connection on \bar{N} .

Proof : Using (2.2), (2.3), (2.6) and (3.1), we have

$$g(\mathcal{T}_U V, X) = -g([U, X], V) - g(\nabla_X \phi^2 U, V) - g(\nabla_X \omega \phi U, V) - g(\nabla_X \omega U, FV)$$

for $U, V \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^\perp)$. Then, it follows from (2.9), (2.12), (3.7) and Theorem (3.7) that

$$g(\mathcal{T}_U V, X) = -g([U, X], V) + \sin 2\theta X[\theta]g(U, V) - \cos^2 \theta g(\nabla_X U, V) - g(\mathcal{A}_X \omega \phi U, V) - g(\mathcal{A}_X \omega U, \phi V) - g'(\nabla_{X_*}^N \pi_*(\omega U), \pi_*(\omega V)).$$

Since \mathcal{T} is skew symmetric, we get

$$\sin^2 \theta g(\mathcal{T}_U V, X) = -\sin^2 \theta g([U, X], V) + \sin 2\theta X[\theta]g(U, V) - g(\mathcal{A}_X \omega \phi U, V) - g(\mathcal{A}_X \omega U, \phi V) - g'(\nabla_{X_*}^N \pi_*(\omega U), \pi_*(\omega V)).$$

Considering the fibers are totally geodesic, we get the equality in hypothesis with simple calculations.

Conversely it is proved by direct calculation. □

Theorem 5 : *Let π be a pointwise slant submersion from locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') . Then π is a totally geodesic map if and only if*

$$g'(\nabla_{X_*} \pi_*(\omega U), \pi_*(\omega V)) = \sin^2 \theta g([U, X], V) + \sin 2\theta X[\theta]g(U, V) - g(\mathcal{A}_X \omega \phi U, V) - g(\mathcal{A}_X \omega U, \phi V)$$

and

$$g(\mathcal{A}_X \omega U, BY) = -g'(\nabla_{X_*}^\pi \pi_*(\omega \phi U), \pi_*(Y)) - g'(\nabla_{X_*}^\pi \pi_*(\omega U), \pi_*(CY))$$

for $U, V \in \Gamma(\ker \pi_*)$, where X and X' are π -related vector fields and ∇^π is the pull-back connection along π .

Proof : We know that π is totally geodesic if and only if

$$(\nabla \pi_*)(X, Y) = 0 \quad (3.12)$$

for any $X, Y \in \Gamma(T\bar{M})$. Using Theorem 4 and (2.13), we have the first equation. Furthermore, by using (2.3), (2.9), (3.1), (3.2) and Theorem 2, we get

$$\begin{aligned} g'((\nabla \pi_*)(X, U), \pi_*(Y)) &= -\sin 2\theta X[\theta]g(U, Y) - \cos^2\theta g(\nabla_X U, Y) \\ &\quad -g(\nabla_X \omega\phi U, Y) - g(\mathcal{A}_X \omega U, \mathcal{B}Y) \\ &\quad -g(\nabla_X \omega U, \mathcal{C}Y). \end{aligned}$$

Considering (2.12) we obtain

$$\begin{aligned} g'((\nabla \pi_*)(X, U), \pi_*(Y)) &= \cos^2\theta g'(\nabla \pi_*(X, U), \pi_*(Y)) - g(\nabla_X \omega\phi U, Y) \\ &\quad -g(\nabla_X \omega U, \mathcal{F}U). \end{aligned}$$

Then we have

$$\begin{aligned} \sin^2 \theta g'((\nabla \pi_*)(X, U), \pi_*(Y)) &= -g'(\nabla_X^\pi \pi_*(\omega\phi U), \pi_*(Y)) \\ &\quad -g'(\nabla_X^\pi \pi_*(\omega U), \pi_*(\mathcal{C}Y)) \\ &\quad -g(\mathcal{A}_X \omega U, \mathcal{B}Y). \end{aligned}$$

Therefore, we arrive the result. Conversely, it can be directly verified. \square

4. Curvature Relations

Let π be a pointwise slant submersion from an almost product Riemannian manifold (\bar{M}, g, F) to a Riemannian manifold (\bar{N}, g') . We know that the sectional curvature K is defined

$$\mathcal{K}(X, Y) = \frac{\mathcal{R}(X, Y, Y, X)}{\|X\|^2 \|Y\|^2} \quad (4.1)$$

for nonzero orthogonal vectors X and Y in \bar{M} [9]. Now, we examine curvature relations between the total space, the base space and the fibers of a pointwise slant submersion.

