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Characterizations for the fractional maximal operator and its commutators on total Morrey spaces

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Abstract

We shall give a characterization for the strong and weak type Adams type boundedness of the fractional maximal operator M_α on total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, respectively. Also we give necessary and sufficient conditions for the boundedness of the fractional maximal commutator operator $M_{b,\alpha}$ and commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$ spaces, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$ spaces are obtained.

Keywords Total Morrey spaces · Fractional maximal operator · Commutator · *BMO*

Mathematics Subject Classification Primary 42B20 · 42B25 · 42B35

1 Introduction

The study of fractional maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [7]). For $0 < \alpha < n$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{n}} \int_{B(x,r)} |f(y)|dy,$$

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where $B(x,r)$ is the ball of radius r centered at $x \in \mathbb{R}^n$, ${}^{\complement}B(x,r)$ is its complement and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$. When $\alpha = 0$, we simply denote by $M = M_0$.

The fractional maximal commutator generated by the operator M_α and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by

$$M_{b,\alpha} f(x) = M_\alpha((b(x) - b) f)(x).$$

The commutator generated by the operator M_α and a suitable function b is defined by

$$[b, M_\alpha] f(x) = b(x) M_\alpha f(x) - M_\alpha(bf)(x).$$

When $\alpha = 0$, we simply denote by $M_b = M_{b,0}$ and $[b, M] = [b, M_0]$. Obviously, the operators $M_{b,\alpha}$ and $[b, M_\alpha]$ essentially differ from each other since $M_{b,\alpha}$ is positive and sublinear and $[b, M_\alpha]$ is neither positive nor sublinear. The operators M_α, $[b, M_\alpha]$ and $M_{b,\alpha}$ play an important role in real and harmonic analysis and applications (see, for instance [2–4, 6, 11–15, 24]).

The boundedness of the Hardy–Littlewood maximal operator M on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of M with BMO functions b. These turn out to be L^p bounded for $1 < p < \infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b, 0\} \in L^\infty(\mathbb{R}^n)$ [4]. This is useful, for instance, when studying the product of an H^1 function with a BMO function [5]. Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [6].

The commutator estimates play an important role in studying the regularity of solutions of elliptic partial differential equations, and their boundedness can be used to characterize certain function spaces (see, for instance [2–4, 7, 11–15, 20–24]).

The classical Morrey spaces were introduced by Morrey [16] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equation, the reader is referred to [18–21]. In [13] introduced a variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Total Morrey spaces generalize the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, respectively. We given necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and the commutator of maximal operator $[b, M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$.

The aim of this paper is to give necessary and sufficient conditions for the boundedness of the fractional maximal commutator operator $M_{b,\alpha}$ and the commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$ spaces. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The structure of the paper is as follows. In Sect. 2 we introduce the total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We give some definitions, auxiliary results and some embeddings into the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. In Sect. 3 we give a characterization for the strong and weak type Adams type boundedness of the fractional maximal operator M_α on $L^{p,\lambda,\mu}(\mathbb{R}^n)$, respectively. In Sect. 4 we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\alpha}$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces. In Sect. 5 we find necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definition and basic properties of total Morrey spaces

In this section we define the total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, and give auxiliary results and some embeddings into the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

Definition 2.1 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^{p,\lambda}(\mathbb{R}^n)$ the classical Morrey space , by $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space [9], and by $L^{p,\lambda,\mu}(\mathbb{R}^n)$ the total Morrey space the set of all classes of locally integrable functions f with the finite norms

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))},$$

$$\|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))},$$

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))},$$

respectively.

Definition 2.2 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ [9] and the weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norms

$$\|f\|_{WL^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{W\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{WL^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL^p(B(x,t))},$$

respectively, where $WL^p(B(x,t))$ denotes the weak L^p-space of measurable functions f for which

$$\|f\|_{WL^p(B(x,t))} = \sup_{r>0} r\, |\{y \in B(x,t) : |f(y)| > r\}|^{\frac{1}{p}}.$$

Lemma 2.1 *If* $0 < p < \infty$, $0 \le \mu \le \lambda \le n$, *then*

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}.$$

Proof Let $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and $0 \le \mu \le \lambda \le n$. Then

$$\begin{aligned}\|f\|_{L^{p,\lambda}} &= \|f\|_{L^{p,\lambda,\lambda}}\\ &= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} t^{-\frac{\mu}{p}} t^{\frac{\mu-\lambda}{p}}\|f\|_{L^p(B(x,t))}\right\}\\ &\le \|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}\end{aligned}$$

and

$$\begin{aligned}\|f\|_{L^{p,\mu}} &= \|f\|_{L^{p,\mu,\mu}}\\ &= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{\frac{\lambda-\mu}{p}} t^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} t^{-\frac{\mu}{p}}\|f\|_{L^p(B(x,t))}\right\}\\ &\le \|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, $f \in L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\} \le \|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}$.

Now let $f \in L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$. Then

$$\begin{aligned}\|f\|_{L^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))}\\ &= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} t^{-\frac{\mu}{p}}\|f\|_{L^p(B(x,t))}\right\}\\ &\le \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}.\end{aligned}$$

Therefore, $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and $\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} \le \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}$. □

Remark 2.1 In [13], Lemma 2 omitted the condition $\mu \le \lambda$, so we proved this lemma again in correct form.

Analogously proved

Lemma 2.2 *If* $0 < p < \infty$, $0 \le \mu \le \lambda \le n$, *then*

$$WL^{p,\lambda,\mu}(\mathbb{R}^n) = WL^{p,\lambda}(\mathbb{R}^n) \cap WL^{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^{p,\lambda}}, \|f\|_{WL^{p,\mu}}\right\}.$$

Lemma 2.3 [13, Lemma 6] *If* $0 \le \lambda < n$, $0 \le \mu < n$, $0 \le \alpha < n - \lambda$ *and* $0 \le \beta < n - \mu$, *then for* $\frac{n-\lambda}{\alpha} \le p \le \frac{n-\mu}{\beta}$

$$L^{p,\lambda,\mu}(\mathbb{R}^n) \hookrightarrow L^{1,n-\alpha,n-\beta}(\mathbb{R}^n)$$

and for $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ *the following inequality*

$$\|f\|_{L^{1,n-\alpha,n-\beta}} \le v_n^{1/p'} \|f\|_{L^{p,\lambda,\mu}}$$

is valid.

