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TWO-WEIGHTED INEQUALITIES FOR THE RIESZ POTENTIAL IN p-CONVEX WEIGHTED MODULAR BANACH FUNCTION SPACES

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We prove the property of two-weight boundedness for the Riesz potential from one weighted Banach function space to another weighted Banach function space. In particular, we establish the two-weight boundedness for the Riesz potential and determine sufficient conditions on weights for the boundedness of the Riesz potential in weighted Musielak–Orlicz spaces.

1. Introduction

The investigations of the Riesz operator in weighted Banach function spaces (BFS) have recent history. The aim of these investigations is closely connected with founding a criterion for the geometry and weights in BFS guaranteeing the validity of boundedness of the Riesz operator in BFS. The mapping properties, such as boundedness and compactness, were characterized in [9, 10, 14, 33], etc. More precisely, the boundedness of certain integral operators in ideal Banach spaces was considered in [9, 10]. In [14], the boundedness of the Hardy operator was proved in Orlicz spaces. Moreover, the compactness and measure of noncompactness of Hardy-type operators in BFS was established in [33]. At the same time, in this paper, we used the property of boundedness of the Hardy operator in p-convex BFS. Note that the notion of BFS was introduced in [35]. Thus, in particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm, Musielak–Orlicz spaces, etc. are BFS.

In the present paper, we establish an integral-type sufficient condition on weights guaranteeing the boundedness of the Riesz operator from one weighted BFS to another weighted BFS.

2. Preliminaries

Let (Ω, μ) be a complete σ-finite measure space. By $L_0 = L_0(\Omega, \mu)$ we denote the collection of all real-valued μ-measurable functions on Ω.

Definition 2.1 [20]. *Let L be a real vector space. A function $\rho\colon L \mapsto [0, \infty]$ is called semimodular on L if the following properties hold:*

(a) $\rho(0) = 0$,

(b) $\rho(\lambda x) = \rho(x)$ *for all* $x \in L$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| = 1$,

(c) ρ *is convex,*

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(d) *ρ is left-continuous,*

(e) *$\rho(\lambda x) = 0$ for all $\lambda > 0$ implies that $x = 0$.*

A semimodular ρ is called modular if

(f) *$\rho(x) = 0$ implies that $x = 0$.*

A semimodular ρ is called continuous if

(g) *the mapping $\lambda \mapsto \rho(\lambda x)$ is continuous on $[0, \infty)$ for every $x \in L$.*

If ρ is semimodular or modular on L, then

$$L_\rho := \left\{ x \in L\colon \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}$$

is called a semimodular space or a modular space, respectively. The limit as $\lambda \to 0$ exists in $\mathbb{R}$.

Theorem 2.1 [20]. *Let ρ be semimodular on L. Then L_ρ is a normed real vector space. The norm in this space is called the Luxemburg norm and defined by the formula*

$$\|x\|_\rho := \inf \left\{ \lambda > 0\colon \rho\left(\frac{1}{\lambda} x\right) \le 1 \right\}.$$

Definition 2.2 [8, 32, 35]. *We say that a real normed space X is a Banach function space (BFS) provided that:*

(P_1) *the norm $\|f\|_X$ is defined for any μ-measurable function f and, moreover, $f \in X$ if and only if $\|f\|_X < \infty$ and $\|f\|_X = 0$ if and only if $f = 0$ a.e.;*

(P_2) *$\|f\|_X = \||f|\|_X$ for all $f \in X$;*

(P_3) *if $0 \le f_n \uparrow f \le g$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$ (Fatou property);*

(P_4) *if E is a measurable subset of Ω such that $\mu(E) < \infty$, then $\|\chi_E\|_X < \infty$, where χ_E is the characteristic function of the set E;*

(P_5) *for any measurable set $E \subset \Omega$ with $\mu(E) < \infty$, there is a constant $C_E > 0$ such that*

$$\int_E f(x)\, dx \le C_E \|f\|_X.$$

Recall that the condition (P_3) immediately yields the following property:

$$\text{if} \quad 0 \le f \le g, \quad \text{then} \quad \|f\|_X \le \|g\|_X.$$

Given a BFS X, we can always consider its associate space X' formed by $g \in L_0$ such that $f \cdot g \in L_1$ for any $f \in X$ with usual order and the norm

$$\|g\|_{X'} = \sup \left\{ \|f \cdot g\|_{L_1}\colon \|g\|_{X'} \le 1 \right\}.$$

Note that X' is a BFS in (Ω, μ) and a closed normed subspace.

Let X be a BFS and let ω be a weight, i.e., a positive Lebesgue measurable and a.e. finite function on Ω. Also let $X_\omega = \{f \in L_0 \colon \, f\omega \in X\}$. This space is a weighted BFS equipped with the norm

$$\|f\|_{X_\omega} = \|f\,\omega\|_X.$$

(For more details and proofs of the results obtained for the BFS, we refer the reader to [8, 32].)

We now recall the notion of p-convexity and p-concavity of BFS.

Definition 2.3 [42]. *Let X be a BFS. Then X is called p-convex for $1 \le p \le \infty$ if there exists a constant $M > 0$ such that, for all $f_1, \dots, f_n \in X$,*

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_X \le M \left(\sum_{k=1}^{n} \|f_k\|_X^p \right)^{1/p} \qquad \text{if} \quad 1 \le p < \infty,$$

or

$$\Big\| \sup_{1 \le k \le n} |f_k| \Big\|_X \le M \max_{1 \le k \le n} \|f_k\|_X \qquad \text{if} \quad p = \infty.$$

Similarly, X is called p-concave for $1 \le p \le \infty$ if there exists a constant $M > 0$ such that, for all $f_1, \dots, f_n \in X$,

$$\left(\sum_{k=1}^{n} \|f_k\|_X^p \right)^{1/p} \le M \left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_X \qquad \text{if} \quad 1 \le p < \infty$$

or

$$\max_{1 \le k \le n} \|f_k\|_X \le M \Big\| \sup_{1 \le k \le n} |f_k| \Big\|_X \qquad \text{if} \quad p = \infty.$$

Remark 2.1. Note that the notion of p-convexity (resp., p-concavity) is closely related to the notion of upper p-estimate (strong ℓ_p-composition property) [or, respectively, to the notion of lower p-estimate (strong ℓ_p-decomposition property)] as can be found in [32].

