Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 54 (6) (2025), 2244 – 2255
DOI : 10.15672/hujms.1472848

Research Article

Boundedness of the maximal operator and the Riesz potential on Musielak-Orlicz-Morrey spaces

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Abstract

In this paper, we investigate the strong and weak type boundedness of the maximal operator in Musielak-Orlicz-Morrey spaces. As an application of this boundedness, we give a sufficient condition for the strong and weak Adams type boundedness of the Riesz potential in these spaces.

Mathematics Subject Classification (2020). 26A33, 42B25, 42B35, 46E30

Keywords. Musielak-Orlicz-Morrey spaces, Hardy-Littlewood maximal operator, Riesz potential

1. Introduction

Generalized Orlicz spaces, also known as Musielak-Orlicz spaces and Nakano spaces, are a class of Banach spaces that include several important spaces in harmonic analysis and partial differential equations (PDEs). These spaces include classical L^p spaces, Orlicz spaces, variable exponent spaces and double phase spaces as special cases. Nakano [23,24] and other researchers introduced them, in accordance with the work of Orlicz [26]. For further details about these spaces, we recommend the reader to [13,20] and the corresponding references.

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ were initially created by C. Morrey [19] to investigate the local behavior of solutions to partial differential equations. Subsequently, they have become extensively employed in the field of regularity theory for partial differential equations (PDEs), such as heat equations and NavierStokes equations. The primary characteristics of these spaces and relevant historical information can be found in the works [1,29,30] and in the comprehensive review [27].

Orlicz-Morrey spaces were introduced in [2] with the aim to unify Orlicz and Morrey spaces into a unified framework. Other definitions of Orlicz-Morrey spaces can be located in references [21] and [28]. According to[10], the Orlicz-Morrey space described in [2] is classified as the third kind, while the spaces mentioned in [21] and [28] are classified as the first and second kinds, respectively. Based on the examples provided in reference [6], it can

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Received: 26.04.2024; Accepted: 14.03.2025

be concluded that the Orlicz-Morrey space of the first kind and the second kind exhibit distinct characteristics. It is important to observe that the definition of the third kind space is based solely on the fact that the Orlicz space L^{Φ} is a normed linear space. This condition is unrelated to the requirement that it is generated by modulars. Conversely, the spaces of the first and second kind are determined by the set of modulars.

This research specifically examines the viewpoint of third kind spaces. Many classical operators from harmonic analysis such as maximal operators, singular operators and potential operators, are known to be bounded in these spaces and their weighted versions. The papers referenced are [2–5, 9, 11, 12].

Musielak-Orlicz-Morrey spaces were introduced in multiple iterations, as documented in references [17, 18, 22]. The space [18, 22] can be interpreted as the Musielak-Orlicz-Morrey spaces of the first kind, while the space [17] can be interpreted as the Musielak-Orlicz-Morrey spaces of the second kind. The Musielak-Orlicz-Morrey spaces of the third kind have not been explored thus far. The aim of this paper is to introduce Musielak-Orlicz-Morrey spaces of the third kind, including all the function spaces mentioned above, and to study the boundedness of the Hardy-Littlewood maximal operator M and the Riesz potential I_α in these spaces.

The rest of the article is organized as follows. In the second section we provide necessary preliminaries and definitions. The third section is devoted to the boundedness of the maximal operator in Musielak-Orlicz-Morrey spaces. In the fourth section, as an application of the results of the third section, we give a sufficient condition for the Adams type boundedness of the Riesz potential. In the proof, we use a pointwise estimate by $Mf(x)$ and the boundedness of M. This method was introduced by Hedberg [15] to give a simple proof of the Hardy-Littlewood-Sobolev theorem. The fifth section is devoted to estimates of the weak type of the maximal operator and the Riesz potential. In the sixth section we consider our conditions and results in some special cases. More precisely, we apply our main theorems to a concrete example of Musielak-Orlicz-Morrey spaces, namely, generalized variable exponent Morrey spaces.

2. Preliminaries

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. The (Lebesgue) measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and χ_E denotes its characteristic function. By $L^0(\mathbb{R}^n)$ we denote the set of (Lebesgue) measurable functions on $\mathbb{R}^n$. We use C as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \le CB$ for some independent constant $C > 0$, and $A \approx B$ means $A \lesssim B \lesssim A$. By almost increasing we mean that a function satisfies the inequality $f(s) \le Cf(t)$ for all $s < t$ and some constant $C \ge 1$ and almost decreasing is defined analogously.

We recall some definitions pertaining to Musielak-Orlicz spaces. For the proofs and further properties see [13] (cf. [16]).

Definition 2.1. We say that $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ is a Musielak-Orlicz (MO) function if the following conditions hold:

- For any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ the function $x \mapsto \varphi(x, |f(x)|)$ is measurable and for every $x \in \mathbb{R}^n$ the function $t \mapsto \varphi(x,t)$ is non-decreasing.
- $\varphi(x,0) = \lim_{t\to 0^+} \varphi(x,t)$ and $\lim_{t\to\infty} \varphi(x,t) = \infty$ for every $x \in \mathbb{R}^n$.
- The function $t \mapsto \frac{\varphi(x,t)}{t}$ is almost increasing uniformly for all $x \in \mathbb{R}^n$.