3 Fractional maximal operator in total Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the fractional maximal operator M_α in the $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

The following local estimate is valid (see also [8, 10]).

Lemma 3.4 [10, Lemma 4.1] *Let* $0 \le \alpha < n$, $1 \le p < \frac{n}{\alpha}$, *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. *Then, for* $p > 1$ *the inequality*

$$\|M_\alpha f\|_{L^q(B(x,r))} \lesssim r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L^p(B(x,t))} \tag{3.1}$$

holds for all $B(x,r)$ *and for all* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover if $p = 1$, *then the inequality*

$$\|M_\alpha f\|_{WL^q(B(x,r))} \lesssim r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L^1(B(x,t))} \tag{3.2}$$

holds for all $B(x,r)$ *and for all* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The following is Spanne's type result for fractional maximal operators in total Morrey spaces (see, for example, [8]).

Theorem 3.1 (Spanne's type result) *Let* $1 \le p < \infty$, $0 \le \lambda, \mu < n$, $0 \le \alpha < \frac{n}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

1. If $p > 1$, $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $M_\alpha f \in L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,\lambda,\mu} \|f\|_{L^{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

2. *If* $p = 1,\ f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$*, then* $Mf \in WL^{q,\lambda q,\mu q}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{WL^{q,\lambda q,\mu q}} \le C_{1,\lambda,\mu}\, \|f\|_{L^{1,\lambda,\mu}},$$

where $C_{1,\lambda,\mu}$ *is independent of* f.

Proof Let $1 < p < \infty$. From the inequality (3.1) we get

$$\begin{aligned}
\|M_\alpha f\|_{L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, \|M_\alpha f\|_{L^q(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^p(B(x,t))} \\
&\lesssim \|f\|_{L^{p,\lambda,\mu}} \sup_{r>0}[r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, r^{-\alpha+\frac{n}{p}} \sup_{t>r} t^{\alpha-\frac{n}{p}}\, [t]_1^{\frac{\lambda}{p}}\, [1/t]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L^{p,\lambda,\mu}} \sup_{r>0}[r]_1^{-\alpha+\frac{n-\lambda}{p}}\, [1/r]_1^{\alpha-\frac{n-\mu}{p}} \sup_{t>r}[t]_1^{\alpha-\frac{n-\lambda}{p}}\, [1/t]_1^{-\alpha+\frac{n-\mu}{p}} \\
&= \|f\|_{L^{p,\lambda,\mu}},
\end{aligned}$$

which implies that the operator $M_\alpha f$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$.

Let $p = 1$. From the inequality (3.2) we get

$$\begin{aligned}
\|M_\alpha f\|_{WL^{q,\lambda q,\mu q}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\lambda}\, [1/r]_1^{\mu}\, \|M_\alpha f\|_{WL^q(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\lambda}\, [1/r]_1^{\mu}\, r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^1(B(x,t))} \\
&\lesssim \|f\|_{L^{1,\lambda,\mu}} \sup_{r>0}[r]_1^{-\lambda}\, [1/r]_1^{\mu}\, r^{-\alpha+n} \sup_{t>r} t^{\alpha-n}\, [t]_1^{\lambda}\, [1/t]_1^{-\mu} \\
&= \|f\|_{L^{1,\lambda,\mu}} \sup_{r>0}[r]_1^{-\alpha+n-\lambda}\, [1/r]_1^{\alpha-(n-\mu)} \sup_{t>r}[t]_1^{\alpha-(n-\lambda)}\, [1/t]_1^{-\alpha+(n-\mu)} \\
&= \|f\|_{L^{1,\lambda,\mu}},
\end{aligned}$$

which implies that the operator $M_\alpha f$ is bounded from $L^{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^{q,\lambda q,\mu q}(\mathbb{R}^n)$. □

From Theorem 3.1 in the case $\alpha = 0$ we get the following corollaries.

Corollary 3.1 [13, Theorem 1] *Let* $1 \le p < \infty$, $0 \le \lambda < n$ *and* $0 \le \mu < n$.

1. *If* $p > 1,\ f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$*, then* $Mf \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{L^{p,\lambda,\mu}} \le C_{p,\lambda,\mu}\, \|f\|_{L^{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

2. *If* $p = 1,\ f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$*, then* $Mf \in WL^{1,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{WL^{1,\lambda,\mu}} \le C_{1,\lambda,\mu}\, \|f\|_{L^{1,\lambda,\mu}},$$

where $C_{1,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

From Theorem 3.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.2 [17, Theorem 5.4] *Let* $1 \le p < \infty$, $0 \le \lambda < n$, $0 \le \alpha < \frac{n}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

1. *If* $p > 1$, $f \in L^{p,\lambda}(\mathbb{R}^n)$, *then* $M_\alpha f \in L^{q,\frac{\lambda q}{p}}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{L^{q,\frac{\lambda q}{p}}} \le C_{p,\lambda} \, \|f\|_{L^{p,\lambda}},$$

where $C_{p,\lambda}$ *depends only on* p, λ *and* n.

2. *If* $p = 1$, $f \in L^{1,\lambda}(\mathbb{R}^n)$, *then* $M_\alpha f \in WL^{q,\lambda}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{WL^{q,\lambda q}} \le C_{1,\lambda} \, \|f\|_{L^{1,\lambda}},$$

where $C_{1,\lambda}$ *is independent of* f.