We now present some examples of p-convex and, respectively, p-concave BFS.

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space of the points $x = \big(x_1, \dots, x_n\big)$, let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, and let

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

The Lebesgue measure of a set Ω is denoted by $|\Omega|$. It is well known that

$$|B(0,1)| = \frac{\pi^{n/2}}{\Gamma\left(\dfrac{n}{2} + 1\right)},$$

where $B(0,1) = \{x \colon \, x \in \mathbb{R}^n \; |x| < 1\}$. Suppose that $\delta \colon \, \Omega \to [1, \infty)$. Throughout the present paper, we assume that

$$\underline{\delta} = \operatorname{ess\,inf}_{x\in\Omega} \delta(x) \qquad \text{and} \qquad \overline{\delta} = \operatorname*{ess\,sup}_{x\in\Omega} \delta(x)$$

and that

$$p' = \frac{p}{p-1}$$

is the conjugate exponent of $p > 1$.

Example 2.1. Let $1 \le q \le \infty$ and let $X = L_q$. Then the space L_q is a p-convex (p-concave) modular BFS if and only if $1 \le p \le q \le \infty$ $(1 \le q \le p \le \infty.)$

The proof follows from the Minkowski inequality in Lebesgue spaces.

Example 2.2. The following lemma shows that the variable Lebesgue space $L_{q(\cdot)}(\Omega)$ is a p-convex modular BFS:

Lemma 2.1 [1]**.** *Let $1 \le p \le q(x) \le \overline{q} < \infty$ for all $x \in \Omega_2 \subset \mathbb{R}^m$. Then the inequality*

$$\big\| \|f\|_{L_p(\Omega_1)} \big\|_{L_{q(\cdot)}(\Omega_2)} \le C_{p,q}^{2/p} \Big\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \Big\|_{L_p(\Omega_1)}$$

is true, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + p\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right)\Big(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty\Big),$$

$$\Delta_1 = \big\{(x,y) \in \Omega_1 \times \Omega_2 \colon\, q(y) = p\big\}, \qquad \Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1$$

and $f \colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ is an arbitrary measurable function such that

$$\big\| \|f\|_{L_p(\Omega_1)} \big\|_{L_{q(\cdot)}(\Omega_2)} = \inf\left\{ \mu > 0 \colon \int\limits_{\Omega_2} \left(\frac{\|f(\cdot,y)\|_{L_p(\Omega_1)}}{\mu} \right)^{q(y)} dy \le 1 \right\} < \infty$$

and

$$\|f(\cdot,y)\|_{L_p(\Omega_1)} = \left(\int\limits_{\Omega_1} |f(x,y)|^p \, dx \right)^{1/p}.$$

Similarly, if $1 \le q(x) \le p < \infty$, then $L_{q(x)}(\Omega)$ is a p-concave BFS.

Definition 2.4 [20, 40]**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. A real function $\varphi \colon \Omega \times [0,\infty) \mapsto [0,\infty)$ is called a generalized φ-function if it satisfies the conditions:*

(a) $\varphi(x,\,\cdot)$ *is a φ-function for all $x \in \Omega$, i.e., $\varphi(x,\,\cdot) \colon [0,\infty) \mapsto [0,\infty)$ is convex and such that*

$$\varphi(x,\,0) = 0, \qquad \lim_{t\to+0} \varphi(x,\,t) = 0;$$

(b) $\psi \colon x \mapsto \varphi(x,\,t)$ *is measurable for all $t \ge 0$.*

If φ is a generalized φ-function on Ω, then we briefly write $\varphi \in \Phi$.

Definition 2.5 [20, 40]. *Let $\varphi \in \Phi$ and let ρ_φ be defined by the expression*

$$\rho_\varphi(f) := \int\limits_\Omega \varphi\big(x, |f(x)|\big)\, dx \qquad \textit{for all} \quad f \in L_0(\Omega).$$

We set

$$L_\varphi = \big\{f \in L_0(\Omega) \colon \rho_\varphi(\lambda_0 f) < \infty \ \textit{for some} \ \lambda_0 > 0\big\}$$

and

$$\|f\|_{L_\varphi} = \inf\left\{\lambda > 0 \colon \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The space L_φ is called the Musielak–Orlicz space.

Let ω be a weight function on Ω, i.e., let ω be a nonnegative almost everywhere positive function on Ω. We denote

$$L_{\varphi,\,\omega} = \big\{f \in L_0(\Omega) \colon f\omega \in L_\varphi\big\}.$$

It is clear that the norm in this space is given by the formula

$$\|f\|_{L_{\varphi,\,\omega}} = \|f\omega\|_{L_\varphi}.$$

Remark 2.2. Let $\varphi(x,t) = t^{q(x)}$ in Definition 2.4, where $1 \le q(x) < \infty$ and $x \in \Omega$. Thus, we get the definition of weighted Lebesgue spaces with variable exponent $L_{q(x)}(\Omega)$. For detailed information about Lebesgue spaces with variable exponent, we refer the reader to [18].

Example 2.3. The following lemma shows that the Musielak–Orlicz space L_φ is a p-convex modular BFS.

Lemma 2.2 [6]. *Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset R^m$. Also let $(x,t) \in \Omega_1 \times [0,\infty)$ and $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \le p < \infty$. Suppose that $f \colon \Omega_1 \times \Omega_2 \mapsto R$. Then the inequality*

$$\left\|\|f(x,\cdot)\|_{L_p(\Omega_2)}\right\|_{L_\varphi} \le 2^{1/p} \left\|\|f(\cdot,y)\|_{L_\varphi}\right\|_{L_p(\Omega_2)}$$

is true.

Note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces, etc. are p-convex (p-concave) modular BFS. We now reduce a more general result connected with Minkowski's integral inequality.