If φ does not depend on the x variable we say that φ is an Orlicz function.

Two functions φ and ψ are equivalent, $\varphi \simeq \psi$, if there exist $L \ge 1$ such that $\psi(x, t/L) \le \varphi(x,t) \le \psi(x, Lt)$ for every $x \in \mathbb{R}^n$ and every $t > 0$. Equivalent functions give rise to the same space with comparable norms. By $\varphi^{-1}(x,t)$ we mean the generalized inverse defined by

$$\varphi^{-1}(x,t) := \inf\{\tau \in \mathbb{R} : \varphi(x,\tau) \geq t\}.$$

Functions φ and ψ are equivalent if and only if $\varphi^{-1}(x,t) \approx \psi^{-1}(x,t)$.

For a MO function φ we define the conjugate function φ^* by

$$\varphi^*(x,t) = \sup_{s\geq 0}(st - \varphi(x,s)).$$

Note that, φ^* is also a MO function. Moreover, $t \mapsto \varphi^*(x,t)$ is convex and left-continuous for almost every x (cf. [13, Lemma 2.5.8]).

We now give a family of hypotheses that guarantee the boundedness of maximal operators.

Definition 2.2. We say that $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfies

(aInc)$_p$ if $t \mapsto \frac{\varphi(x,t)}{t^p}$ is almost increasing uniformly for a.e. $x \in \mathbb{R}^n$,

(aDec)$_q$ if $t \mapsto \frac{\varphi(x,t)}{t^q}$ is almost decreasing uniformly for a.e. $x \in \mathbb{R}^n$,

(A0) if there exists $\beta \in (0,1]$ such that $\beta \leq \varphi^{-1}(x,1) \leq \frac{1}{\beta}$ for a.e. $x \in \mathbb{R}^n$,

(A1) if there exists $\beta \in (0,1)$ such that

$$\beta\varphi^{-1}(x,t) \leq \varphi^{-1}(y,t)$$

for every $t \in [1, \frac{1}{|B|}]$, a.e. $x, y \in B$ and every ball B with $|B| \leq 1$.

(A2) if there exists a MO function $\varphi_\infty(t) := \limsup_{|x|\to\infty} \varphi(x,t)$, $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\beta \in (0,1]$ and $s > 0$ such that

$$\varphi(x,\beta t) \leq \varphi_\infty(t) + h(x) \quad \text{and} \quad \varphi_\infty(\beta t) \leq \varphi(x,t) + h(x)$$

for a.e. $x \in \mathbb{R}^n$ when $\varphi_\infty(t) \in [0,s]$ and $\varphi(x,t) \in [0,s]$, respectively.

We say φ satisfies (aInc) if it satisfies (aInc)$_p$ for some $p > 1$ and similarly (aDec) if it satisfies (aDec)$_q$ for some $q < \infty$. Conditions (aInc) and (aDec) correspond to the ∇_2 and Δ_2 conditions respectively from the classical Orlicz space theory and rule out the often problematic L^1 and L^∞ spaces.

Definition 2.3. (Musielak-Orlicz space) Let φ be a MO function. The generalized Orlicz space $L^\varphi(\mathbb{R}^n)$, also called Musielak-Orlicz space, comprises of measurable functions f that satisfy

$$\int_{\mathbb{R}^n} \varphi(x, \lambda|f(x)|)dx < \infty$$

for some $\lambda > 0$. The space $L^\varphi_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all measurable functions f such that $f\chi_B \in L^\varphi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

The space $L^\varphi(\mathbb{R}^n)$ is a (quasi)Banach space when equipped with a (quasi)norm

$$\|f\|_{L^\varphi} \equiv \|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi\Big(x, \frac{|f(x)|}{\lambda}\Big)dx \leq 1\Big\}.$$

The definition is readily restricted to the subsets in $\mathbb{R}^n$;

$$\|f\|_{L^\varphi(E)} \equiv \|f\chi_E\|_{L^\varphi}.$$

We extend the classical Hölders inequality to Musielak-Orlicz spaces.

Lemma 2.4. *[13, Lemma 3.2.11] Let φ be a MO function, $f \in L^\varphi(\mathbb{R}^n)$ and $g \in L^{\varphi^*}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f(x)||g(x)|dx \leq 2\|f\|_{L^\varphi}\|g\|_{L^{\varphi^*}}.$$

Here the constant cannot be lower than 2 in general.

The following lemma is valid.

Lemma 2.5 ([13, Proposition 4.4.8]). *Let φ is a MO function satisfying (A0),(A1) and (A2). Then for every ball $B \subset \mathbb{R}^n$ we have*

$$\|\chi_B\|_{L^\varphi}\|\chi_B\|_{L^{\varphi^*}} \approx |B|.$$

Here the implicit constant is independent of the ball B.

