

Commutator of the Maximal Function in Total Morrey Spaces for the Dunkl Operator on the Real Line

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Abstract. On the real line, the Dunkl operators D_ν are differential-difference operators associated with the reflection group \mathbb{Z}_2 on \mathbb{R} . In this paper, in the setting \mathbb{R} we study the commutators of the maximal operator associated with the Dunkl operator $[b, M_\nu]$ in the total D_ν -Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. We give necessary and sufficient conditions for the boundedness of the operator $[b, M_\nu]$ on total D_ν -Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ when b belongs to $BMO(\mathbb{R}, dm_\nu)$ spaces, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}, dm_\nu)$ spaces are obtained.

Key Words and Phrases: Maximal operator; total D_ν -Morrey space; Dunkl operator; commutator; BMO

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1. Introduction

On the real line, the Dunkl operators Λ_ν are differential-difference operators introduced in 1989 by Dunkl [11]. For a real parameter $\nu \geq -1/2$, we consider the *Dunkl operator*, associated with the reflection group \mathbb{Z}_2 on \mathbb{R} :

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu + 1) \frac{f(x) - f(-x)}{2x}, \quad x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on \mathbb{R} , given by

$$dm_\nu(x) := (2^{\nu+1}\Gamma(\nu + 1))^{-1} |x|^{2\nu+1} dx, \quad x \in \mathbb{R}.$$

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For any $x \in \mathbb{R}$ and $r > 0$, let $B(x, r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x| - r\}, |x| + r[\}$ be a Dunkl-ball in \mathbb{R} . Then $B(0, r) =]-r, r[$ and $m_\nu B(0, r) = c_\nu r^{2\nu+2}$, where $c_\nu := [2^{\nu+1}(\nu+1)\Gamma(\nu+1)]^{-1}$.

The *maximal operator* M_ν associated with the Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0} (m_\nu(B(x, r)))^{-1} \int_{B(x, r)} |f(y)| dm_\nu(y), \quad x \in \mathbb{R},$$

and *sharp maximal operator* M_ν^\sharp associated with the Dunkl operator on the real line is given by

$$M_\nu^\sharp f(x) := \sup_{r>0} (m_\nu(B(x, r)))^{-1} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dm_\nu(y), \quad x \in \mathbb{R},$$

where $f_{B(x, r)} := (m_\nu(B(x, r)))^{-1} \int_{B(x, r)} f(y) dm_\nu(y)$. For a fixed $q \in (0, 1)$, any suitable function h and $x \in \mathbb{R}$, let $M_{\nu, q}^\sharp h(x) = (M_\nu^\sharp(|h|^q)(x))^{1/q}$ and $M_{\nu, q} h(x) = (M_\nu(|h|^q)(x))^{1/q}$.

The *maximal commutator* $M_{b, \nu}$ associated with the Dunkl operator on the real line and with a locally integrable function $b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ is defined by

$$M_{b, \nu} f(x) := \sup_{r>0} (m_\nu(B(x, r)))^{-1} \int_{B(x, r)} |b(x) - b(y)| |f(y)| dm_\nu(y), \quad x \in \mathbb{R}.$$

We can define the (nonlinear) commutator of the maximal operator M_ν with a locally integrable function b by

$$[b, M_\nu](f)(x) = b(x)M_\nu(f)(x) - M_\nu(bf)(x).$$

For more details about the operators $M_{b, \nu}$ and $[b, M_\nu]$, we refer to [6, 22] and references therein.

It is well known that maximal and fractional maximal operators play an important role in harmonic analysis (see [34]). Also, the fractional maximal function and the fractional integral, associated with differential-difference Dunkl operators D_ν play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [4, 5, 24]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and

physical applications (see, for example, [12]). The maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [2], V.S. Guliyev and Y.Y. Mammadov [4, 5, 6], Y.Y. Mammadov [21], L. Kamoun [16], M.A. Mourou [25], F. Soltani [32, 33], K. Trimeche [35] and others. Moreover, the results on $L_\Phi(\mathbb{R}, dm_\nu)$ -boundedness of fractional maximal operator and its commutators associated with D_ν were obtained in [6, 22].