Let X and Y be BFS on (Ω_1, μ) and (Ω_2, ν), respectively. By $X[Y]$ and $Y[X]$ we denote the spaces with mixed norm and formed by all functions $g \in L_0\left(\Omega_1 \times \Omega_2,\, \mu \times \nu\right)$ such that $\left\|g(x,\cdot)\right\|_Y \in X$ and $\left\|g(\cdot,y)\right\|_X \in Y$. The norms in these spaces are defined as follows:

$$\|g\|_{X[Y]} = \left\|\|g(x,\,\cdot)\|_Y\right\|_X, \qquad \|g\|_{Y[X]} = \left\|\|g(\cdot,\,y)\|_X\right\|_Y.$$

It is known that $X[Y]$ and $Y[X]$ are BFS on $\Omega_1 \times \Omega_2$ (see [32]).

Definition 2.6 [40]. *We say that a modular BFS X satisfies the Δ_2-condition if there exists $K \geq 2$ such that*

$$\rho(2f) \leq K\,\rho(f)$$

for all $f \in X$ and all $t > 0$. The smallest K of this kind is called the Δ_2-constant of ρ.

Lemma 2.3. *Let X be a modular BFS and let $\gamma \geq 1$ and $1 \leq q(x) \leq \overline{q} < \infty$. Further, let*

$$\min_{s>0}\{s, s^{\gamma}\}\,\rho(f) \leq \rho(sf) \leq \max_{s>0}\{s, s^{q(x)}\}\rho(f) \tag{2.1}$$

for almost all $x \in \Omega$ and all $f \in X_\rho$. Then

$$\rho\left(\frac{f}{\|f\|_\rho}\right) = 1$$

and

$$\min_{\|f\|_\rho}\left\{\|f\|_\rho, \|f\|_\rho^{\gamma}\right\} \leq \rho(f) \leq \max_{\|f\|_\rho}\left\{\|f\|_\rho, \|f\|_\rho^{q(x)}\right\}$$

for any $x \in \Omega$.

Proof. Let

$$0 < \|f\|_\rho < \infty \qquad \text{and} \qquad \rho\left(\frac{f}{\|f\|_\rho}\right) < 1.$$

We choose a positive number $\lambda \leq \|f\|_\rho$ such that

$$\rho\left(\frac{f}{\lambda}\right) < 1.$$

Indeed, we set

$$\lambda = \|f\|_\rho \rho^{1/\overline{q}}\left(\frac{f}{\|f\|_\rho}\right).$$

Then $\lambda < \|f\|_\rho$ and, by virtue of condition (2.1) with $s > 1$, we get

$$\rho\left(\frac{f}{\lambda}\right) = \rho\left(\frac{f}{\|f\|_\rho \rho^{1/\overline{q}}\left(\frac{f}{\|f\|_\rho}\right)}\right) \leq \rho^{-q(x)/\overline{q}}\left(\frac{f}{\|f\|_\rho}\right)\rho\left(\frac{f}{\|f\|_\rho}\right)$$

$$\leq \rho^{-1}\left(\frac{f}{\|f\|_\rho}\right)\rho\left(\frac{f}{\|f\|_\rho}\right) = 1.$$

Lemma 2.3 is proved.

We consider a multidimensional Hardy-type operator and its dual operator

$$Hf(x) = \int\limits_{|y|<|x|} f(y)\,dy \qquad \text{and} \qquad H^*f(x) = \int\limits_{|y|>|x|} f(y)\,dy,$$

where $f \geq 0$ and $x \in \mathbb{R}^n$.

We now reduce a two-weight criterion for a multidimensional Hardy-type operator acting from a p-concave weighted BFS to a weighted Lebesgue space. Suppose that $M > 0$ is a constant from Definition 2.3.

Theorem 2.2 [7]. *Let $v(x)$ and $w(x)$ be weights on $\mathbb{R}^n$. Suppose that X_w is a p-convex weighted BFS with $1 \leq p < \infty$ on $\mathbb{R}^n$. Then the inequality*

$$\|Hf\|_{X_w} \leq C\|f\|_{L_{p,\,v}} \tag{2.2}$$

holds for every $f \geq 0$ if and only if there is an $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} v(y)^{-p'}\,dy \right)^{\alpha/p'} \left\| \chi_{\{|z|>t\}}(\cdot) \left(\int\limits_{|y|<|\cdot|} v(y)^{-p'}\,dy \right)^{(1-\alpha)/p'} \right\|_{X_w} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (2.2), then

$$\sup_{0<\alpha<1} \frac{p'\,A(\alpha)}{(1-\alpha)\left(\left(\dfrac{p'}{1-\alpha}\right)^p + \dfrac{1}{\alpha\,(p-1)}\right)^{1/p}} \leq C \leq M \inf_{0<\alpha<1} \frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

For the dual operator, the theorem formulated below is proved similarly:

Theorem 2.3 [7]. *Let $v(x)$ and $w(x)$ be weights on $\mathbb{R}^n$. Suppose that X_w is a p-convex weighted BFS with $1 \leq p < \infty$ on $\mathbb{R}^n$. Then the inequality*

$$\|H^*f\|_{X_w} \leq C\,\|f\|_{L_{p,\,v}} \tag{2.3}$$

holds for any $f \geq 0$ if and only if there is a $\gamma \in (0,1)$ such that

$$B(\gamma) = \sup_{t>0} \left(\int\limits_{|y|>t} v(y)^{-p'}\,dy \right)^{\gamma/p'} \left\| \chi_{\{|z|<t\}}(\cdot) \left(\int\limits_{|y|>|\cdot|} v(y)^{-p'}\,dy \right)^{(1-\gamma)/p'} \right\|_{X_w} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (2.3), then

$$\sup_{0<\gamma<1} \frac{p'\,B(\gamma)}{(1-\gamma)\left(\left(\dfrac{p'}{1-\gamma}\right)^p + \dfrac{1}{\gamma\,(p-1)}\right)^{1/p}} \leq C \leq M \inf_{0<\gamma<1} \frac{B(\gamma)}{(1-\gamma)^{1/p'}}.$$