The (HardyLittlewood) maximal operator is defined for $f \in L^0(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

The boundedness of HardyLittlewood maximal operator in Musielak-Orlicz spaces with the current framework was proven by Hästö [14]. Note that, this boundedness was also discussed from a different perspective, more precisely, using interpolation techniques in [31, Theorem 2.1.1 and Corollary 2.1.2]. The result formulated in the next theorem is given in [14, Theorem 4.6].

Theorem 2.6 ([14, Theorem 4.6]). *Let φ be a MO function. If φ satisfies (A0), (A1), (A2) and (aInc), we have*

$$\|Mf\|_{L^\varphi} \le C\|f\|_{L^\varphi}$$

for all $f \in L^\varphi(\mathbb{R}^n)$, where $C > 0$ is independent of f.

3. Boundedness of the maximal operator

Now, we introduce the Musielak-Orlicz-Morrey space $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ of the third kind.

Definition 3.1. Let φ be a *MO* function and ϕ a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We denote by $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ the Musielak-Orlicz-Morrey space, the space of all functions $f \in L^\varphi_{\text{loc}}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{\mathcal{M}^{\varphi,\phi}} \equiv \|f\|_{\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|f\|_{L^\varphi(B(x,r))}}{\phi(x,r)\|\chi_{B(x,r)}\|_{L^\varphi}}.$$

Example 3.2. Let $\Phi : [0,\infty) \to [0,\infty]$ be an Orlicz function, $p : \mathbb{R}^n \to [1,\infty]$, $a : \mathbb{R}^n \to (1,\infty)$ be measurable functions and w be a weight function.

- If $\varphi(x,t) = \Phi(t)$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = M^{\Phi,\phi}(\mathbb{R}^n)$, where $M^{\Phi,\phi}(\mathbb{R}^n)$ denotes Orlicz-Morrey space (cf. [2]).
- If $\varphi(x,t) = w(x)\Phi(t)$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = M^{\Phi,\phi}_w(\mathbb{R}^n)$, where $M^{\Phi,\phi}_w(\mathbb{R}^n)$ denotes weighted Orlicz-Morrey space (cf. [3]).
- If $\varphi(x,t) = t^{p(x)}$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = M^{p(\cdot),\phi}(\mathbb{R}^n)$, where $M^{p(\cdot),\phi}(\mathbb{R}^n)$ denotes the generalized variable exponent Morrey spaces (cf. [8]).
- If $\phi(x,t) = \|\chi_{B(x,t)}\|^{-1}_{L^\varphi}$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = L^\varphi(\mathbb{R}^n)$.
- If $\phi(x,t) = \|\chi_{B(x,t)}\|^{-1}_{L^\varphi}$ and $\varphi(x,t) = t^r + a(x)t^q$, $1 \le r < q < \infty$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ equal to so-called double phase space (cf. [13, Chapter 7]).
- If $\phi(x,t) = \|\chi_{B(x,t)}\|^{-1}_{L^\varphi}$ and $\varphi(x,t) = t^{p(x)}w(x)$, then $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = L^{p(\cdot)}_w(\mathbb{R}^n)$, where $L^{p(\cdot)}_w(\mathbb{R}^n)$ denotes the weighted variable exponent Lebesgue spaces (cf. [13, Chapter 7]).

We have the following Guliyev type local estimate, see, for example, [7].

Lemma 3.3. *Let φ be a MO function. If φ satisfies (A0), (A1), (A2) and (aInc), then the inequality*

$$\|Mf\|_{L^\varphi(B(x,r))} \lesssim \|\chi_{B(x,r)}\|_{L^\varphi} \sup_{t>2r} \|\chi_{B(x,t)}\|^{-1}_{L^\varphi} \|f\|_{L^\varphi(B(x,t))} \tag{3.1}$$

holds uniformly for any $f \in L^\varphi_{\text{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$.

Proof. We split f with $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement_{B(x,2r)}}$ and have

$$\|Mf\|_{L^\varphi(B(x,r))} \lesssim \|Mf_1\|_{L^\varphi(B(x,r))} + \|Mf_2\|_{L^\varphi(B(x,r))}.$$

Estimation of Mf_1: By the boundedness of the operator M on $L^\varphi(\mathbb{R}^n)$ (see Theorem 2.6), we have

$$\|Mf_1\|_{L^\varphi(B(x,r))} \le \|Mf_1\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f_1\|_{L^\varphi(\mathbb{R}^n)} = \|f\|_{L^\varphi(B(x,2r))}.$$