It is well known that the maximal operator plays an important role in harmonic analysis (see [34]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting \mathbb{R} , we study the boundedness of the maximal commutator $M_{b,\nu}$ and the commutator of the maximal operator $[b, M_\nu]$ on total D_ν -Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, when b belongs to the space $BMO(\mathbb{R}, dm_\nu)$, by which some new characterizations of the space $BMO(\mathbb{R}, dm_\nu)$ are given.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries in the Dunkl setting on \mathbb{R}

Definition 1. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ the Morrey space [26] ($\equiv D_\nu$ -Morrey space), by $\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ the modified Morrey space [26] (\equiv modified D_ν -Morrey space), and by $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ the total Morrey space [27, 28] (\equiv total D_ν -Morrey space) associated with the Dunkl operator, the set of all classes of locally integrable functions f , with the finite norms

$$\begin{aligned} \|f\|_{L_{p,\lambda}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}, t > 0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t), dm_\nu)}, \\ \|f\|_{\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}, t > 0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t), dm_\nu)}, \\ \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,t), dm_\nu)}, \end{aligned}$$

respectively, see also [7, 9, 10, 29, 30].

Definition 2. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ [26] (\equiv weak D_ν -Morrey space), the weak modified Morrey space $\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ [26] (\equiv weak modified D_ν -Morrey space), and the weak total

Morrey space $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ [27] (\equiv weak total D_ν -Morrey space) associated with the Dunkl operator, the set of all classes of locally integrable functions f , with the finite norms

$$\begin{aligned}\|f\|_{WL_{p,\lambda}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}, t > 0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t), dm_\nu)}, \\ \|f\|_{W\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}, t > 0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t), dm_\nu)}, \\ \|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(B(x,t), dm_\nu)},\end{aligned}$$

respectively.

Lemma 1. [23, 28] If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq 2\nu + 2$, then

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = L_{p,\lambda}(\mathbb{R}, dm_\nu) \cap L_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} = \max \left\{ \|f\|_{L_{p,\lambda}(\mathbb{R}, dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}, dm_\nu)} \right\}.$$

Lemma 2. [23, 28] If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq 2\nu + 2$, then

$$WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda}(\mathbb{R}, dm_\nu) \cap WL_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} = \max \left\{ \|f\|_{WL_{p,\lambda}(\mathbb{R}, dm_\nu)}, \|f\|_{WL_{p,\mu}(\mathbb{R}, dm_\nu)} \right\}.$$

Remark 1.. If $0 < p < \infty$, and $\lambda > 2\nu + 2$ or $\mu < 0$, then

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = \Theta(\mathbb{R}),$$

where $\Theta \equiv \Theta(\mathbb{R})$ is the set of all functions equivalent to 0 on \mathbb{R} .

Lemma 3. [23] If $0 < p < \infty$, $0 \leq \lambda_2 \leq \lambda_1 \leq 2\nu + 2$ and $0 \leq \mu_1 \leq \mu_2 \leq 2\nu + 2$, then

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}, dm_\nu) \subset_{\succ} L_{p,\lambda_2,\mu_2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}, dm_\nu)} \leq \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R}, dm_\nu)}.$$

Lemma 4. [23] If $0 < p < \infty$, $0 \leq \lambda \leq 2\nu + 2$ and $0 \leq \mu \leq 2\nu + 2$, then

$$L_{p,2\nu+2,\mu}(\mathbb{R}, dm_\nu) \subset_{\succ} L_\infty(\mathbb{R}, dm_\nu) \subset_{\succ} L_{p,\lambda,2\nu+2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R}, dm_\nu)} \leq c_\nu^{1/p} \|f\|_{L_\infty(\mathbb{R}, dm_\nu)} \leq \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}, dm_\nu)}.$$

Lemma 5. [23] *If $0 \leq \lambda < 2\nu + 2$, $0 \leq \mu < 2\nu + 2$, $0 \leq \alpha < 2\nu + 2 - \lambda$ and $0 \leq \beta < 2\nu + 2 - \mu$, then for $\frac{2\nu+2-\lambda}{\alpha} \leq p \leq \frac{2\nu+2-\mu}{\beta}$*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) \subset_{\succ} L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R}, dm_\nu)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ the inequality

$$\|f\|_{L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R}, dm_\nu)} \leq c_\nu^{1/p'} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)}$$

is valid.

3. Maximal commutators $M_{b,\alpha,\nu}$ in total Morrey spaces

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$$

In this section, we investigate the boundedness of the maximal commutator $M_{b,\nu}$ in total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

The following Guliyev type local estimates are valid (see also [3]).