Theorem 6 : *Let π be a pointwise slant submersion from a locally product Riemannian manifold (\bar{M}, g, F) onto Riemannian manifold (\bar{N}, g') . Suppose that*

\mathcal{K} , $\hat{\mathcal{K}}$ and \mathcal{K}' be the sectional curvatures of the total space \bar{M} , fibers and the base space \bar{N} , respectively. If X, Y are horizontal and U, V vertical vectors, then

$$\begin{aligned} \mathcal{K}(U, V) = & \mathcal{K}'(\pi_*\omega U, \pi_*\omega V) + \hat{\mathcal{K}}(\phi U, \phi V) - g(\omega[\phi U, U], \omega[\phi V, V]) \\ & - g(\omega[\phi U, U], \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) - \cos^4 \theta g(\mathcal{T}_U U, \mathcal{T}_V V) \\ & - g(\omega[\phi V, V], \mathcal{H}\nabla_U \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega U) - 3 \| \mathcal{A}_{\omega U} \omega V \|^2 \\ & - \cos^2 \theta \{ g(\mathcal{T}_U U, \omega[\phi V, V]) + \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V \} \\ & + g(\mathcal{T}_V V, \omega[\phi U, U]) + \mathcal{H}\nabla_U \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega U \} \\ & - g(\mathcal{H}\nabla_U \omega\phi U, \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) + \| \mathcal{A}_{\omega V} \phi U \|^2 \\ & + g(\mathcal{H}\nabla_{\phi U} \omega U, \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) \\ & + \| \omega[\phi U, V] + \cos^2 \theta \mathcal{T}_V U + \mathcal{H}\nabla_V \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega V \|^2 \\ & - g((\nabla_{\omega U} \mathcal{T})_{\phi V} \phi V, \omega U) + g((\nabla_{\phi V} \mathcal{A})_{\omega U} \omega U, \phi V) \\ & - \| \phi[\phi V, U] - \sin 2\theta U[\theta]V + \cos^2 \theta \hat{\nabla}_U V + \mathcal{T}_U \omega\phi V - \hat{\nabla}_{\phi V} \phi U \|^2 \\ & + \| \mathcal{A}_{\omega U} \phi V \|^2 - g((\nabla_{\omega V} \mathcal{T})_{\phi U} \phi U, \omega V) + g((\nabla_{\phi U} \mathcal{A})_{\omega V} \omega V, \phi U) \\ & - \| \phi[\phi U, V] - \sin 2\theta V[\theta]U + \cos^2 \theta \hat{\nabla}_V U + \mathcal{T}_V \omega\phi U - \hat{\nabla}_{\phi U} \phi V \|^2, \end{aligned}$$

$$\begin{aligned} \mathcal{K}(X, Y) = & \mathcal{K}'(\pi_*CX, \pi_*CY) + \hat{\mathcal{K}}(BX, BY) - g(\mathcal{T}_{BX} BX, \mathcal{T}_{BY} BY) \\ & - g((\nabla_{CY} \mathcal{T})_{BX} BX, CY) + g((\nabla_{BX} \mathcal{A})_{CY} CY, BX) \\ & - \| \mathcal{T}_{BX} CY \|^2 + \| \mathcal{A}_{CY} BX \|^2 + \| \mathcal{T}_{BY} BY \|^2 \\ & - g((\nabla_{CX} \mathcal{T})_{BY} BY, CX) + g((\nabla_{BY} \mathcal{A})_{CX} CX, BY) \\ & - \| \mathcal{T}_{BY} CX \|^2 + \| \mathcal{A}_{CX} BY \|^2 - 3 \| \mathcal{A}_{CX} CY \|^2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(X, U) = & \mathcal{K}'(\pi_*CX, \pi_*\omega U) + \hat{\mathcal{K}}(BX, \phi U) - 3 \| \mathcal{A}_{CX} \omega U \|^2 \\ & - g(\mathcal{T}_{BX} BX, \omega[\phi U, U]) + \mathcal{H}\nabla_U \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega U \\ & + \| \omega[\phi U, X] + \cos^2 \theta \mathcal{A}_X U + \mathcal{H}\nabla_X \omega\phi U - \mathcal{H}\nabla_{\phi U} CX \|^2 \\ & - g((\nabla_{\omega U} \mathcal{T})_{BX} BX, \omega U) + g((\nabla_{BX} \mathcal{A})_{\omega U} \omega U, BX) \\ & - \| \mathcal{T}_{BX} \omega U \|^2 + \| \mathcal{A}_{\omega U} BX \|^2 - g((\nabla_{CX} \mathcal{T})_{\phi U} \phi U, CX) \\ & + g((\nabla_{\omega U} \mathcal{A})_{CX} CX, \phi U) + \| \mathcal{A}_{CX} \phi U \|^2 \\ & - \| \phi[\phi U, X] + \mathcal{A}_X \omega\phi U - \sin 2\theta X[\theta]U + \cos^2 \theta \mathcal{V}_{XU} \\ & - \hat{\nabla}_{\phi U} BX \|^2 - \cos^2 \theta g(\mathcal{T}_{BX} BX, \mathcal{T}_U U). \end{aligned}$$