Corollary 3.3 *Let* $1 \le p < \infty$, $0 \le \lambda < n$, $0 \le \alpha < \frac{n}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

1. *If* $p > 1$, $f \in \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, *then* $M_\alpha f \in \widetilde{L}^{q,\frac{\lambda q}{p}}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{\widetilde{L}^{q,\frac{\lambda q}{p}}} \le C_{p,\lambda} \, \|f\|_{\widetilde{L}^{p,\lambda}},$$

where $C_{p,\lambda}$ *depends only on* p, λ *and* n.

2. *If* $p = 1$, $f \in \widetilde{L}^{1,\lambda}(\mathbb{R}^n)$, *then* $M_\alpha f \in W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{W\widetilde{L}^{q,\lambda q}} \le C_{1,\lambda} \, \|f\|_{\widetilde{L}^{1,\lambda}},$$

where $C_{1,\lambda}$ *is independent of* f.

The following is Adams type result for fractional maximal operators in total Morrey spaces (see, for example, [8]).

Theorem 3.2 *(Adams type result) Let* $1 \le p < \infty$, $0 \le \mu \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{p}$.

(1) *If* $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

(2) *If* $p = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n-\mu} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $L^{1,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

(3) *If* $\frac{n-\lambda}{\alpha} \le p \le \frac{n-\mu}{\alpha}$, *then the operator* M_α *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^\infty(\mathbb{R}^n)$.

Proof *Sufficiency.* Let $1 \le p < \infty, 0 \le \mu \le \lambda < n, 0 \le \alpha < \frac{n-\lambda}{p}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ and $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$.

$$M_\alpha f(x) \approx \sup_{r>0} r^{\alpha-n} \, \|f\|_{L^1(B(x,r))}$$

$$
\begin{aligned}
&\le \sup_{r>0} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n}{p}}\, \|f\|_{L^p(B(x,r))}\}\\
&\le \sup_{r>0} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n}{p}}\, [r]_1^{\frac{\lambda}{p}}\, [1/r]_1^{-\frac{\mu}{p}}\, \|f\|_{L^{p,\lambda,\mu}}\}\\
&\le \sup_{r>0} \min\{r^{\alpha}\, Mf(x), [r]_1^{\alpha-\frac{n-\lambda}{p}}\, [1/r]_1^{-\alpha+\frac{n-\mu}{p}}\, \|f\|_{L^{p,\lambda,\mu}}\}\\
&\le \max\Big\{ \sup_{0<r\le 1} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n-\lambda}{p}}\, \|f\|_{L^{p,\lambda,\mu}}\},\\
&\qquad \sup_{r>1} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n-\mu}{p}}\, \|f\|_{L^{p,\lambda,\mu}}\}\Big\}.
\end{aligned}
$$

Minimizing with respect to r, at

$$
r = \Big(\frac{\|f\|_{L^{p,\lambda,\mu}}}{Mf(x)}\Big)^{\frac{p}{n-\lambda}} \quad \text{and} \quad r = \Big(\frac{\|f\|_{L^{p,\lambda,\mu}}}{Mf(x)}\Big)^{\frac{p}{n-\mu}}
$$

we have

$$
\begin{aligned}
M_\alpha f(x) \le \max\Big\{&\big(Mf(x)\big)^{1-\frac{\alpha p}{n-\lambda}}\, \|f\|_{L^{p,\lambda,\mu}}^{\frac{\alpha p}{n-\lambda}},\\
&\big(Mf(x)\big)^{1-\frac{\alpha p}{n-\mu}}\, \|f\|_{L^{p,\lambda,\mu}}^{\frac{\alpha p}{n-\mu}}\Big\},
\end{aligned} \tag{3.3}
$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 3.1 and inequality (3.3), we get

$$
\begin{aligned}
\|M_\alpha f\|_{L^{q,\lambda,\mu}} &\lesssim \|f\|_{L^{p,\lambda,\mu}}^{1-\frac{p}{q}}\, \|(Mf)^{\frac{p}{q}}\|_{L^{q,\lambda,\mu}}\\
&= \|f\|_{L^{p,\lambda,\mu}}^{1-\frac{p}{q}}\, \|Mf\|_{L^{p,\lambda,\mu}}^{\frac{p}{q}} \lesssim \|f\|_{L^{p,\lambda,\mu}},
\end{aligned}
$$

if $1 < p < q < \infty$ and

$$
\|M_\alpha f\|_{WL^{q,\lambda,\mu}} \lesssim \|f\|_{L^{1,\lambda,\mu}}^{1-\frac{1}{q}}\, \|Mf\|_{WL^{1,\lambda,\mu}}^{\frac{1}{q}} \lesssim \|f\|_{L^{1,\lambda,\mu}},
$$

if $p = 1 < q < \infty$.

Necessity. Let $1 \le p < \frac{n-\lambda}{\alpha}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$, $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that M_α is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1, t\}$. Then

$$
\begin{aligned}
\big\|f_t\big\|_{L^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, \|f_t\|_{L^p(B(x,r))}\\
&= t^{-\frac{n}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, \|f\|_{L^p(B(x,tr))}
\end{aligned}
$$

$$= t^{-\frac{n}{p}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\frac{\lambda}{p}} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\frac{\mu}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} [1/(tr)]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,tr))}$$
$$= t^{-\frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p}} [1/t]_{1,+}^{-\frac{\mu}{p}} \|f\|_{L^{p,\lambda,\mu}},$$

and

$$M_\alpha f_t(x) = t^{-\alpha} M_\alpha f(tx),$$

$$\Big\|M_\alpha f_t\Big\|_{L^{q,\lambda,\mu}} = t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|M_\alpha f(t\,\cdot)\|_{L^q(B(x,r))}$$
$$= t^{-\alpha-\frac{n}{q}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q}$$
$$\sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} [1/(tr)]_1^{\frac{\mu}{p}} \|M_\alpha f\|_{L^q(B(tx,tr))}$$
$$= t^{-\alpha-\frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{q}} \Big\|M_\alpha f\Big\|_{L^{q,\lambda,\mu}}.$$