By using the monotonicity of the function $\|f\|_{L^\varphi(B(x,t))}$ with respect to t we get

$$\begin{aligned}
&\|\chi_{B(x,r)}\|_{L^\varphi} \sup_{t>2r} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \|f\|_{L^\varphi(B(x,t))} \\
&\gtrsim \|f\|_{L^\varphi(B(x,2r))} \|\chi_{B(x,r)}\|_{L^\varphi} \sup_{t>2r} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \\
&= \|f\|_{L^\varphi(B(x,2r))} \frac{\|\chi_{B(x,r)}\|_{L^\varphi}}{\|\chi_{B(x,2r)}\|_{L^\varphi}} \\
&\gtrsim \|f\|_{L^\varphi(B(x,2r))},
\end{aligned} \tag{3.2}$$

where we use the well-known pointwise estimate $\chi_{2B}(z) \lesssim M\chi_B(z)$, for all $z \in \mathbb{R}^n$ (cf. [29, Proposition 129]) and Theorem 2.6 in the last inequality. Consequently we have

$$\|Mf_1\|_{L^\varphi(B(x,r))} \lesssim \|\chi_{B(x,r)}\|_{L^\varphi} \sup_{t>2r} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \|f\|_{L^\varphi(B(x,t))}. \tag{3.3}$$

Estimation of Mf_2: Let y be an arbitrary point from $B(x,r)$. A geometric observation shows that $t > r$ provided $B(y,t) \cap {}^\complement(B(x,2r)) \neq \emptyset$. Indeed, if $z \in B(y,t) \cap {}^\complement(B(x,2r))$, then $t > |y-z| \ge |x-z| - |x-y| > 2r - r = r$.

Meanwhile, a geometric observation shows $B(y,t) \cap {}^\complement(B(x,2r)) \subset B(x,2t)$. Indeed, if $z \in B(y,t) \cap {}^\complement(B(x,2r))$, then we get $|x-z| \le |y-z| + |x-y| < t + r < 2t$.

Hence, it follows that

$$\begin{aligned}
Mf_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|} \int_{B(y,t)\cap {}^\complement(B(x,2r))} |f(z)|dz \\
&\le 2^n \sup_{t>r} \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |f(z)|dz \\
&= 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz.
\end{aligned}$$

Therefore, for all $y \in B(x,r)$ we have

$$Mf_2(y) \lesssim \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz.$$

We proceed with Hölders inequality and Lemma 2.5

$$\begin{aligned}
Mf_2(y) &\lesssim \sup_{t>2r} |B(x,t)|^{-1} \|f\|_{L^\varphi(B(x,t))} \|\chi_{B(x,t)}\|_{L^{\varphi^*}} \\
&\lesssim \sup_{t>2r} \|f\|_{L^\varphi(B(x,t))} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1}.
\end{aligned} \tag{3.4}$$

Thus the function $Mf_2(y)$, with fixed x and r, is dominated by the expression not depending on y. Then we integrate the obtained estimate for $Mf_2(y)$ in y over $B(x,r)$, we get

$$\|Mf_2\|_{L^\varphi(B(x,r))} \lesssim \|\chi_{B(x,r)}\|_{L^\varphi} \sup_{t>2r} \|f\|_{L^\varphi(B(x,t))} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1}. \tag{3.5}$$

Gathering the estimates (3.3) and (3.5) we arrive at (3.1). □

Theorem 3.4. *Suppose that φ is a MO function satisfying (A0), (A1), (A2) and (aInc). If the functions (ϕ_1, ϕ_2) and φ satisfy the condition*

$$\sup_{r<t<\infty} \Big(\operatorname*{ess\,inf}_{0<t<s<\infty} \phi_1(x,s) \|\chi_{B(x,s)}\|_{L^\varphi} \Big) \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \le C\, \phi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Then M is bounded from $\mathcal{M}^{\varphi,\phi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\varphi,\phi_2}(\mathbb{R}^n)$.

Proof. Note that

$$\Big(\operatorname*{ess\,inf}_{x\in A} f(x) \Big)^{-1} = \operatorname*{ess\,sup}_{x\in A} \frac{1}{f(x)}$$

is true for any real-valued nonnegative function f and measurable on A and the fact that $\|f\|_{L^\varphi(B(x,t))}$ is a nondecreasing function of t

$$\begin{aligned} \frac{\|f\|_{L^\varphi(B(x,t))}}{\operatorname*{ess\,inf}_{0<t<s<\infty} \phi_1(x,s)\|\chi_{B(x,s)}\|_{L^\varphi}} &= \operatorname*{ess\,sup}_{0<t<s<\infty} \frac{\|f\|_{L^\varphi(B(x,t))}}{\phi_1(x,s)\|\chi_{B(x,s)}\|_{L^\varphi}} \\ &\le \sup_{s>0, x\in\mathbb{R}^n} \frac{\|f\|_{L^\varphi(B(x,s))}}{\phi_1(x,s)\|\chi_{B(x,s)}\|_{L^\varphi}} = \|f\|_{\mathcal{M}^{\varphi,\phi_1}}. \end{aligned}$$