Proof : Since (\bar{M}, g, F) is a locally product Riemannian manifold , we have

$$\mathcal{K}(U, V) = \mathcal{K}(FU, FV) = \mathcal{K}(\varphi U, \varphi V) + \mathcal{K}(\varphi U, \omega V) + \mathcal{K}(\omega U, \varphi V) + \mathcal{K}(\omega U, \omega V)$$

for $U, V \in \Gamma(\ker \pi_*)$. Then using the equations (2.16), (2.17), (2.18) and (4.1), we get

$$\begin{aligned} \mathcal{K}(\varphi U, \varphi V) &= \hat{\mathcal{K}}(\varphi U, \varphi V) - g(\omega[\varphi U, U], \omega[\varphi V, V]) \\ &\quad - \cos^4 \theta g(\mathcal{T}_U U, \mathcal{T}_V V) - g(\omega[\varphi U, U], \mathcal{H}\nabla_V \omega \varphi V - \mathcal{H}\nabla_{\varphi V} \omega V) \\ &\quad - g(\omega[\varphi V, V], \mathcal{H}\nabla_U \omega \varphi U - \mathcal{H}\nabla_{\varphi U} \omega U) \\ &\quad - \cos^2 \theta \{g(\mathcal{T}_U U, \omega[\varphi V, V]) + \mathcal{H}\nabla_V \omega \varphi V - \mathcal{H}\nabla_{\varphi V} \omega V\} \\ &\quad + g(\mathcal{T}_V V, \omega[\varphi U, U]) + \mathcal{H}\nabla_U \omega \varphi U - \mathcal{H}\nabla_{\varphi U} \omega U \\ &\quad - g(\mathcal{H}\nabla_U \omega \varphi U, \mathcal{H}\nabla_V \omega \varphi V - \mathcal{H}\nabla_{\varphi V} \omega V) \\ &\quad + g(\mathcal{H}\nabla_{\varphi U} \omega U, \mathcal{H}\nabla_V \omega \varphi V - \mathcal{H}\nabla_{\varphi V} \omega V) \\ &\quad + \|\omega[\varphi U, V] + \cos^2 \theta \mathcal{T}_V U + \mathcal{H}\nabla_V \omega \varphi U - \mathcal{H}\nabla_{\varphi U} \omega V\|^2, \end{aligned}$$

$$\begin{aligned} \mathcal{K}(\omega U, \varphi V) &= -g((\nabla_{\omega U} \mathcal{T})_{\varphi V} \varphi V, \omega U) + g((\nabla_{\varphi V} \mathcal{A})_{\omega U} \omega U, \varphi V) \\ &\quad - \|\varphi[\varphi V, U] - \sin 2\theta U[\theta]V + \cos^2 \theta \hat{\mathcal{N}}_U V + \mathcal{T}_U \omega \varphi V \\ &\quad - \hat{\nabla}_{\varphi V} \varphi U\|^2 + \|\mathcal{A}_{\omega U} \varphi V\|^2, \end{aligned}$$

$$\begin{aligned} \mathcal{K}(\varphi U, \omega V) &= -g((\nabla_{\omega V} \mathcal{T})_{\varphi U} \varphi U, \omega V) + g((\nabla_{\varphi U} \mathcal{A})_{\omega V} \omega V, \varphi U) \\ &\quad - \|\varphi[\varphi U, V] - \sin 2\theta V[\theta]U + \cos^2 \theta \hat{\mathcal{N}}_V U + \mathcal{T}_V \omega \varphi U \\ &\quad - \hat{\nabla}_{\varphi U} \varphi V\|^2 + \|\mathcal{A}_{\omega V} \varphi U\|^2 \end{aligned}$$

and

$$\mathcal{K}(\omega U, \omega V) = \mathcal{K}(\pi_* \omega U, \pi_* \omega V) - 3 \|\mathcal{A}_{\omega U} \omega V\|^2.$$