By the boundedness of M_α from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$ we have

$$\Big\|M_\alpha f\Big\|_{L^{q,\lambda,\mu}} = t^{\alpha+\frac{n}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \big\|M_\alpha f_t\big\|_{L^{q,\lambda,\mu}}$$
$$\lesssim t^{\alpha+\frac{n}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \big\|f_t\big\|_{L^{p,\lambda,\mu}}$$
$$= t^{\alpha+\frac{n}{q}-\frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{p}+\frac{\mu}{q}} \|f\|_{L^{p,\lambda,\mu}}$$
$$= t^{\alpha} [t]_{1,+}^{-\frac{n-\lambda}{p}+\frac{n-\lambda}{q}} [1/t]_{1,+}^{\frac{n-\mu}{p}-\frac{n-\mu}{q}} \|f\|_{L^{p,\lambda,\mu}}.$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n-\mu}$, then by letting $t \to 0$ we have $\|M_\alpha f\|_{L^{q,\lambda,\mu}} = 0$ for all $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|M_\alpha f\|_{L^{q,\lambda,\mu}} = 0$ for all $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\lambda} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\mu}$.

Let $p = 1 < \frac{n-\mu}{\alpha}$, $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that M_α is bounded from $L^{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^{q,\lambda,\mu}(\mathbb{R}^n)$. Then

$$\big\|f_t\big\|_{L^{1,\lambda,\mu}} = t^{-n} [t]_{1,+}^{\lambda} [1/t]_{1,+}^{-\mu} \|f\|_{L^{1,\lambda,\mu}}$$

and

$$\begin{aligned}\left\|M_\alpha f_t\right\|_{WL^{q,\lambda,\mu}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, \|M_\alpha f(t\,\cdot)\|_{WL^q(B(x,r))} \\ &= t^{-\alpha-\frac{n}{q}} \sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q} \\ &\sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}}\, [1/(tr)]_1^{\frac{\mu}{p}}\, \|M_\alpha f\|_{WL^q(B(tx,tr))} \\ &= t^{-\alpha-\frac{n}{q}}\, [t]_{1,+}^{\frac{\lambda}{q}}\, [1/t]_{1,+}^{-\frac{\mu}{q}} \left\|M_\alpha f\right\|_{WL^{q,\lambda,\mu}}.\end{aligned}$$

By the boundedness of M_α from $L^{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^{q,\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}\left\|M_\alpha f\right\|_{WL^{q,\lambda,\mu}} &= t^{\alpha+\frac{n}{q}}\, [t]_{1,+}^{-\frac{\lambda}{q}}\, [1/t]_{1,+}^{\frac{\mu}{q}} \left\|M_\alpha f_t\right\|_{WL^{q,\lambda,\mu}} \\ &\lesssim t^{\alpha+\frac{n}{q}}\, [t]_{1,+}^{-\frac{\lambda}{q}}\, [1/t]_{1,+}^{\frac{\mu}{q}} \left\|f_t\right\|_{L^{1,\lambda,\mu}} \\ &= t^{\alpha+\frac{n}{q}-n}\, [t]_{1,+}^{\lambda-\frac{\lambda}{q}}\, [1/t]_{1,+}^{-\mu+\frac{\mu}{q}} \|f\|_{L^{1,\lambda,\mu}} \\ &= t^{\alpha}\, [t]_{1,+}^{-n+\lambda+\frac{n-\lambda}{q}}\, [1/t]_{1,+}^{n-\mu-\frac{n-\mu}{q}} \|f\|_{L^{1,\lambda,\mu}}.\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then by letting $t \to 0$ we have $\|M_\alpha f\|_{WL^{q,\lambda,\mu}} = 0$ for all $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $1 > \frac{1}{q} + \frac{\alpha}{n-\mu}$, then at $t \to \infty$ we obtain $\|M_\alpha f\|_{WL^{q,\lambda,\mu}} = 0$ for all $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\lambda} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\mu}$.

3) Let us show that, if $\frac{n-\mu}{\alpha} \le p \le \frac{n-\lambda}{\alpha}$, then the operator M_α is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Let $\frac{n-\mu}{\alpha} \le p \le \frac{n-\lambda}{\alpha}$ and $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$. Then

$$\begin{aligned}M_\alpha f(x) &\approx \sup_{r>0} r^{\alpha-n}\, \|f\|_{L^1(B(x,r))} \le \sup_{r>0} r^{\alpha-\frac{n}{p}}\, \|f\|_{L^p(B(x,r))} \\ &\le \sup_{r>0} r^{\alpha-\frac{n}{p}}\, [r]_1^{\frac{\lambda}{p}}\, [1/r]_1^{-\frac{\mu}{p}}\, \|f\|_{L^{p,\lambda,\mu}} \le \sup_{r>0} [r]_1^{\alpha-\frac{n-\lambda}{p}}\, [1/r]_1^{-\alpha+\frac{n-\mu}{p}}\, \|f\|_{L^{p,\lambda,\mu}} \\ &\le \max\Big\{\sup_{0<r\le 1} r^{\alpha-\frac{n-\lambda}{p}}\, \|f\|_{L^{p,\lambda,\mu}},\, \sup_{r>1} r^{\alpha-\frac{n-\mu}{p}}\, \|f\|_{L^{p,\lambda,\mu}}\Big\} \lesssim \|f\|_{L^{p,\lambda,\mu}} \\ &\Longleftrightarrow \frac{n-\lambda}{p} \le \alpha \le \frac{n-\mu}{p},\end{aligned}$$

which implies that the operator M_α is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. □

From Theorem 3.2 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.4 [1, Theorem 3.1] *(Adams result) Let* $1 \le p < \infty$, $0 \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{p}$.

1) *If* $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

2) *If* $p = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $L^{1,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

3) *If* $p = \frac{n-\lambda}{\alpha}$, *then the operator* M_α *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^\infty(\mathbb{R}^n)$.