Corollary 2.1. *Note that, in the case*

$$X_w = L_{\varphi,\,w}, \quad \varphi(x, t^{1/p}) \in \Phi \quad \textit{for some} \quad 1 \le p < \infty, \quad x \in \mathbb{R}^n,$$

Theorems 2.2 and 2.3 were proved in [6]. In the case

$$X_w = L_{q,\,w}, \;\; 1 < p \le q < \infty, \quad \textit{for} \quad x \in (0,\infty), \quad \alpha = \frac{s-1}{p-1}, \quad \textit{and} \quad s \in (1,\,p),$$

Theorems 2.2 and 2.3 were proved in [44]. For $x \in \mathbb{R}^n$, in the case

$$X_w = L_{q(x),\,w} \quad \textit{and} \quad 1 < p \le q(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x) < \infty,$$

Theorems 2.2 and 2.3 were proved in [3] (see also [2]).

Remark 2.3. In the case

$$n = 1, \quad X_w = L_{q,\,w}, \;\; 1 < p \le q \le \infty, \quad \text{for} \quad x \in (0,\infty)$$

and the classical Lebesgue spaces, various versions of Theorems 2.2 and 2.3 were proved in [12, 23, 25–28, 30, 31, 34, 38, 39, 43], etc. In particular, in the Lebesgue spaces with variable exponent, the property of boundedness of the Hardy-type operator was proved in [15–17, 19, 21, 24, 29, 36, 37], etc. For

$$X_w = L_{q(x),\,w}, \quad 1 < p \le q(x) \le \operatorname*{ess\,sup}_{x\in[0,1]} q(x) < \infty, \quad \text{and} \quad x \in [0,1],$$

the two-weighted criterion was proved for a one-dimensional Hardy operator in [29]. Moreover, a two-weighted criterion of a different type was proved in [36] for a multidimensional Hardy-type operator in the case where

$$X_w = L_{q(x),\,w}, \quad 1 < p \le q(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x) < \infty, \quad \text{and} \quad x \in \mathbb{R}^n$$

(see also [37] and [17]). For

$$L_{q(x),\,w} \quad \text{with} \quad 0 < \underline{q} \le \overline{q} < 1,$$

the property of boundedness of the classical Hardy operator was proved in [5]. The inequalities of modular type were proved for more general operators in [11] and [41]. In addition, the Hardy-type inequalities with special power-type weights in Orlicz spaces were proved in [13].

3. Main Result

We now consider the Riesz potential

$$\mathcal{R}^s f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}\, dy,$$

where $0 < s < n$.

The sufficient conditions for the general weights guaranteeing the validity of two-weight strong-type inequalities for the Riesz potential in the BFS are established by the following theorem:

Theorem 3.1. *Suppose that $v(x)$ and $w(x)$ are weight functions on $\mathbb{R}^n$. Let Y_w be a modular p-convex weighted BFS for $1 \le p < \infty$ and $x \in \mathbb{R}^n$. Let $0 < s < n$, let $\mathcal{R}^s$ be bounded from X into Y, and let $L_{p,v}(\mathbb{R}^n) \hookrightarrow X_v$. Assume that there exists*

$$r(x)\colon\ 1 < p \le r(x) \le \overline{r} < \infty$$

such that

$$\rho(Cf) \le C_1(r)\,\rho(f)$$

for all $C > 0$, where $C_1(r) = \max\left\{C^{\underline{r}}, C^{\overline{r}}\right\}$.

Moreover, suppose that $v(x)$ and $w(x)$ satisfy the following three conditions:

(1)

$$A = \sup_{t>0}\left(\int\limits_{|y|<t} v(y)^{-p'}\,dy\right)^{\alpha/p'}\left\|\frac{\chi_{\{|x|>t\}}}{|x|^{n-s}}\left(\int\limits_{|y|<|x|} v(y)^{-p'}\,dy\right)^{(1-\alpha)/p'}\right\|_{Y_w} < \infty; \tag{3.1}$$

(2)

$$B = \sup_{t>0}\left(\int\limits_{|y|>t}\left(v(y)|y|^{n-s}\right)^{-p'}dy\right)^{\beta/p'}\left\|\chi_{\{|x|<t\}}\left(\int\limits_{|y|>|x|}\left(v(y)|y|^{n-s}\right)^{-p'}dy\right)^{(1-\beta)/p'}\right\|_{Y_w} < \infty, \tag{3.2}$$

where $0 < \alpha, \beta < 1$;

(3) there exists $M > 0$ such that

$$\sup_{|x|/2<|y|\le 4\,|x|} w(y) \le M \inf_{|x|/2<|y|\le 4\,|x|} v(y). \tag{3.3}$$

Then there exists a positive constant C independent of f and such that for all $f \in X_v$

$$\|\mathcal{R}^s f\|_{Y_w} \le C\|f\|_{X_v}.$$

Proof. Let $Z = \{0, \pm1, \pm2, \ldots\}$. For $k \in Z$, we define

$$E_k = \left\{x \in \mathbb{R}^n\colon\ 2^k < |x| \le 2^{k+1}\right\},$$

$$E_{k,1} = \left\{x \in \mathbb{R}^n\colon\ |x| \le 2^{k-1}\right\},$$

$$E_{k,2} = \{x \in \mathbb{R}^n : \, 2^{k-1} < |x| \le 2^{k+2}\},$$

$$E_{k,3} = \{x \in \mathbb{R}^n : \, |x| > 2^{k-1}\}.$$

Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to three.