Similarly we can write

$$\mathcal{K}(FX, FY) = \mathcal{K}(BX, BY) + \mathcal{K}(BX, CY) + \mathcal{K}(CX, BY) + \mathcal{K}(CX, CY)$$

for $X, Y \in \Gamma((\ker \pi_*)^\perp)$. Then we obtain

$$\mathcal{K}(BX, BY) = \hat{\mathcal{K}}(BX, BY) - g(\mathcal{T}_{BX} BX, \mathcal{T}_{BY} BY) + \|\mathcal{T}_{BX} BY\|^2,$$

$$\begin{aligned} \mathcal{K}(BX, CY) &= -((\nabla_{CY} \mathcal{T})_{BX} BX, CY) + g((\nabla_{BX} \mathcal{A})_{CY} CY, BX) \\ &\quad - \|\mathcal{T}_{BX} CY\|^2 + \|\mathcal{A}_{CY} BX\|^2, \end{aligned}$$

$$\begin{aligned} \mathcal{K}(CX, BY) &= -((\nabla_{CX} \mathcal{T})_{BY} BY, CX) + g((\nabla_{BY} \mathcal{A})_{CX} CX, BY) \\ &\quad - \| \mathcal{T}_{CX} BY \|^2 + \| \mathcal{A}_{CX} BY \|^2 \end{aligned}$$

and

$$\mathcal{K}(CX, CY) = \mathcal{K}'(\pi_* CX, \pi_* CY) - 3 \| \mathcal{A}_{CX} CY \|^2 .$$

Furthermore we have

$$\mathcal{K}(FX, FU) = \mathcal{K}(BX, \varphi U) + \mathcal{K}(BX, \omega U) + \mathcal{K}(CX, \varphi U) + \mathcal{K}(CX, \omega U)$$

for unit vectors $X \in \Gamma((ker \pi_*)^\perp)$ and $U \in \Gamma(ker \pi_*)$. Therefore we arrive

$$\begin{aligned} \mathcal{K}(BX, \varphi U) &= \hat{\mathcal{K}}(BX, \varphi U) - g(\mathcal{T}_{BX} BX, \omega[\varphi U, U]) + \mathcal{H}\nabla_U \omega \varphi U - \mathcal{H}\nabla_{\varphi U} \omega U \\ &\quad + \| \omega[\varphi U, X] + \cos^2 \theta \mathcal{A}_X U + \mathcal{H}\nabla_X \omega \varphi U - \mathcal{H}\nabla_{\varphi U} CX \|^2 \\ &\quad - \cos^2 \theta g(\mathcal{T}_{BX} BX, \mathcal{T}_U U), \end{aligned}$$

$$\begin{aligned} \mathcal{K}(BX, \omega U) &= -g((\nabla_{\omega U} \mathcal{T})_{BX} BX, \omega U) + g((\nabla_{BX} \mathcal{A})_{\omega U} \omega U, BX) \\ &\quad - \| \mathcal{T}_{BX} \omega U \|^2 + \| \mathcal{A}_{\omega U} BX \|^2, \end{aligned}$$

$$\begin{aligned} \mathcal{K}(CX, \varphi U) &= -g((\nabla_{CX} \mathcal{T})_{\varphi U} \varphi U, CX) + g((\nabla_{\varphi U} \mathcal{A})_{CX} CX, \varphi U) \\ &\quad + \| \mathcal{A}_{CX} \varphi U \|^2 - \| \varphi[\varphi U, X] - \sin 2\theta X[\theta] U + \cos^2 \theta \mathcal{V}_X U \\ &\quad + \mathcal{A}_X \omega \varphi U - \hat{\nabla}_{\varphi U} BX \|^2 \end{aligned}$$

and

$$\mathcal{K}(CX, \omega U) = \mathcal{K}'(\pi_* CX, \pi_* \omega U) - 3 \| \mathcal{A}_{CX} \omega U \|^2$$

We obtain the equations given in the theorem with direct calculations. □

By using Theorem 6 we get the following results.