Lemma 6. *Let $1 \leq p < \infty$ and $B(x, r)$ be any Dunkl-ball in \mathbb{R} . If $p > 1$, then the inequality*

$$\|M_\nu f\|_{L_p(B(x,r), dm_\nu)} \lesssim r^{\frac{2\nu+2}{p}} \sup_{t>2r} t^{-\frac{2\nu+2}{p}} \|f\|_{L_p(B(x,t), dm_\nu)} \quad (1)$$

holds for all $f \in L_p^{\text{loc}}(\mathbb{R}, dm_\nu)$.

Moreover, if $p = 1$, then the inequality

$$\|M_\nu f\|_{WL_1(B(x,r), dm_\nu)} \lesssim r^{2\nu+2} \sup_{t>2r} t^{-2\nu-2} \|f\|_{L_1(B(x,t), dm_\nu)} \quad (2)$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$.

Proof. Let $1 \leq p < \infty$. For arbitrary Dunkl-ball $B = B(x, r)$, let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathfrak{c}(2B)}$.

$$\|M_\nu f\|_{L_p(B, dm_\nu)} \leq \|M_\nu f_1\|_{L_p(B, dm_\nu)} + \|M_\nu f_2\|_{L_p(B, dm_\nu)}.$$

By the continuity of the operator $M_\nu : L_p(\mathbb{R}, dm_\nu) \rightarrow L_p(\mathbb{R}, dm_\nu)$ (see, for example, [26]), we have

$$\|M_\nu f_1\|_{L_p(B, dm_\nu)} \lesssim \|f\|_{L_p(2B, dm_\nu)}.$$

Let y be an arbitrary point from B . If $B(y, \tau) \cap \mathfrak{c}(2B) \neq \emptyset$, then $\tau > r$. Indeed, if $z \in B(y, \tau) \cap \mathfrak{c}(2B)$, then $\tau > |y - z| \geq |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y, \tau) \cap \mathring{c}(2B) \subset B(x, 2\tau)$. Indeed, $z \in B(y, \tau) \cap \mathring{c}(2B)$. Then we get $|x - z| \leq |y - z| + |x - y| < \tau + r < 2\tau$.

Hence,

$$\begin{aligned} M_\nu f_2(y) &= \sup_{\tau > 0} \frac{1}{m_\nu(B(y, \tau))} \int_{B(y, \tau) \cap \mathring{c}(2B)} |f(z)| dm_\nu(z) \\ &\leq 2^{2\nu+2} \sup_{\tau > r} \frac{1}{m_\nu(B(x, 2\tau))} \int_{B(x, 2\tau)} |f(z)| dm_\nu(z) \\ &= 2^{2\nu+2} \sup_{\tau > 2r} \frac{1}{m_\nu(B(x, \tau))} \int_{B(x, \tau)} |f(z)| dm_\nu(z). \end{aligned}$$

Therefore, for all $y \in B$ we have

$$M_\nu f_2(y) \leq 2^{2\nu+2} \sup_{\tau > 2r} \frac{1}{m_\nu(B(x, \tau))} \int_{B(x, \tau)} |f(z)| dm_\nu(z). \quad (3)$$

Applying Hölder's inequality, we get

$$M_\nu f_2(y) \lesssim \sup_{\tau > 2r} \frac{1}{m_\nu(B(x, \tau))^{\frac{1}{p}}} \int_{B(x, \tau)} |f(z)|^p dm_\nu(z). \quad (4)$$

Thus,

$$\begin{aligned} \|M_\nu f\|_{L_p(B, dm_\nu)} &\lesssim \|f\|_{L_p(2B, dm_\nu)} \\ &\quad + m_\nu(B(x, \tau))^{\frac{1}{p}} \left(\sup_{\tau > 2r} \frac{1}{m_\nu(B(x, \tau))} \int_{B(x, \tau)} |f(z)| dm_\nu(z) \right). \end{aligned}$$

Let $p = 1$. It is obvious that for any ball $B = B(x, r)$

$$\|M_\nu f\|_{WL_p(B, dm_\nu)} \leq \|M_\nu f_1\|_{WL_p(B, dm_\nu)} + \|M_\nu f_2\|_{WL_p(B, dm_\nu)}.$$

By the continuity of the operator $M_\nu : L_1(\mathbb{R}, dm_\nu) \rightarrow WL_1(\mathbb{R}, dm_\nu)$, we have

$$\|M_\nu f_1\|_{WL_1(B, dm_\nu)} \lesssim \|f\|_{L_1(2B, dm_\nu)}.$$

Then by (4) we get the inequality (2). ◀

The following result completely characterizes the boundedness of M_ν on total Morrey spaces $L_{p, \lambda, \mu}(\mathbb{R}, dm_\nu)$.