Given $f \in L_{p,v}(\mathbb{R}^n)$, we can write

$$\begin{aligned}\big|\mathcal{R}^s f(x)\big| &= \sum_{k\in Z} |\mathcal{R}^s f(x)| \, \chi_{E_k}(x) \\ &\le \sum_{k\in Z} |\mathcal{R}^s f_{k,1}(x)| \, \chi_{E_k}(x) \\ &\qquad + \sum_{k\in Z} |\mathcal{R}^s f_{k,2}(x)| \, \chi_{E_k}(x) + \sum_{k\in Z} |\mathcal{R}^s f_{k,3}(x)| \, \chi_{E_k}(x) \\ &= \mathcal{R}^s_1 f(x) + \mathcal{R}^s_2 f(x) + \mathcal{R}^s_3 f(x),\end{aligned}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1, 2, 3$.

First, we estimate $\|\mathcal{R}^s_1 f\|_{Y_w}$. Note that, for $x \in E_k$, $y \in E_{k,1}$ we get

$$|y| < 2^{k-1} \le |x|/2.$$

Moreover,

$$E_k \cap \operatorname{supp} f_{k,1} = \varnothing \qquad \text{and} \qquad |x-y| \ge |x| - |y| \ge |x| - |x|/2 = |x|/2.$$

Hence, we conclude that

$$\begin{aligned}\big|\mathcal{R}^s_1 f(x)\big| &\le C \sum_{k\in Z} \left(\int_{\mathbb{R}^n} \frac{|f_{k,1}(y)|}{|x-y|^{n-s}} \, dy \right) \chi_{E_k} \\ &\le C \int_{|y|<|x|/2} \frac{|f(y)|}{|x-y|^{n-s}} \, dy \\ &\le C \int_{|y|<|x|} \frac{|f(y)|}{|x-y|^{n-s}} \, dy \\ &\le 2^n \, C \, \frac{1}{|x|^{n-s}} \int_{|y|<|x|} |f(y)| \, dy\end{aligned}$$

for any $x \in E_k$. Thus, we get

$$\|\mathcal{R}^s_1 f\|_{Y_w} \le 2^n \, C \left\| \frac{1}{|x|^{n-s}} \int_{|y|<|x|} |f(y)| \, dy \right\|_{Y_w} = \left\| \int_{|y|<|x|} |f(y)| \, dy \right\|_{Y_{w/|x|^{n-s}}}.$$

By virtue of condition (3.1) and Theorem 2.2, we obtain

$$\|\mathcal{R}_1^s f\|_{Y_w} \le C_1 \|f\|_{L_{p,v}(\mathbb{R}^n)} \le C_2 \|f\|_{X_v}, \tag{3.4}$$

where $C_1 > 0$ is independent of f and $x \in \mathbb{R}^n$.

Further, we estimate $\|\mathcal{R}_3^s f\|_{Y_w}$. It is clear that, for $x \in E_k,\ y \in E_{k,3}$, we find

$$|y| > 2\,|x| \qquad \text{and} \qquad |x-y| \ge |y| - |x| \ge |y| - |y|/2 = |y|/2.$$

Since $E_k \cap \operatorname{supp} f_{k,3} = \varnothing$ for $x \in E_k$, we can write

$$|\mathcal{R}_3^s f(x)| \le C \int\limits_{|y|>2|x|} \frac{|f(y)|}{|x-y|^{n-s}}\, dy \le 2^n\, C \int\limits_{|y|>2|x|} \frac{|f(y)|}{|y|^{n-s}}\, dy.$$

Therefore, we get

$$\begin{aligned}\|\mathcal{R}_3^s f\|_{Y_w} &\le 2^n\, C \left\| \int\limits_{|y|>2|x|} \frac{|f(y)|}{|y|^{n-s}}\, dy \right\|_{Y_w} \\ &\le 2^n\, C \left\| \int\limits_{|y|>|x|} \frac{|f(y)|}{|y|^{n-s}}\, dy \right\|_{Y_w}.\end{aligned}$$

By condition (3.2) and Theorem 2.3, we obtain

$$\left\|\mathcal{R}_3^s f\right\|_{Y_w} \le C_2\, \|f\|_{L_{p,v}(\mathbb{R}^n)} \le C_3 \|f\|_{X_v}, \tag{3.5}$$

where $C_2 > 0$ is independent of f and $x \in \mathbb{R}^n$.

Finally, we estimate $\|\mathcal{R}^s f_{k,2}\|_{Y_w}$, where

$$\|\mathcal{R}^s f_{k,2}\|_{Y_w} = \left\| \sum_{k \in Z} |\mathcal{R}^s f_{k,2}|\ \chi_{E_k} \right\|_{Y_w}.$$

By virtue of Lemma 2.3, it suffices to prove that the inequality $\|f\|_{X_v} \le 1$ implies that

$$\rho\left(w \sum_{k \in Z} |\mathcal{R}^s f_{k,2}|\, \chi_{E_k} \right) \le C,$$

where $C > 0$ is independent of $k \in Z$.

By virtue of the boundedness of $\mathcal{R}^s$ from X to Y and condition (3.3), we conclude that

$$\rho\left(w(y) \sum_{k \in Z} |\mathcal{R}^s f_{k,2}(y)|\ \chi_{E_k}(y) \right) = \sum_{m \in Z} \rho\left(w(y) \sum_{k \in Z} |\mathcal{R}^s f_{k,2}(y)|\ \chi_{E_k}(y) \right)$$

$$= \sum_{k\in Z} \rho\left(w(y)\ |\mathcal{R}^s f_{k,2}(y)|\right)$$

$$= \sum_{k\in Z} \rho\bigg(C\,w(y)\,\|f_{k,2}\|_X\ \frac{|\mathcal{R}^s f_{k,2}|}{C\,\|f_{k,2}\|_X}\bigg)$$

$$\le \sum_{k\in Z} \left(C\,w(y)\,\|f_{k,2}\|_X\right)^{r(y)} \rho\bigg(\frac{|\mathcal{R}^s f_{k,2}|}{C\,\|f_{k,2}\|_X}\bigg)$$

$$\le C_2 \sum_{k\in Z} \sup_{y\in E_k} \Big(w(y)\|f\|_{X(E_{k,2})}\Big)^{r(y)}\ \rho\bigg(\frac{|\mathcal{R}^s f_{k,2}|}{C\,\|f_{k,2}\|_X}\bigg)$$