Corollary 1 : *Let π be a pointwise slant submersion from a locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') and $\omega[\varphi U, V] = \mathcal{H}\nabla_{\varphi U} \omega V - \mathcal{H}\nabla_V \omega \varphi U$. Then we find following inequality*

$$\begin{aligned} \mathcal{K}(U, V) &\geq \mathcal{K}'(\pi_* \omega U, \pi_* \omega V) + \hat{\mathcal{K}}(\varphi U, \varphi V) - g(\omega[\varphi U, U], \omega[\varphi V, V]) \\ &\quad - \cos^4 \theta g(\mathcal{T}_U U, \mathcal{T}_V V) - 3 \| \mathcal{A}_{\omega U} \omega V \|^2 + \| \mathcal{A}_{\omega V} \varphi U \|^2 \\ &\quad - \cos^2 \theta \{ g(\mathcal{T}_U U, \omega[\varphi V, V]) + \mathcal{H}\nabla_V \omega \varphi V - \mathcal{H}\nabla_{\varphi V} \omega V \\ &\quad + g(\mathcal{T}_V V, \omega[\varphi U, U]) + \mathcal{H}\nabla_U \omega \varphi U - \mathcal{H}\nabla_{\varphi U} \omega U \} \end{aligned}$$

$$\begin{aligned}
& -g(\mathcal{H}\nabla_U \omega\phi U, \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) \\
& +g(\mathcal{H}\nabla_{\phi U} \omega U, \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) + \|\mathcal{A}_{\omega U} \phi V\|^2 \\
& -g((\nabla_{\omega U} \mathcal{T})_{\phi V} \phi V, \omega U) + g((\nabla_{\phi V} \mathcal{A})_{\omega U} \omega U, \phi V) \\
& -\|\phi[\phi V, U] - \sin 2\theta U[\theta]V + \cos^2 \theta \hat{\nabla}_U V + \mathcal{T}_U \omega\phi V - \hat{\nabla}_{\phi V} \phi U\|^2 \\
& -\|\phi[\phi U, V] - \sin 2\theta V[\theta]U + \cos^2 \theta \hat{\nabla}_V U + \mathcal{T}_V \omega\phi U - \hat{\nabla}_{\phi U} \phi V\|^2 \\
& -g((\nabla_{\omega V} \mathcal{T})_{\phi U} \phi U, \omega V) + g((\nabla_{\phi U} \mathcal{A})_{\omega V} \omega V, \phi U)
\end{aligned}$$

The equality case is satisfied if and only if the fibers are totally geodesic or π is an anti-invariant submersion.

Corollary 2 : Let π be a pointwise slant submersion from a locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') and $\cos^2 \theta \nabla_V U = \hat{\nabla}_{\phi U} \phi U - \mathcal{T}_V \omega\phi U$. Then we obtain

$$\begin{aligned}
\mathcal{K}(U, V) & \leq \mathcal{K}'(\pi_* \omega U, \pi_* \omega V) + \hat{\mathcal{K}}(\phi U, \phi V) - g_M(\omega[\phi U, U], \omega[\phi V, V]) \\
& - \cos^4 \theta g(\mathcal{T}_U U, \mathcal{T}_V V) - 3 \|\mathcal{A}_{\omega U} \omega V\|^2 + \|\mathcal{A}_{\omega V} \phi U\|^2 \\
& - \cos^2 \theta \{g(\mathcal{T}_U U, \omega[\phi V, V]) + \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V\} \\
& + g(\mathcal{T}_V V, \omega[\phi U, U]) + \mathcal{H}\nabla_U \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega U\} \\
& - g(\mathcal{H}\nabla_U \omega\phi U, \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) - \|\phi[\phi U, V]\|^2 \\
& + g(\mathcal{H}\nabla_{\phi U} \omega U, \mathcal{H}\nabla_V \omega\phi V - \mathcal{H}\nabla_{\phi V} \omega V) + \|\mathcal{A}_{\omega U} \phi V\|^2 \\
& - g((\nabla_{\omega U} \mathcal{T})_{\phi V} \phi V, \omega U) + g((\nabla_{\phi V} \mathcal{A})_{\omega U} \omega U, \phi V) \\
& - \|\phi[\phi V, U] - \sin 2\theta U[\theta]V + \cos^2 \theta \hat{\nabla}_U V + \mathcal{T}_U \omega\phi V - \hat{\nabla}_{\phi V} \phi U\|^2 \\
& - g((\nabla_{\omega V} \mathcal{T})_{\phi U} \phi U, \omega V) + g((\nabla_{\phi U} \mathcal{A})_{\omega V} \omega V, \phi U) \\
& - \|\omega[\phi U, V] + \cos^2 \theta \mathcal{T}_V U + \mathcal{H}\nabla_V \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega V\|^2 \\
& + 2 \sin 2\theta V[\theta]g(\phi[\phi U, V], U).
\end{aligned}$$