Corollary 3.5 [9, Corollary 1] *Let* $1 \le p < \infty$, $0 \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{p}$.

1) *If* $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

2) *If* $p = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}^{1,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

3) *If* $\frac{n-\lambda}{\alpha} \le p \le \frac{n}{\alpha}$, *then the operator* M_α *is bounded from* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *to* $L^\infty(\mathbb{R}^n)$.

4 Fractional maximal commutator operator in total Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\alpha}$ in the $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

Definition 4.3 We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm

$$\|f\|_* = \sup_{x \in \mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy < \infty,$$

where $f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y) dy$.

The following property of the space $BMO(\mathbb{R}^n)$ is valid:

If $b \in BMO(\mathbb{R}^n)$ and $\lambda > 0$, then the function b_λ is defined by $b_\lambda(x) = b(\lambda x)$, is also in $BMO(\mathbb{R}^n)$ and

$$\|b_\lambda\|_* = \|b\|_*. \tag{4.4}$$

See, for example, [7, Proposition 7.1.2 (6)].

The following local estimate is valid (see also [8, 10]).

Lemma 4.5 [10, Lemma 3.2] *Let* $0 \le \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then the inequality*

$$\|M_{b,\alpha} f\|_{L^q(B(x,r))} \lesssim \|b\|_* r^{\frac{n}{q}} \sup_{t>2r} \log\left(e + \frac{t}{r}\right) t^{-\frac{n}{q}} \|f\|_{L^p(B(x,t))} \tag{4.5}$$

holds for all $B(x,r)$ *and for all* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

The following is Spanne's type result for fractional maximal commutator operators in total Morrey spaces.

Theorem 4.3 (Spanne's type result) *Let* $1 < p < \infty$, $0 \le \lambda, \mu < n$, $0 \le \alpha < \frac{n}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$.

If $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $M_{b,\alpha} f \in L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ *and*

$$\|M_{b,\alpha} f\|_{L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,\lambda,\mu} \, \|b\|_* \, \|f\|_{L^{p,\lambda,\mu}}, \tag{4.6}$$

where $C_{p,\lambda,\mu}$ *depends only on* p,λ,μ *and* n.

Proof Let $1 < p < \infty$. From the inequality (4.5) we get

$$\begin{aligned}
\|M_{b,\alpha} f\|_{L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|M_{b,\alpha} f\|_{L^q(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, r^{\frac{n}{q}} \sup_{t>2r} \log\Big(e + \frac{t}{r}\Big) \, t^{-\frac{n}{q}} \, \|f\|_{L^p(B(x,t))} \\
&\lesssim \|f\|_{L^{p,\lambda,\mu}} \sup_{r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, r^{-\alpha+\frac{n}{p}} \sup_{t>r} \log\Big(e + \frac{t}{r}\Big) \, t^{\alpha-\frac{n}{p}} \, [t]_1^{\frac{\lambda}{p}} \, [1/t]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L^{p,\lambda,\mu}} \sup_{r>0} [r]_1^{-\alpha+\frac{n-\lambda}{p}} \, [1/r]_1^{\alpha-\frac{n-\mu}{p}} \sup_{t>r} \log\Big(e + \frac{t}{r}\Big) \, [t]_1^{\alpha-\frac{n-\lambda}{p}} \, [1/t]_1^{-\alpha+\frac{n-\mu}{p}} \\
&= \|f\|_{L^{p,\lambda,\mu}},
\end{aligned}$$

which implies that the operator M_α is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$. □

From Theorem 4.3 in the case $\alpha = 0$ we get the following corollaries.

Corollary 4.6 [13, Theorem 1] *Let* $1 \le p < \infty$, $0 \le \lambda < n$, $0 \le \mu < n$ *and* $b \in BMO(\mathbb{R}^n)$. *If* $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $M_b f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|M_b f\|_{L^{p,\lambda,\mu}} \le C_{p,\lambda,\mu} \, \|b\|_* \, \|f\|_{L^{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

Lemma 4.6 [11, Lemma 5.3] *Let* $0 < \alpha < n$ *and* $b \in BMO(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that*

$$M_{b,\alpha} f(x) \le C\|b\|_* \Big(M(M_\alpha f)(x) + M_\alpha(Mf)(x) \Big) \tag{4.7}$$

for almost every $x \in \mathbb{R}^n$ *and for all* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The following is a result of Adams type for the fractional maximal commutators (see, for example, [8]).

Theorem 4.4 (Adams type result) *Let* $1 < p < \infty$, $0 \le \mu \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{p}$ *and* $b \in BMO(\mathbb{R}^n)$.

Then condition $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $M_{b,\alpha}$ *from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

Proof *Sufficiency.* Let $1 < p < \frac{n-\lambda}{\alpha}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ and $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$.

$$\begin{aligned}
M_{b,\alpha} f(x) &\approx \sup_{r>0} r^{\alpha-n} \, \|(b(x)-b)f\|_{L^1(B(x,r))} \\
&\le \sup_{r>0} \min\{r^{\alpha} \, M_b f(x), r^{\alpha-\frac{n}{p}} \, \|f\|_{L^p(B(x,r))}\} \\
&\le \sup_{r>0} \min\{r^{\alpha} \, M_b f(x), r^{\alpha-\frac{n}{p}} \, [r]_1^{\frac{\lambda}{p}} \, [1/r]_1^{-\frac{\mu}{p}} \, \|f\|_{L^{p,\lambda,\mu}}\} \\
&\le \sup_{r>0} \min\{r^{\alpha} \, M_b f(x), [r]_1^{\alpha-\frac{n-\lambda}{p}} \, [1/r]_1^{-\alpha+\frac{n-\mu}{p}} \, \|f\|_{L^{p,\lambda,\mu}}\} \\
&\le \max\Big\{ \sup_{0<r\le 1} \min\{r^{\alpha} \, M_b f(x), r^{\alpha-\frac{n-\lambda}{p}} \, \|f\|_{L^{p,\lambda,\mu}}\}, \\
&\qquad \sup_{r>1} \min\{r^{\alpha} \, M_b f(x), r^{\alpha-\frac{n-\mu}{p}} \, \|f\|_{L^{p,\lambda,\mu}}\}\Big\}.
\end{aligned}$$