$$\le C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(w(y)\|f\|_X\right)^{r(y)} = C_2 \sum_{k\in Z} \sup_{y\in E_k} \Big(\|f\,w\|_{X(E_{k,2})}\Big)^{r(y)}$$

$$\le C_3 \sum_{k\in Z} \sup_{y\in E_k} \bigg(\|f\ \inf_{y\in E_{k,2}}\ v(y)\|_{X(E_{k,2})}\bigg)^{r(y)}$$

$$\le C_3 \sum_{k\in Z} \sup_{y\in E_k} \Big(\|f\,v\|_{X(E_{k,2})}\Big)^{r(y)} = C_3 \sum_{k\in Z} \Big(\|f\|_{X_v(E_{k,2})}\Big)^{\inf\limits_{y\in E_k} r(y)}$$

$$\le C_3 \sum_{k\in Z} \Big(\|f\|_{X_v(E_{k,2})}\Big)^{\underline{r}} \le C_3 \sum_{k\in Z} \rho\left(|f(y)|v(y)\chi_{E_{k,2}}\right)^{\underline{r}/\gamma}$$

$$= C_3 \sum_{k\in Z} \left[\rho\left(|f(y)|v(y)\left(\chi_{E_{k-1}} + \chi_{E_k} + \chi_{E_{k+1}}\right)\right)\right]^{\underline{r}/\gamma}$$

$$\le C_3\ \left[\rho\left(|f(y)|v(y)\right)\right]^{\underline{r}/\gamma} \Bigg(\sum_{k\in Z} \chi_{E_{k-1}} + \sum_{k\in Z} \chi_{E_k} + \sum_{k\in Z} \chi_{E_{k+1}}\Bigg)^{\underline{r}/\gamma}$$

$$= C_3\ \left(3\,\rho\left(|f(y)|v(y)\right)\right)^{\underline{r}/\gamma}\ \le\ 3^{\underline{r}/\gamma} C_3\ \le\ C_4.$$

Thus,

$$\|\mathcal{R}^s_2 f\|_{Y_w} \le C_5, \tag{3.6}$$

where $C > 0$ is independent of f and $x \in \mathbb{R}^n$.

Combining inequalities (3.4), (3.5), and (3.6), we obtain the proof of Theorem 3.1.

Theorem 3.2 [40]. *Let $\psi \in \Phi$ and $\delta \ge 1$. Then $L_\psi(\mathbb{R}^n) \hookrightarrow L_\delta(\mathbb{R}^n)$ if and only if there exists $C > 0$ and $h \in L_1(\mathbb{R}^n)$ with $\|h\|_{L_1(\mathbb{R}^n)} \le 1$ such that*

$$\left(\frac{t}{C}\right)^{\delta} \le \psi(x,t) + h(x) \tag{3.7}$$

for almost all $x \in \mathbb{R}^n$ and all $t \ge 0$.

Lemma 3.1. *Let $\psi \in \Phi,\ \gamma \geq 1$ and $1 \leq q(x) \leq \overline{q} < \infty$. Further, let*

$$\min_{s>0}\{s,\ s^{\gamma}\}\,\psi(x,t) \leq \psi(x,st) \leq \max_{s>0}\{s,\ s^{q(x)}\}\psi(x,t) \tag{3.8}$$

for almost all $x \in \Omega$ and all $t \geq 0$. Then

$$\rho_\psi\left(\frac{f}{\|f\|_{L_\psi}}\right) = 1$$

and

$$\min_{\|f\|_{L_\psi}}\left\{\|f\|_{L_\psi},\ \|f\|^{\gamma}_{L_\psi}\right\} \leq \rho_\psi(f) \leq \max_{\|f\|_{L_\psi}}\left\{\|f\|_{L_\psi},\ \|f\|^{q(x)}_{L_\psi}\right\}.$$

Theorem 3.1 has the following corollary:

Corollary 3.1. *Assume that, for some $1 < p < \infty$, $\varphi(x,t^{1/p}) \in \Phi$ and a function $\psi \in \Phi$ satisfy conditions (3.7) and (3.8), where $x \in \mathbb{R}^n$. Suppose that $v(x)$ and $w(x)$ are weight functions on $\mathbb{R}^n$. Let $\mathcal{R}^s$ be bounded from $L_\psi(\mathbb{R}^n)$ into $L_\varphi(\mathbb{R}^n)$. Assume that there exists $r(x)\colon 1 < \theta \leq r(x) \leq \overline{r} < \infty$ such that*

$$\varphi(x,Ct) \leq C^{r(x)}\,\varphi(x,t)$$

for all $C > 0$.

Moreover, suppose that $v(x)$ and $w(x)$ satisfy the following three conditions:

(1)
$$\sup_{t>0}\left(\int_{|y|<t} v(y)^{-p'}\,dy\right)^{\alpha/p'}\left\|\frac{w(\cdot)}{|\cdot|^{n-s}}\left(\int_{|y|<|\cdot|} v(y)^{-p'}\,dy\right)^{(1-\alpha)/p'}\right\|_{L_\varphi(|\cdot|>t)} < \infty;$$

(2)
$$\sup_{t>0}\left(\int_{|y|>t}\left(v(y)|y|^{n-s}\right)^{-p'}dy\right)^{\beta/p'}\left\|w(\cdot)\left(\int_{|y|>|\cdot|}\left(v(y)|y|^{n-s}\right)^{-p'}dy\right)^{(1-\beta)/p'}\right\|_{L_\varphi(|\cdot|<t)} < \infty,$$

where $0 < \alpha, \beta < 1$;

(3) there exists $M > 0$ such that

$$\sup_{|x|/2<|y|\leq 4\,|x|} w(y) \leq M \inf_{|x|/2<|y|\leq 4\,|x|} v(y).$$

Then there exists a positive constant C independent of f such that, for all $f \in L_{\psi,v}(\mathbb{R}^n)$,

$$\|\mathcal{R}^s f\|_{L_{\varphi,w}(\mathbb{R}^n)} \leq C\|f\|_{L_{\psi,v}(\mathbb{R}^n)}.$$