The equality case is satisfied if and only if π is a slant submersion.

Corollary 3 : Let π be a pointwise slant submersion from a locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') and $\omega[\phi U, X] = \mathcal{H}\nabla_{\phi U} CX - \mathcal{H}\nabla_X \omega\phi U$. Then we find the inequality

$$\begin{aligned}
\mathcal{K}(X, U) & \geq \mathcal{K}'(\pi_* CX, \pi_* \omega U) + \hat{\mathcal{K}}(BX, \phi U) - 3 \|\mathcal{A}_{CX} \omega U\|^2 \\
& - g(\mathcal{T}_{BX} BX, \omega[\phi U, U]) + \mathcal{H}\nabla_U \omega\phi U - \mathcal{H}\nabla_{\phi U} \omega U \\
& - g((\nabla_{\omega U} \mathcal{T})_{BX} BX, \omega U) + g((\nabla_{BX} \mathcal{A})_{\omega U} \omega U, BX)
\end{aligned}$$

$$\begin{aligned}
 &+ \|\psi[\psi U, X] + \mathcal{A}_X \omega \psi U - \sin 2\theta X[\theta]U + \cos^2 \theta \nabla_X U \\
 &- \hat{\nabla}_{\psi U} BX \|^2 - \cos^2 \theta g(\mathcal{T}_{BX} BX, \mathcal{T}_U U) + \|\mathcal{A}_{CX} \phi U \|^2 \\
 &- \|\mathcal{T}_{BX} \omega U \|^2 + \|\mathcal{A}_{\omega U} BX \|^2 - g((\nabla_{CX} \mathcal{T})_{\phi U} \phi U, CX) \\
 &+ g((\nabla_{\omega U} \mathcal{A})_{CX} CX, \phi U)
 \end{aligned}$$

The equality case is satisfied if and only if π is an anti-invariant submersion or $\Gamma((\ker \pi_*)^\perp)$ is integrable.

Corollary 4 : Let π be a pointwise slant submersion from a locally product Riemannian manifold (\bar{M}, g, F) onto a Riemannian manifold (\bar{N}, g') and $\cos^2 \theta \nabla_X U = \hat{\nabla}_{\psi U} BX - \mathcal{A}_X \omega \psi U$. Then we have

$$\begin{aligned}
 \mathcal{K}(X, U) \leq & \mathcal{K}'(\pi_* CX, \pi_* \omega U) + \hat{\mathcal{K}}(BX, \phi U) - 3 \|\mathcal{A}_{CX} \omega U \|^2 \\
 & + g(\mathcal{T}_{BX} BX, \omega[\phi U, U]) + \mathcal{H}\nabla_U \omega \phi U - \mathcal{H}\nabla_{\phi U} \omega U \\
 & - g((\nabla_{\omega U} \mathcal{T})_{BX} BX, \omega U) + g((\nabla_{BX} \mathcal{A})_{\omega U} \omega U, BX) \\
 & - \|\mathcal{T}_{BX} \omega U \|^2 + \|\mathcal{A}_{\omega U} BX \|^2 - g((\nabla_{CX} \mathcal{T})_{\phi U} \phi U, CX) \\
 & + g((\nabla_{\omega U} \mathcal{A})_{CX} CX, \phi U) + \|\mathcal{A}_X \phi U \|^2 + \cos^2 \theta g(\mathcal{T}_{BX} BX, \mathcal{T}_U U) \\
 & + \|\omega[\phi U, X] + \mathcal{H}\nabla_X \omega \phi U + \cos^2 \theta \mathcal{A}_X U - \mathcal{H}\nabla_{\phi U} CX \|^2 \\
 & - \|\phi[\phi U, X] \|^2 + 2 \sin 2\theta X[\theta]g(\phi[\phi U, X], U)
 \end{aligned}$$

The equality case is satisfied if and only if π is a slant submersion.