Theorem 1. *1. If $f \in L_{1, \lambda, \mu}(\mathbb{R}, dm_\nu)$, $0 \leq \lambda < 2\nu + 2$ and $0 \leq \mu < 2\nu + 2$, then $M_\nu f \in WL_{1, \lambda, \mu}(\mathbb{R}, dm_\nu)$ and*

$$\|M_\nu f\|_{WL_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)} \leq C_{1,\lambda,\mu} \|f\|_{L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)}, \quad (5)$$

where $C_{1,\lambda,\mu}$ is independent of f .

2. If $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$, $0 \leq \lambda < 2\nu + 2$ and $0 \leq \mu < 2\nu + 2$, then $M_\nu f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ and

$$\|M_\nu f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \leq C_{p,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)}, \quad (6)$$

where $C_{p,\lambda,\mu}$ depends only on p, λ, μ and n .

Proof. Let $p = 1$. From the inequality (2) we get

$$\begin{aligned} \|M_\nu f\|_{WL_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\lambda} [1/t]_1^\mu \|M_\nu f\|_{WL_1(B(x,t), dm_\nu)} \\ &\lesssim \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\lambda} [1/t]_1^\mu t^{2\nu+2} \sup_{\tau > 2t} \tau^{-2\nu+2} \|f\|_{L_1(B(x,\tau))} \\ &\lesssim \|f\|_{L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)} \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\lambda} [1/t]_1^\mu t^{2\nu+2} \sup_{\tau > t} \tau^{-2\nu+2} [\tau]_1^\lambda [1/\tau]_1^{-\mu} \\ &= \|f\|_{L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)} \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{2\nu+2-\lambda} [1/t]_1^{\mu-2\nu+2} \sup_{\tau > t} [\tau]_1^{\lambda-2\nu+2} [1/\tau]_1^{2\nu+2-\mu} \\ &= \|f\|_{L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)}, \end{aligned}$$

which implies that the operator M_ν is bounded from $L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$ to $WL_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Let $1 < p < \infty$. From the inequality (1) we get

$$\begin{aligned} \|M_\nu f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} &= \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|M_\nu f\|_{L_p(B(x,t), dm_\nu)} \\ &\lesssim \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} t^{\frac{2\nu+2}{den}} \sup_{\tau > 2t} \tau^{-\frac{2\nu+2}{p}} \|f\|_{L_p(B(x,\tau))} \\ &\lesssim \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} t^{\frac{2\nu+2}{p}} \sup_{\tau > t} \tau^{-\frac{2\nu+2}{p}} [\tau]_1^{\frac{\lambda}{p}} [1/\tau]_1^{-\frac{\mu}{p}} \\ &= \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{\frac{2\nu+2-\lambda}{p}} [1/t]_1^{\frac{\mu-2\nu+2}{p}} \sup_{\tau > t} [\tau]_1^{\frac{\lambda-2\nu+2}{p}} [1/\tau]_1^{\frac{2\nu+2-\mu}{p}} \\ &= \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)}, \end{aligned}$$

which implies that the operator M_ν is bounded in $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. ◀

From Theorem 1, in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 1. [2, 20, 32] 1. If $f \in L_{1,\lambda}(\mathbb{R}, dm_\nu)$ and $0 \leq \lambda < 2\nu + 2$, then $M_\nu f \in WL_{1,\lambda}(\mathbb{R}, dm_\nu)$ and

$$\|M_\nu f\|_{WL_{1,\lambda}(\mathbb{R}, dm_\nu)} \leq C_{1,\lambda} \|f\|_{L_{1,\lambda}(\mathbb{R}, dm_\nu)},$$

where $C_{1,\lambda}$ is independent of f .

2. If $f \in L_{p,\lambda}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$ and $0 \leq \lambda < 2\nu + 2$, then $M_\nu f \in L_{p,\lambda}(\mathbb{R}, dm_\nu)$ and

$$\|M_\nu f\|_{L_{p,\lambda}(\mathbb{R}, dm_\nu)} \leq C_{p,\lambda} \|f\|_{L_{p,\lambda}(\mathbb{R}, dm_\nu)},$$

where $C_{p,\lambda}$ depends only on p , λ and n .