Minimizing with respect to r, at

$$r = \left(\frac{\|f\|_{L^{p,\lambda,\mu}}}{M_b f(x)}\right)^{\frac{p}{n-\lambda}} \quad \text{and} \quad r = \left(\frac{\|f\|_{L^{p,\lambda,\mu}}}{M_b f(x)}\right)^{\frac{p}{n-\mu}}$$

we have

$$\begin{aligned}
M_{b,\alpha} f(x) \le \max\Big\{ &\big(M_b f(x)\big)^{1-\frac{\alpha p}{n-\lambda}} \, \|f\|_{L^{p,\lambda,\mu}}^{\frac{\alpha p}{n-\lambda}}, \\
&\big(M_b f(x)\big)^{1-\frac{\alpha p}{n-\mu}} \, \|f\|_{L^{p,\lambda,\mu}}^{\frac{\alpha p}{n-\mu}}\Big\},
\end{aligned} \tag{4.8}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 4.6 and inequality (4.8), we get

$$\begin{aligned}
\|M_{b,\alpha} f\|_{L^{q,\lambda,\mu}} &\lesssim \|b\|_* \, \|f\|_{L^{p,\lambda,\mu}}^{1-\frac{p}{q}} \, \|(M_b f)^{\frac{p}{q}}\|_{L^{q,\lambda,\mu}} \\
&= \|b\|_* \, \|f\|_{L^{p,\lambda,\mu}}^{1-\frac{p}{q}} \, \|M_b f\|_{L^{p,\lambda,\mu}}^{\frac{p}{q}} \lesssim \|b\|_* \, \|f\|_{L^{p,\lambda,\mu}},
\end{aligned}$$

Necessity. Let $1 < p < \frac{n-\lambda}{\alpha}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$, $b \in BMO(\mathbb{R}^n)$, $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that $M_{b,\alpha}$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

Note that

$$M_{b,\alpha} f_t(x) = t^{-\alpha} \, M_{b_{\frac{1}{t}},\alpha} f(tx),$$

$$\begin{aligned}\left\|M_{b,\alpha} f_t\right\|_{L^{q,\lambda,\mu}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}\, \|M_{b_{\frac{1}{t}},\alpha} f(t\,\cdot)\|_{L^q(B(x,r))} \\ &= t^{-\alpha-\frac{n}{q}} \sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q} \\ &\quad \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}}\, [1/(tr)]_1^{\frac{\mu}{p}}\, \|M_{b_{\frac{1}{t}},\alpha} f\|_{L^q(B(tx,tr))} \\ &= t^{-\alpha-\frac{n}{q}}\, [t]_{1,+}^{\frac{\lambda}{q}}\, [1/t]_{1,+}^{-\frac{\mu}{q}} \left\|M_{b_{\frac{1}{t}},\alpha} f\right\|_{L^{q,\lambda,\mu}}.\end{aligned}$$

By the boundedness of M_α from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$ and from the equality (4.4) we get

$$\begin{aligned}\left\|M_{b,\alpha} f\right\|_{L^{q,\lambda,\mu}} &= t^{\alpha+\frac{n}{q}}\, [t]_{1,+}^{-\frac{\lambda}{q}}\, [1/t]_{1,+}^{\frac{\mu}{q}} \left\|M_{b_{\frac{1}{t}},\alpha} f_t\right\|_{L^{q,\lambda,\mu}} \\ &\lesssim t^{\alpha+\frac{n}{q}}\, [t]_{1,+}^{-\frac{\lambda}{q}}\, [1/t]_{1,+}^{\frac{\mu}{q}} \left\|f_t\right\|_{L^{p,\lambda,\mu}} \\ &= t^{\alpha+\frac{n}{q}-\frac{n}{p}}\, [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\, [1/t]_{1,+}^{-\frac{\mu}{p}+\frac{\mu}{q}}\, \|f\|_{L^{p,\lambda,\mu}} \\ &= t^{\alpha}\, [t]_{1,+}^{-\frac{n-\lambda}{p}+\frac{n-\lambda}{q}}\, [1/t]_{1,+}^{\frac{n-\mu}{p}-\frac{n-\mu}{q}}\, \|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n-\mu}$, then by letting $t \to 0$ we have $\left\|M_{b,\alpha} f\right\|_{L^{q,\lambda,\mu}} = 0$ for all $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\left\|M_{b,\alpha} f\right\|_{L^{q,\lambda,\mu}} = 0$ for all $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\lambda} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\mu}$. □

Theorem 4.5 *Let* $1 < p < \infty$, $0 \le \mu \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{p}$ *and* $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. *The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator $M_{b,\alpha}$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$.