Since (ϕ_1, ϕ_2) and φ satisfy the condition (3.6),

$$\begin{aligned} &\sup_{r<t<\infty} \|f\|_{L^\varphi(B(x,t))} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \\ &\le \sup_{r<t<\infty} \frac{\|f\|_{L^\varphi(B(x,t))}}{\operatorname*{ess\,inf}_{0<t<s<\infty} \phi_1(x,s)\|\chi_{B(x,s)}\|_{L^\varphi}} \operatorname*{ess\,inf}_{0<t<s<\infty} \phi_1(x,s)\|\chi_{B(x,s)}\|_{L^\varphi} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \\ &\le \|f\|_{\mathcal{M}^{\varphi,\phi_1}} \sup_{r<t<\infty} \Big(\operatorname*{ess\,inf}_{0<t<s<\infty} \phi_1(x,s)\|\chi_{B(x,s)}\|_{L^\varphi} \Big) \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \\ &\lesssim \phi_2(x,r) \|f\|_{\mathcal{M}^{\varphi,\phi_1}}. \end{aligned}$$

Then by (3.1)

$$\|Mf\|_{\mathcal{M}^{\varphi,\phi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \phi_2(x,r)^{-1} \sup_{t>r} \|f\|_{L^\varphi(B(x,t))} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \lesssim \|f\|_{\mathcal{M}^{\varphi,\phi_1}}.$$

□

For a MO function φ, we denote by $\mathcal{G}_\varphi$ the set of all functions $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ decreasing in the second variable such that $t \in (0,\infty) \mapsto \phi(x,t)\|\chi_{B(x,t)}\|_{L^\varphi}$ is almost increasing uniformly over the first variable x.

Corollary 3.5. *Suppose that φ is a MO function satisfying (A0), (A1), (A2) and (aInc). If $\phi \in \mathcal{G}_\varphi$ and the functions (ϕ_1, ϕ_2) satisfy the condition*

$$\sup_{r<t<\infty} \phi_1(x,t) \le C\, \phi_2(x,r),$$

where C does not depend on x and r. Then M is bounded from $\mathcal{M}^{\varphi,\phi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\varphi,\phi_2}(\mathbb{R}^n)$.

Corollary 3.6. *Suppose that φ is a MO function satisfying (A0), (A1), (A2) and (aInc). If $\phi \in \mathcal{G}_\varphi$, then M is bounded on $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$.*

4. Boundedness of the Riesz potential

In this section we prove Adams type boundedness properties of the Riesz potential in Musielak-Orlicz-Morrey spaces.

Theorem 4.1. *Suppose that φ is a MO function satisfying (A0), (A1), (A2) and (aInc). Let $\beta \in (0,1)$ and define $\eta(x,t) \equiv \phi(x,t)^\beta$ and $\psi(x,t) \equiv \varphi(x,t^{1/\beta})$. If the conditions* (3.6) *and*

$$t^\alpha \phi(x,t) + \int_t^\infty r^\alpha \, \phi(x,r) \frac{dr}{r} \lesssim \eta(x,t) \tag{4.1}$$

hold uniformly in $x \in \mathbb{R}^n$ and $t > 0$, then the operator I_α is bounded from $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ to $\mathcal{M}^{\psi,\eta}(\mathbb{R}^n)$.

Proof. Fix an $x \in \mathbb{R}^n$. Without loss of generality we may assume that f is non-negative and $I_\alpha f(x) < \infty$. Choose $t_0 > 0$ so that

$$I_\alpha f(x) = 2 \int_{B(x,t_0)} \frac{f(y)}{|x-y|^{n-\alpha}} dy = 2 \int_{\complement_{(B(x,t_0))}} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Following Hedberg's trick, we obtain

$$I_\alpha f(x) = 2 \int_{B(x,t_0)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \lesssim t_0^\alpha M f(x).$$

If we proceed as in (3.4), we have

$$\begin{aligned} I_\alpha f(x) &= 2 \int_{\complement_{(B(x,t_0))}} \frac{f(y)}{|x-y|^{n-\alpha}} dy \\ &\approx \int_{\complement_{(B(x,t_0))}} f(y) \int_{|x-y|}^\infty \frac{dr}{r^{n+1-\alpha}} dy \\ &\approx \int_{t_0}^\infty \int_{t_0 \le |x-y| < r} f(y) dy \frac{dr}{r^{n+1-\alpha}} \\ &\lesssim \int_{t_0}^\infty \|\chi_{B(x,r)}\|_{L^\varphi}^{-1} r^{\alpha-1} \|f\|_{L^\varphi(B(x,r))} dr. \end{aligned}$$

Consequently we have from (4.1)

$$I_\alpha f(x) \lesssim t_0^\alpha M f(x) \lesssim \phi(x,t_0)^{\beta-1} \, M f(x) \tag{4.2}$$

and

$$\begin{aligned} I_\alpha f(x) &\lesssim \int_{t_0}^\infty \|\chi_{B(x,r)}\|_{L^\varphi}^{-1} r^{\alpha-1} \|f\|_{L^\varphi(B(x,r))} dr \\ &\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}} \int_{t_0}^\infty r^\alpha \phi(x,r) \frac{dr}{r} \\ &\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}} \phi(x,t_0)^\beta. \end{aligned} \tag{4.3}$$