Further, we assume that the exponent $p(x)$ satisfies the standard conditions

$$\left|p(x) - p(y)\right| \leq \frac{M_1}{-\ln|x-y|}, \qquad 0 < |x-y| \leq \frac{1}{2}, \quad x,y \in \mathbb{R}^n, \tag{3.9}$$

together with the following conditions imposed at infinity:

$$|p(x) - p(y)| \le \frac{M_2}{\ln(e + |x|)}, \qquad |x| \ge |y|, \quad x, y \in \mathbb{R}^n, \tag{3.10}$$

where the positive constants M_1 and M_2 are independent of x and y. Note that it follows from condition (3.10) that there is a number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$ and this limit is uniform in all directions. It is known that if $p(x)$ satisfies (3.10),

$$p_\infty = \underline{p}, \qquad \text{and} \qquad \frac{1}{r(x)} = \frac{1}{\underline{p}} - \frac{1}{p(x)},$$

then $\dfrac{1}{r(x)}$ satisfies (3.10),

$$\lim_{|x|\to\infty} r(x) = \infty, \qquad \text{and} \qquad L_{p(x)}(\mathbb{R}^n) \hookrightarrow L_{\underline{p}}(\mathbb{R}^n).$$

In particular, for $X_v = L_{p(x),v}(\mathbb{R}^n)$ and $Y_w = L_{q(x),w}(\mathbb{R}^n)$, Theorem 3.1 yields the following corollary:

Corollary 3.2. *Let*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{s}{n}, \qquad \underline{p} > 1, \quad \overline{p} < n/s, \quad \underline{q} \ge \overline{p},$$

and let $p(x)$ satisfy conditions (3.9) and (3.10) with $p_\infty = \underline{p}$. Moreover, let $v(x)$ and $w(x)$ be weight functions on $\mathbb{R}^n$ satisfying the following three conditions:

(1)

$$\sup_{t>0} \left(\int_{|y|<t} v(y)^{-\overline{p}'}\, dy \right)^{\alpha/\overline{p}'} \left\| \frac{w(\cdot)}{|\cdot|^{n-s}} \left(\int_{|y|<|\cdot|} v(y)^{-\overline{p}'}\, dy \right)^{(1-\alpha)/\overline{p}'} \right\|_{L_{q(\cdot)}(|\cdot|>t)} < \infty;$$

(2)

$$\sup_{t>0} \left(\int_{|y|>t} \left(v(y)|y|^{n-s}\right)^{-\overline{p}'}\, dy \right)^{\beta/\overline{p}'} \left\| w(\cdot) \left(\int_{|y|>|\cdot|} \left(v(y)|y|^{n-s}\right)^{-\overline{p}'} dy \right)^{(1-\beta)/\overline{p}'} \right\|_{L_{q(\cdot)}(|\cdot|<t)} < \infty,$$

where $0 < \alpha, \beta < 1$;

(3) there exists a constant $M > 0$ such that

$$\sup_{|x|/4<|y|\le 4\,|x|} w(y) \le M \inf_{|x|/4<|y|\le 4\,|x|} v(y) \qquad \text{for a.e.} \quad x \in \mathbb{R}^n.$$

Then there exists a positive constant C independent of f such that, for all $f \in L_{p(x),v}(\mathbb{R}^n)$,

$$\left\| \mathcal{R}^s f \right\|_{L_{q(\cdot),w}(\mathbb{R}^n)} \le C \|f\|_{L_{p(\cdot),v}(\mathbb{R}^n)}.$$

Remark 3.1. In the case where $X_v = L_{p,\,v},\ Y_w = L_{q,\,w},\ 1 < p \le q \le \infty$, various versions of Theorem 3.1 were proved for the classical Lebesgue spaces in [4, 22, 45], etc.