Proof $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Corollary 3.1, Theorems 3.2 and 4.6, we get

$$\begin{aligned}\|M_{b,\alpha} f\|_{L^{q,\lambda,\mu}} &\lesssim \|b\|_* \|M(M_\alpha f) + M_\alpha(Mf)\|_{L^{q,\lambda,\mu}} \\ &\lesssim \|b\|_* \|M(M_\alpha f)\|_{L^{q,\lambda,\mu}} + \|M_\alpha(Mf)\|_{L^{q,\lambda,\mu}} \\ &\lesssim \|b\|_* \|M_\alpha f\|_{L^{q,\lambda,\mu}} + \|Mf\|_{L^{p,\lambda,\mu}} \\ &\lesssim \|b\|_* \|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $M_{b,\alpha}$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\begin{aligned}\|\chi_B\|_{L^{p,\lambda,\mu}} &\approx \sup_{y\in\mathbb{R}^n,t>0}\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(y,t)}\chi_B(z)dz\right)^{\frac{1}{p}}\\ &= \sup_{y\in\mathbb{R}^n,t>0}\left(|B(y,t)\cap B|\,[t]_1^{-\lambda}\,[1/t]_1^{\mu}\right)^{\frac{1}{p}}\\ &= \sup_{B(y,t)\subseteq B}\left(|B(y,t)|\,[t]_1^{-\lambda}\,[1/t]_1^{\mu}\right)^{\frac{1}{p}} = r^{\frac{n}{p}}\,[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}. \qquad (4.9)\end{aligned}$$

On the one hand, since

$$M_{b,\alpha}(\chi_B)(x) \gtrsim \frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |b(z)-b_B|dz \quad \text{for all} \quad x\in B,$$

we have

$$\begin{aligned}\|M_{b,\alpha}(\chi_B)\|_{L^{q,\lambda,\mu}} &\approx \sup_{B(y,t)\subseteq B}\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(y,t)}|M_{b,\alpha}(\chi_B)(z)|^q dz\right)^{\frac{1}{q}}\\ &\gtrsim r^{\frac{n}{q}}\,[r]_1^{-\frac{\lambda}{q}}\,[1/r]_1^{\frac{\mu}{q}}\,\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |b(z)-b_B|dz\\ &= r^{\alpha+\frac{n}{q}}\,[r]_1^{-\frac{\lambda}{q}}\,[1/r]_1^{\frac{\mu}{q}}\,\frac{1}{|B|}\int_B |b(z)-b_B|dz. \qquad (4.10)\end{aligned}$$

On the other hand, by assumption

$$\|M_{b,\alpha}(\chi_B)\|_{L^{q,\lambda,\mu}} \lesssim \|\chi_B\|_{L^{p,\lambda,\mu}},$$

by (4.9) and (4.10), we get that

$$\begin{aligned}\frac{1}{|B|}\int_B |b(z)-b_B|dz &\lesssim r^{-\alpha-\frac{n}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,\|M_{b,\alpha}(\chi_B)\|_{L^{q,\lambda,\mu}}\\ &\lesssim r^{-\alpha-\frac{n}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,\|\chi_B\|_{L^{p,\lambda,\mu}}\\ &\lesssim r^{-\alpha-\frac{n}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,r^{\frac{n}{p}}\,[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\\ &\lesssim [r]_1^{-\alpha+\frac{n-\lambda}{p}-\frac{n-\lambda}{q}}\,[1/r]_1^{\alpha-\frac{n-\mu}{p}+\frac{n-\mu}{q}}\\ &\lesssim 1.\end{aligned}$$

□

From Theorem 4.5 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 4.7 *Let* $1<p<\infty$, $0\le\alpha<\frac{n-\lambda}{p}$, $0\le\lambda<n-\alpha$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. *The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$.

(ii) The operator $M_{b,\alpha}$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Corollary 4.8 *Let* $1 < p < \infty$, $0 \le \alpha < \frac{n-\lambda}{p}$ *and* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. *The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator $M_{b,\alpha}$ *is bounded from* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

From Corollaries 4.7 and 4.8 in the case $\alpha = 0$ we get the following corollaries, respectively.

Corollary 4.9 [2, Theorem 1.4] *Let* $1 < p < \infty$ *and* $0 \le \lambda \le n$. *The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator M_b *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$.

Corollary 4.10 [3, Theorem 4] *Let* $1 < p < \infty$ *and* $0 \le \lambda \le n$. *The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator M_b *is bounded on* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

Remark 4.2 Note that Corollary 4.8 is new.

5 Commutator of fractional maximal operator in total Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$ in the $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} \big|[b, M_\alpha]f(x)\big| &= \big|b(x)M_\alpha f(x) - M_\alpha(bf)(x)\big| \\ &= \big|M_\alpha(b(x)f)(x) - M_\alpha(bf)(x)\big| \le M_\alpha(|b(x) - b|f)(x) = M_{b,\alpha} f(x). \end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \le M_{b,\alpha} f(x) + 2b^-(x)\, M_\alpha f(x), \qquad x \in \mathbb{R}^n \tag{5.11}$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (see, for example [12, 24]).

Applying Theorem 4.5, we obtain the following result.

Theorem 5.6 *Let* $1 < p < \infty$, $0 \le \mu \le \lambda < n$, $0 \le \alpha < \frac{n-\mu}{p}$ *and* $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. *Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) *The operator* $[b, M_\alpha]$ *is bounded from* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{q,\lambda,\mu}(\mathbb{R}^n)$.
(iii) *There exists a constant* $C > 0$ *such that*

$$\sup_B \frac{\|\big(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\big)\chi_B\|_{L^{q,\lambda,\mu}}}{\|\chi_B\|_{L^{q,\lambda,\mu}}} \le C. \tag{5.12}$$

Proof (i)$\Rightarrow$(ii). Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 3.2 and 4.5, and inequality (5.11), we get

$$\begin{aligned}\|[b, M_\alpha]f\|_{L^{q,\lambda,\mu}} &\le \|M_{b,\alpha}f + 2b^- M_\alpha f\|_{L^{q,\lambda,\mu}}\\ &\le \|M_{b,\alpha}f\|_{L^{q,\lambda,\mu}} + \|b^-\|_{L^\infty}\|M_\alpha f\|_{L^{q,\lambda,\mu}}\\ &\lesssim \big(\|b\|_* + \|b^-\|_{L^\infty}\big)\|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

(ii) $\Rightarrow$ (iii). Assume that $[b, M_\alpha]$ is bounded from $L^{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^{q,\lambda,\mu}(\mathbb{R}^n)$. For a given ball B and $0 \le \alpha < n$, we define the following local fractional maximal function:

$$M_{\alpha,B}f(x) := \sup_{B \supseteq B' \ni x} |B'|^{-1+\frac{\alpha}{n}} \int_{B'} |f(y)|\,dy,$$

where the supremum is taken over all balls B' such that $x \in B' \subseteq B$. Moreover, we denote by $M_B = M_{0,B}$ when $\alpha = 0$.