Thus, from (4.2) and (4.3) we obtain

$$\begin{aligned} I_\alpha f(x) &\lesssim \min\left\{\phi(x,t_0)^{\beta-1} M f(x), \phi(x,t_0)^\beta \|f\|_{\mathcal{M}^{\varphi,\phi}}\right\} \\ &\lesssim \sup_{s>0} \min\left\{s^{\beta-1} M f(x), s^\beta \|f\|_{\mathcal{M}^{\varphi,\phi}}\right\} \\ &= (M f(x))^\beta \, \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta}. \end{aligned}$$

Hence for every $x \in \mathbb{R}^n$ we have

$$I_\alpha f(x) \lesssim (M f(x))^\beta \, \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta}. \tag{4.4}$$

By using the inequality (4.4) we have for an arbitrary ball B

$$\|I_\alpha f\|_{L^\psi(B)} \lesssim \|(M f)^\beta\|_{L^\psi(B)} \, \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta}.$$

It follows directly from the definition of the norm that (cf. [13, p.108])

$$\|(M f)^\beta\|_{L^\psi(B)} = \|M f\|_{L^\varphi(B)}^\beta. \tag{4.5}$$

Consequently by this scaling property we have

$$\|I_\alpha f\|_{L^\psi(B)} \lesssim \|Mf\|_{L^\varphi(B)}^{\beta} \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta}. \tag{4.6}$$

From Theorem 3.4 and (4.6), we get

$$\begin{aligned}\|I_\alpha f\|_{\mathcal{M}^{\psi,\eta}} &= \sup_{x\in\mathbb{R}^n, t>0} \eta(x,t)^{-1} \|\chi_{B(x,t)}\|_{L^\psi}^{-1} \|I_\alpha f\|_{L^\psi(B(x,t))} \\ &\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta} \sup_{x\in\mathbb{R}^n, t>0} \eta(x,t)^{-1} \|\chi_{B(x,t)}\|_{L^\psi}^{-1} \|Mf\|_{L^\varphi(B(x,t))}^{\beta} \\ &= \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta} \left(\sup_{x\in\mathbb{R}^n, t>0} \phi(x,t)^{-1} \|\chi_{B(x,t)}\|_{L^\varphi}^{-1} \|Mf\|_{L^\varphi(B(x,t))} \right)^{\beta} \\ &= \|f\|_{\mathcal{M}^{\varphi,\phi}}^{1-\beta} \|Mf\|_{\mathcal{M}^{\varphi,\phi}}^{\beta} \lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}}.\end{aligned}$$

□

5. Weak type results

We also deal with the weak type maximal operator estimates in this paper. We give the definition of weak Musielak-Orlicz space now.

Definition 5.1. Let φ be a MO function. The weak Musielak-Orlicz space is defined as

$$WL^\varphi(\mathbb{R}^n) := \{f \in L^0(\mathbb{R}^n) : \|f\|_{WL^\varphi} < \infty\},$$

where

$$\|f\|_{WL^\varphi} = \sup_{t>0} \|t\chi_{\{|f|>t\}}\|_{L^\varphi}.$$

For any MO function φ we have

$$L^\varphi(\mathbb{R}^n) \subset WL^\varphi(\mathbb{R}^n) \qquad \text{with} \quad \|f\|_{WL^\varphi} \le \|f\|_{L^\varphi}. \tag{5.1}$$

The following weak type boundedness of the maximal operator was proved in [13, Theorem 4.3.10].

Theorem 5.2. *Let φ be a MO function. If φ satisfies (A0), (A1) and (A2), we have*

$$\|Mf\|_{WL^\varphi} \le C\|f\|_{L^\varphi}$$

for all $f \in L^\varphi(\mathbb{R}^n)$, where $C > 0$ is independent of f.

Now, we introduce the weak Musielak-Orlicz-Morrey space $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$.

Definition 5.3. Let φ be a MO function and let $\phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. By $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ we denote the weak Musielak-Orlicz-Morrey space of all measurable functions for which

$$\|f\|_{W\mathcal{M}^{\varphi,\phi}} \equiv \|f\|_{W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|f\|_{WL^\varphi(B(x,r))}}{\phi(x,r)\|\chi_{B(x,r)}\|_{L^\varphi}} < \infty.$$

Example 5.4. Let $\Phi : [0,\infty) \to [0,\infty]$ be an Orlicz function, $p : \mathbb{R}^n \to [1,\infty]$, $a : \mathbb{R}^n \to (1,\infty)$ be measurable functions and w be a weight function.