References

- [1] M.A. Akyol and B. Sahin, Conformal anti-invariant submersions from almost Hermitian manifolds, *Turk. J. Math.* 40(2016), 43–70.
- [2] M.A. Akyol and B. Sahin, Conformal semi-invariant submersions, *Commun. Contemp. Math.* 19(2017), 43–70.
- [3] M. Atceken, Slant submanifolds of a Riemannian product manifold, *Acta Math. Sci.* 30(2010), 215–224.
- [4] S. Aykurt Sepet and M. Ergut, Pointwise slant submersions from co-symplectic manifolds, *Turk. J. Math.* 40(2016), 582–593.
- [5] G. Baditoiu and S. Ianus, Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces, *Differ. Geom. Appl.* 16(2002), 79–94.

- [6] P. Baird and J.C. Wood, Harmonic morphisms between Riemannian manifolds, Oxford University Press, The Clarendon Press, Oxford, 2003.
- [7] B.Y. Chen and O. Garay, Pointwise slant submanifolds in almost Hermitian manifolds, *Turk. J. Math.* 36(2012), no.4, 630–640.
- [8] F. Etayo, On quasi-slant submanifolds of an almost Hermitian manifold, *Publ. Math. Debrecen* 53(1998), 217–223.
- [9] M. Falcitelli, S. Ianus and A.M Pastore, Riemannian Submersions and Related Topics, World Scientific, River Edge, NJ, 2004.
- [10] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.* 16(1967), 715–737.
- [11] Y. Gunduzalp, Slant submersions from almost product Riemannian manifolds, *Turk. J. Math.* 37(2013), 863–873.
- [12] Y. Gunduzalp, Semi-slant submersions from almost product Riemannian manifolds, *Demonstratio Mathematica* 49(2016), 345–356.
- [13] S. Ianus, R. Mazocco and G.E. Vilcu, Riemannian submersions from quaternionic manifolds, *Acta Appl. Math.* 104(2008), 83–89.
- [14] I. Kupeli Erken and C. Murathan, On Slant Riemannian submersions for cosymplectic manifolds, *Bull. Korean Math. Soc.*, 51(2014), 1749–1771.
- [15] J.W. Lee and B. Sahin, Pointwise slant submersions, *Bull. Korean Math. Soc.* 51(2014) 1115–1126.
- [16] J.C. Marrero and J. Rocha, Locally conformal Kahler submersions, *Geom. Dedicata* 52(1994), 271–289.
- [17] B. O'Neill, The fundamental equations of a submersions, *Mich. Math. J.* 13(1966), 459–469.
- [18] K.S. Park, H-Semi-Slant Submersions from almost quaternionic hermitian manifolds, *Taiwan. J. Math.* 18(2014), 1909–1926.
- [19] K.S. Park, Semi-slant submersion, *Bull. Korean Math. Soc.* 50(2013), 951–962.
- [20] S.K. Porwal and S.K. Mishra, Geodesic semilocal E-convex functions on Riemannian manifolds, *Journal of Information and Optimization Sciences*, 39(2018), 1717–1733, DOI: 10.1080/02522667.2018.1424091.
- [21] B. Sahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, *Cent. Eur. J. Math.*, 8(2010), 437–447.

- [22] B. Sahin, Riemannian Submersions from almost Hermitian manifolds, *Taiwanese J. Math.* 17(2013), 629–659.
- [23] B. Sahin, Slant submersions from almost Hermitian manifolds, *Bull. Math. Soc. Sci. Math. Roumanie* 54(2011), 93–105.
- [24] H.M. Tastan, On Lagrangian submersions, *Hacettepe J. Math. Stat.* 43(2014), 993–1000.
- [25] H.M. Tastan, Sahin B. and Yanan S., Hemi-slant submersions, *Mediterr. J. Math.* DOI 10.1007/s00009-015-0602-7.
- [26] B. Watson, Almost Hermitian submersions, *J. Differential Geom.* 11(1976), 147–165.
- [27] Y. Yano and M. Kon, Structures on manifolds, Singapore. World Scientific 1984.

Received July, 2018

Revised May, 2019