Since

$$M_\alpha(b\chi_B)\chi_B = M_{\alpha,B}(b)\,\chi_B \quad \text{and} \quad M_\alpha(\chi_B)\chi_B = M_{\alpha,B}\chi_B = |B|^{\frac{\alpha}{n}}\chi_B,$$

we have

$$\begin{aligned}\big(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\big)\chi_B &= |B|^{-\frac{\alpha}{n}}\big(|B|^{\frac{\alpha}{n}} b - M_{\alpha,B}(b)\big)\chi_B\\ &= |B|^{-\frac{\alpha}{n}}\big(bM_\alpha(\chi_B) - M_\alpha(b\chi_B)\big)\chi_B\\ &= |B|^{-\frac{\alpha}{n}}[b, M_\alpha](\chi_B).\end{aligned}$$

By this and $[b, M_\alpha] : L^{p,\lambda,\mu}(\mathbb{R}^n) \to L^{q,\lambda,\mu}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\|\big(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\big)\chi_B\|_{L^{q,\lambda,\mu}} &\le |B|^{-\frac{\alpha}{n}}\|[b, M_\alpha]\chi_B\|_{L^{q,\lambda,\mu}(\mathbb{R}^n)}\\ &\lesssim |B|^{-\frac{\alpha}{n}}\|\chi_B\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}.\end{aligned}$$

Thus from (4.9) we get

$$\frac{\|\big(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\big)\chi_B\|_{L^{q,\lambda,\mu}}}{\|\chi_B\|_{L^{q,\lambda,\mu}}} \le |B|^{-\frac{\alpha}{n}} \frac{\|\chi_B\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q,\lambda,\mu}(\mathbb{R}^n)}}$$

$$\begin{aligned}
&\approx r^{-\alpha+\frac{n}{p}-\frac{n}{q}}\,[r]_1^{\frac{\lambda}{q}-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}-\frac{\mu}{q}}\\
&\approx [r]_1^{-\alpha+\frac{n-\lambda}{p}-\frac{n-\lambda}{q}}\,[1/r]_1^{\alpha-\frac{n-\mu}{p}+\frac{n-\mu}{q}}\\
&\lesssim 1.
\end{aligned}$$

(iii) ⇒ (i). Assume that (5.12) is valid.

Now, let us prove $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. For any ball B, let $E = \{y \in B : b(y) \le b_B\}$ and $F = \{y \in B : b(y) > b_B\}$. The following equality is true (see [4, page 3331]):

$$\int_E |b(y) - b_B|dy = \int_F |b(y) - b_B|dy.$$

Since $b(y) \le b_B \le |b_B| \le |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_B| \le \left|b(y) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(y)\right|, \ \ y \in E.$$

Then from Hölder's inequality and (5.12) we have

$$\begin{aligned}
\frac{1}{|B|}\int_B |b(y) - b_B|dy &= \frac{2}{|B|}\int_E |b(y) - b_B|dy\\
&\le \frac{2}{|B|}\int_E \left|b(y) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(y)\right|dy\\
&\le \frac{2}{|B|}\int_B \left|b(y) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(y)\right|dy\\
&\lesssim \frac{1}{|B|^{\frac{1}{q}}}\,\left\|b(\cdot) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(\cdot)\right\|_{L^q(B)}\\
&\le |B|^{-\frac{1}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,\|b\chi_B - M_{b,\alpha}(b)\|_{L^{q,\lambda,\mu}(\mathbb{R}^n)}\\
&\lesssim r^{-\frac{n}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,\|\chi_B\|_{L^{q,\lambda,\mu}}\\
&\lesssim r^{-\frac{n}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,r^{\frac{n}{q}}\,[r]_1^{-\frac{\lambda}{q}}\,[1/r]_1^{\frac{\mu}{q}} \approx 1.
\end{aligned}$$

□

From Theorem 5.6 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 5.11 *Let* $1 < p < \infty$, $0 \le \alpha < \frac{n-\lambda}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Suppose that* b *is a real valued locally integrable function in* $\mathbb{R}^n$. *Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) The operator $[b, M_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

(iii) There exists a constant $C > 0$ *such that*

$$\sup_B \frac{\|\big(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\big)\,\chi_B\|_{L^{q,\lambda}}}{\|\chi_B\|_{L^{q,\lambda}}} \le C.$$

Corollary 5.12 *Let* $1 < p < \infty$, $0 \le \alpha < \frac{n-\lambda}{p}$ *and* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. *Suppose that* b *is a real valued locally integrable function in* $\mathbb{R}^n$. *Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) The operator $[b, M_\alpha]$ *is bounded from* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.
(iii) There exists a constant $C > 0$ *such that*

$$\sup_B \frac{\|\big(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\big)\,\chi_B\|_{\widetilde{L}^{q,\lambda}}}{\|\chi_B\|_{\widetilde{L}^{q,\lambda}}} \le C.$$

From Corollaries 5.11 and 5.12 in the case $\alpha = 0$ we get the following corollaries, respectively.

Corollary 5.13 [2, Theorem 1.2] *Let* $1 < p < \infty$, $0 \le \lambda \le n$. *Suppose that* b *is a real valued locally integrable function in* $\mathbb{R}^n$. *Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) The operator $[b, M]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$.

Corollary 5.14 [3, Theorem 6] *Let* $1 < p < \infty$, $0 \le \lambda \le n$. *Suppose that* b *is a real valued locally integrable function in* $\mathbb{R}^n$. *Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) The operator $[b, M]$ *is bounded on* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

Remark 5.3 Note that Corollaries 5.11 and 5.12 are new.

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Declarations

Conflict of interest The author have no conflict of interest to declare.

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