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REFERENCES

1. R. A. Bandaliev, "On an inequality in Lebesgue space with mixed norm and with variable summability exponent," *Math. Notes*, **84**, No. 3, 303–313 (2008) (correction in *Math. Notes*, **99**, No. 2, 319–320 (2016)).
2. R. A. Bandaliev, "The boundedness of certain sublinear operator in the weighted variable Lebesgue spaces," *Czechoslovak Math. J.*, **60**, No. 2, 327–337 (2010) (corrections in *Czechoslovak Math. J.*, **63**, No. 4, 1149–1152 (2013)).
3. R. A. Bandaliev, "The boundedness of multidimensional Hardy operator in the weighted variable Lebesgue spaces," *Lith. Math. J.*, **50**, No. 3, 249–259 (2010).
4. R. A. Bandaliev and K. K. Omarova, "Two-weight norm inequalities for certain singular integrals," *Taiwanese J. Math.*, **16**, No. 2, 713–732 (2012).
5. R. A. Bandaliev, "On Hardy-type inequalities in weighted variable Lebesgue space $L_{p(x),\omega}$ for $0 < p(x) < 1$," *Eurasian Math. J.*, **4**, 5–16 (2013).
6. R. A. Bandaliev, "Criteria of two-weighted inequalities for multidimensional Hardy type operator in weighted Musielak–Orlicz spaces and some application," *Math. Stat.*, **1**, No. 3, 144–156 (2013).
7. R. A. Bandaliev, "On the two-weight boundedness of multidimensional Hardy operator in p-convex Banach function spaces and some applications," *Ukr. Math. J.*, **67**, No. 3, 357–371 (2015).
8. C. Bennett and R. Sharpley, "Interpolation of operators," *Pure Appl. Math.*, **129** (1988).
9. E. I. Berezhnoi, "Sharp estimates of operators on the cones of ideal spaces," *Proc. Steklov Inst. Math.*, **204**, 3–36 (1994).
10. E. I. Berezhnoi, "Two-weighted estimations for the Hardy–Littlewood maximal function in ideal Banach spaces," *Proc. Amer. Math. Soc.*, **127**, 79–87 (1999).
11. S. Bloom and R. Kerman, "Weighted L_Φ integral inequalities for operators of Hardy type," *Studia Math.*, **110**, No. 1, 35–52 (1994).
12. J. Bradley, "Hardy inequalities with mixed norms," *Canad. Math. Bull.*, **21**, 405–408 (1978).
13. A. Cianchi, "Hardy inequalities in Orlicz spaces," *Trans. Amer. Math. Soc.*, **351**, 2459–2478 (1999).
14. J. A. Cochran and C. S. Lee, "Inequalities related to Hardy's and Heinig's," *Math. Proc. Cambridge Phil. Soc.*, **96**, 1–7 (1984).
15. D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, "The maximal function on variable L^p spaces," *Ann. Acad. Sci. Fenn. Math.*, **28**, No. 1, 223–238 (2003) (corrections in *Ann. Acad. Sci. Fenn. Math.*, **29**, No. 2, 247–249 (2004)).
16. D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Perez, "The boundedness of classical operators on variable L^p spaces," *Ann. Acad. Sci. Fenn. Math.*, **31**, 239–264 (2006).
17. D. Cruz-Uribe and F. I. Mamedov, "On a general weighted Hardy type inequality in the variable exponent Lebesgue spaces," *Rev. Mat. Complut.*, **25**, No. 2, 335–367 (2012).
18. D. Cruz-Uribe and A. Fiorenza, "Variable Lebesgue spaces," *Found. Harmon. Anal., Ser. Appl. Numer. Harmon. Anal.* (2013).
19. L. Diening and S. Samko, "Hardy inequality in variable exponent Lebesgue spaces," *Fract. Calc. Appl. Anal.*, **10**, No. 1, 1–18 (2007).
20. L. Diening, P. Harjulehto, P. Hasto, and M. Ruzicka, "Lebesgue and Sobolev spaces with variable exponents," *Springer Lect. Notes*, **2017** (2011).
21. D. E. Edmunds, V. Kokilashvili, and A. Meskhi, "On the boundedness and compactness of weighted Hardy operators in spaces $L^{p(x)}$," *Georgian Math. J.*, **12**, No. 1, 27–44 (2005).
22. V. S. Guliev, "Two-weight inequalities for integral operators in L_p-spaces and their applications," *Proc. Steklov Inst. Math.*, **204**, 97–116 (1994).
23. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press (1988).
24. P. Harjulehto, P. Hasto, and M. Koskenoja, "Hardy's inequality in a variable exponent Sobolev space," *Georgian Math. J.*, **12**, No. 1, 431–442 (2005).
25. H. P. Heinig, "Some extensions of inequalities," *J. SIAM Math. Anal.*, **6**, 698–713 (1975).
26. P. Jain, L. E. Persson, and A. Wedestig, "From Hardy to Carleman and general mean-type inequalities, in: *Function Spaces and Applications*, CRC Press, New York, etc. (2000), pp. 117–130.
27. P. Jain, L. E. Persson, and A. Wedestig, "Carleman–Knopp type inequalities via Hardy's inequality," *Math. Inequal. Appl.*, **4**, No. 3, 343–355 (2001).
28. K. Knopp, "Uber Reihen mit positiven Gliedern," *J. Lond. Math. Soc. (2)*, **3**, 205–211 (1928).
29. T. S. Kopaliani, "On some structural properties of Banach function spaces and boundedness of certain integral operators, *Czechoslovak Math. J.*, **54**, No. 129, 791–805 (2004).
30. M. Krbec, B. Opic, L. Pick, and J. Rakosnik, "Some recent results on Hardy type operators in weighted function spaces and related topics, in: *Function Spaces, Differential Operators, and Nonlinear Analysis*, Teubner, Stuttgart (1993), pp. 158–184.
31. A. Kufner and L. E. Persson, *Integral Inequalities with Weights*, World Scientific Publ., Singapore (2002).
32. J. Lindenstrauss and L. Tzafriri, "Classical Banach spaces II," *Ergeb. Math. Grenzgeb. (3)*, **97** (1979).

33. E. Lomakina and V. Stepanov, "On the Hardy-type integral operator in Banach function spaces," *Publ. Mat.*, **42**, 165–194 (1998).
34. E. R. Love, "Inequalities related to those of Hardy and of Cochran and Lee," *Math. Proc. Cambridge Philos. Soc.*, **99**, 395–408 (1986).
35. W. A. J. Luxemburg, *Banach Function Spaces: Thesis*, Delfi (1955).
36. F. I. Mamedov and A. Harman, "On a weighted inequality of Hardy type in spaces $L^{p(\cdot)}$," *J. Math. Anal. Appl.*, **353**, No. 2, 521–530 (2009).
37. R. A. Mashiyev, B. Cekic, F. I. Mamedov, and S. Ogras, "Hardy's inequality in power-type weighted $L^{p(\cdot)}(0,\infty)$ spaces," *J. Math. Anal. Appl.*, **334**, No. 1, 289–298 (2007).
38. V. G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin (1985).
39. B. Muckenhoupt, "Hardy's inequality with weights," *Studia Math.*, **44**, 31–38 (1972).
40. J. Musielak, "Orlicz spaces and modular spaces," *Lect. Notes Math.*, **1034** (1983).
41. L. Quinsheng, "Two weight Φ-inequalities for the Hardy operator, Hardy–Littlewood maximal operator, and fractional integrals," *Proc. Amer. Math. Soc.*, **118**, No. 1, 129–142 (1993).
42. A. Schep, "Minkowski's integral inequality for function norms," *Oper. Theory: Adv. Appl.*, **75**, 299–308 (1995).
43. G. Tomaselli, "A class of inequalities," *Boll. Unione Mat. Ital.*, **2**, 622–631 (1969).
44. A. Wedestig, "Some new Hardy type inequalities and their limiting inequalities," *J. Inequal. Pure Appl. Math.*, **61**, No. 4, 1–33 (2003).
45. Y. Zeren and V. S. Guliyev, "Two-weight norm inequalities for some anisotropic sublinear operators," *Turkish Math. J.*, **30**, 329–355 (2006).