- If $\varphi(x,t) = \Phi(t)$, then $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = WM^{\Phi,\phi}(\mathbb{R}^n)$, where $WM^{\Phi,\phi}(\mathbb{R}^n)$ denotes weak Orlicz-Morrey space.
- If $\varphi(x,t) = w(x)\Phi(t)$, then $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = WM_w^{\Phi,\phi}(\mathbb{R}^n)$, where $WM_w^{\Phi,\phi}(\mathbb{R}^n)$ denotes weak weighted Orlicz-Morrey space.
- If $\varphi(x,t) = t^{p(x)}$, then $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = WM^{p(\cdot),\phi}(\mathbb{R}^n)$, where $WM^{p(\cdot),\phi}(\mathbb{R}^n)$ denotes the weak generalized variable exponent Morrey spaces.
- If $\phi(x,t) = \|\chi_{B(x,t)}\|_{WL^\varphi}^{-1}$, then $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = WL^\varphi(\mathbb{R}^n)$.

- If $\phi(x,t) = \|\chi_{B(x,t)}\|^{-1}_{WL^\varphi}$ and $\varphi(x,t) = t^r + a(x)t^q$, $1 \le r < q < \infty$, then $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ equals to so-called weak double phase space.
- If $\phi(x,t) = \|\chi_{B(x,t)}\|^{-1}_{WL^\varphi}$ and $\varphi(x,t) = t^{p(x)}w(x)$, then $W\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n) = WL^{p(\cdot)}_w(\mathbb{R}^n)$, where $WL^{p(\cdot)}_w(\mathbb{R}^n)$ denotes the weak weighted variable exponent Lebesgue spaces.

The following lemma is valid.

Lemma 5.5. *Suppose that φ is a MO function satisfying (A0), (A1) and (A2). Then the inequality*

$$\|Mf\|_{WL^\varphi(B(x,r))} \lesssim \|\chi_{B(x,r)}\|_{L^\varphi} \sup_{t>2r} \|\chi_{B(x,t)}\|^{-1}_{L^\varphi} \|f\|_{L^\varphi(B(x,t))}, \tag{5.2}$$

holds uniformly for any $f \in WL^\varphi_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$.

Proof. We split f with $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement_{B(x,2r)}}$ and have

$$\|Mf\|_{WL^\varphi(B)} \lesssim \|Mf_1\|_{WL^\varphi(B)} + \|Mf_2\|_{WL^\varphi(B)}.$$

By the boundedness of the operator M from $L^\varphi(\mathbb{R}^n)$ to $WL^\varphi(\mathbb{R}^n)$ (see Theorem 5.2), we have

$$\|Mf_1\|_{WL^\varphi(B)} \le \|Mf_1\|_{WL^\varphi(\mathbb{R}^n)} \lesssim \|f_1\|_{L^\varphi(\mathbb{R}^n)} = \|f\|_{L^\varphi(B(x,2r))}.$$

By using (3.2), (3.5) and (5.1) we arrive at (5.2). □

Theorem 5.6. *Suppose that φ is a MO function satisfying (A0), (A1) and (A2). If the functions (ϕ_1, ϕ_2) and φ satisfy the condition (3.6), then M is bounded from $\mathcal{M}^{\varphi,\phi_1}(\mathbb{R}^n)$ to $W\mathcal{M}^{\varphi,\phi_2}(\mathbb{R}^n)$.*

Proof. The proof is same as the proof of Theorem 3.4, but now using Lemma 5.5 instead of Lemma 3.3. □

Theorem 5.7. *Suppose that φ is a MO function satisfying (A0), (A1) and (A2). Let $\beta \in (0,1)$ and define $\eta(x,t) \equiv \phi(x,t)^\beta$ and $\psi(x,t) \equiv \varphi(x,t^{1/\beta})$. If the conditions (3.6) and (4.1) hold, then the operator I_α is bounded from $\mathcal{M}^{\varphi,\phi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\psi,\eta}(\mathbb{R}^n)$.*

Proof. By using the inequality (4.4) we have

$$\|I_\alpha f\|_{WL^\psi(B)} \lesssim \|(Mf)^\beta\|_{WL^\psi(B)} \|f\|^{1-\beta}_{\mathcal{M}^{\varphi,\phi}},$$

where $B = B(x,t)$. Note that from (4.5) we get

$$\|(Mf)^\beta\|_{WL^\psi(B)} = \|Mf\|^\beta_{WL^\varphi(B)}.$$

Consequently by this scaling property we have

$$\|I_\alpha f\|_{WL^\psi(B)} \lesssim \|Mf\|^\beta_{WL^\varphi(B)} \|f\|^{1-\beta}_{\mathcal{M}^{\varphi,\phi}}. \tag{5.3}$$

From Theorem 5.6 and (5.3), we get

$$\begin{aligned}
\|I_\alpha f\|_{W\mathcal{M}^{\psi,\eta}} &= \sup_{x\in\mathbb{R}^n, t>0} \eta(x,t)^{-1} \|\chi_{B(x,t)}\|^{-1}_{L^\psi} \|I_\alpha f\|_{WL^\psi(B)} \\
&\lesssim \|f\|^{1-\beta}_{\mathcal{M}^{\varphi,\phi}} \sup_{x\in\mathbb{R}^n, t>0} \eta(x,t)^{-1} \|\chi_{B(x,t)}\|^{-1}_{L^\psi} \|Mf\|^\beta_{WL^\varphi(B)} \\
&= \|f\|^{1-\beta}_{\mathcal{M}^{\varphi,\phi}} \left(\sup_{x\in\mathbb{R}^n, t>0} \phi(x,t)^{-1} \|\chi_{B(x,t)}\|^{-1}_{L^\varphi} \|Mf\|_{WL^\varphi(B)} \right)^\beta \\
&\lesssim \|f\|_{\mathcal{M}^{\varphi,\phi}}.
\end{aligned}$$

□

6. Special cases

In this section, we apply Theorems 3.4, 4.1, 5.6 and 5.7, respectively, to a concrete example of Musielak-Orlicz-Morrey spaces, namely, generalized variable exponent Morrey spaces.

Let $p : \Omega \to [1, \infty]$ be a measurable function. Now, we examine the variable exponent case, $\varphi(x,t) = t^{p(x)}$. Here we understand $t^{\infty} := \infty\chi_{(1,\infty]}(t)$. After that, it is possible to verify that $\varphi^{-1}(x,t) = t^{1/p(x)}$, where $t^{1/\infty} := \chi_{(0,\infty)}(t)$.

Retrieve familiar notions from variable exponent spaces. Let us provide a clear and concise definition;

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

We state that $\frac{1}{p}$ is log-*Hölder continuous*, $\frac{1}{p} \in C^{\log}$, if

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le \frac{c}{\log\left(e + \frac{1}{|x-y|}\right)}$$

for each distinct $x, y \in \Omega$.

Similar to the Musielak-Orlicz scenario, variable exponents necessitate both a local condition and a decay condition. For variable exponent spaces, Nekvinda's decay condition can be expressed as follows: $1 \in L^{s(\cdot)}(\Omega)$, with $\frac{1}{s(x)} = |\frac{1}{p(x)} - \frac{1}{p_\infty}|$ and $p_\infty \in [1,\infty]$. Equivalently, this means that there exists $c > 0$ such that

$$\int_{\{p(x)\neq p_\infty\}} c^{s(x)} dx < \infty.$$

The originator of this condition is Nekvinda, as stated in [25]. We note the presence of the log-Hölder decay condition,

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \le \frac{c}{\log(e + |x|)},$$

implies Nekvinda's decay condition.

The results for the conditions are summarized in the table below [13].

Table 1. Conditions in variable exponent case

$\varphi(x,t)$	$(A0)$	$(A1)$	$(A2)$	$(aInc)$	$(aDec)$
$t^{p(x)}$	True	$\frac{1}{p} \in C^{\log}$	Nekvinda	$p^- > 1$	$p^+ < \infty$

If we take $\varphi(x,t) = t^{p(x)}$ in Theorems 3.4, 4.1, 5.6 and 5.7 we get the following corollaries for generalized variable exponent Morrey space (cf. [8]):

Corollary 6.1. *Suppose that* $\frac{1}{p}$ *is* log-*Hölder continuous and satisfies Nekvindas decays condition. If the functions* (ϕ_1, ϕ_2) *satisfy the condition*

$$\sup_{r<t<\infty} \left(\operatorname*{ess\,inf}_{0<t<s<\infty} \phi_1(x,s)\|\chi_{B(x,s)}\|_{L^{p(\cdot)}}\right) \|\chi_{B(x,t)}\|^{-1}_{L^{p(\cdot)}} \le C\,\phi_2(x,r), \tag{6.1}$$

where C *does not depend on* x *and* r*. Then* M *is bounded from* $\mathcal{M}^{p(\cdot),\phi_1}(\mathbb{R}^n)$ *to* $W\mathcal{M}^{p(\cdot),\phi_2}(\mathbb{R}^n)$*. Moreover, if* $p^- > 1$*, then* M *is bounded from* $\mathcal{M}^{p(\cdot),\phi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{p(\cdot),\phi_2}(\mathbb{R}^n)$*.*

Corollary 6.2. *Suppose that* $\frac{1}{p}$ *is* log-*Hölder continuous and satisfies Nekvindas decays condition. Suppose also that* $q(x) > p(x)$ *and define* $\eta(x,t) \equiv \phi(x,t)^{\frac{p(x)}{q(x)}}$*. If the conditions* (4.1) *and* (6.1) *hold, then the operator* I_α *is bounded from* $\mathcal{M}^{p(\cdot),\phi}(\mathbb{R}^n)$ *to* $W\mathcal{M}^{q(\cdot),\eta}(\mathbb{R}^n)$*. Moreover, if* $p^- > 1$*, then* I_α *is bounded from* $\mathcal{M}^{p(\cdot),\phi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q(\cdot),\eta}(\mathbb{R}^n)$*.*

Acknowledgment. The authors thank the anonymous reviewers for constructive and helpful remarks.

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