Corollary 2. [21] 1. If $f \in \tilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)$ and $0 \leq \lambda < 2\nu + 2$, then $M_\nu f \in W\tilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)$ and

$$\|M_\nu f\|_{W\tilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)} \leq C_{1,\lambda} \|f\|_{\tilde{L}_{1,\lambda}(\mathbb{R}, dm_\nu)},$$

where $C_{1,\lambda}$ is independent of f .

2. If $f \in \tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$, $1 < p < \infty$ and $0 \leq \lambda < 2\nu + 2$, then $M_\nu f \in \tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ and

$$\|M_\nu f\|_{\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)} \leq C_{p,\lambda} \|f\|_{\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)},$$

where $C_{p,\lambda}$ depends only on p , λ and n .

We recall the definition of the space $BMO(\mathbb{R}, dm_\nu)$.

Definition 3. Suppose that $b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$. Let

$$\|b\|_{BMO(\mathbb{R}, dm_\nu)} := \sup_{x \in \mathbb{R}, r > 0} \frac{1}{m_\nu(B(x, r))} \int_{B(x, r)} |b(y) - b_{B(x, r)}(x)| dm_\nu(y),$$

where

$$b_{B(x, r)} := \frac{1}{m_\nu(B(x, r))} \int_{B(x, r)} b(y) dm_\nu(y).$$

Define

$$BMO(\mathbb{R}, dm_\nu) := \{b \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu) : \|b\|_{BMO(\mathbb{R}, dm_\nu)} < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}, dm_\nu)$ is a Banach space with respect to the norm $\|\cdot\|_{BMO(\mathbb{R}, dm_\nu)}$.

We will need the following properties of BMO -functions (see [13]):

$$\|b\|_{BMO(\mathbb{R}, dm_\nu)} \approx \sup_{x \in \mathbb{R}, r > 0} \left(\frac{1}{m_\nu(B(x, r))} \int_{B(x, r)} |b(y) - b_{B(x, r)}|^p dm_\nu(y) \right)^{\frac{1}{p}}, \quad (7)$$

where $1 \leq p < \infty$ and the positive equivalence constants are independent of b , and

$$|b_{B(x,r)} - b_{B(x,t)}| \leq C \|b\|_{BMO(\mathbb{R}, dm_\nu)} \ln \frac{t}{r} \quad \text{for any } 0 < 2r < t, \quad (8)$$

where the positive constant C does not depend on b , x , r and t .

For any measurable set E with $m_\nu(E) < \infty$ and any suitable function f , the norm $\|f\|_{L(\log L), E}$ is defined by

$$\|f\|_{L(\log L), E} = \inf \left\{ \lambda > 0 : \frac{1}{m_\nu(E)} \int_E \frac{|f(x)|}{\lambda} \left(2 + \frac{|f(x)|}{\lambda} \right) dm_\nu(x) \leq 1 \right\}.$$

The norm $\|f\|_{\exp L, E}$ is defined by

$$\|f\|_{\exp L, E} = \inf \left\{ \lambda > 0 : \frac{1}{m_\nu(E)} \int_E \exp \left(\frac{|f(x)|}{\lambda} \right) dm_\nu(x) \leq 2 \right\}.$$

Then, for any suitable functions f and g , the generalized Hölder's inequality holds (see [31]):

$$\frac{1}{m_\nu(E)} \int_E |f(x)| |g(x)| dm_\nu(x) \lesssim \|f\|_{\exp L, E} \|g\|_{L(\log L), E}. \quad (9)$$

The following John-Nirenberg inequalities on spaces of homogeneous type come from [17, Propositions 6, 7].

Lemma 7. *Let $b \in BMO(\mathbb{R}, dm_\nu)$. Then there exist constants $C_1, C_2 > 0$ such that for every ball $B \subset \mathbb{R}$ and every $\alpha > 0$ we have*

$$m_\nu(\{x \in B : |b(x) - b_B| > \alpha\}) \leq C_1 m_\nu(B) \exp \left\{ - \frac{C_2}{\|b\|_{BMO(\mathbb{R}, dm_\nu)}} \alpha \right\}.$$

By the generalized Hölder's inequality in Orlicz spaces (see [31, page 58]) and John-Nirenberg's inequality, we get (see also [18, (2.14)])

$$\frac{1}{|B|} \int_B |b(x) - b_B| |g(x)| dm_\nu(x) \lesssim \|b\|_{BMO(\mathbb{R}, dm_\nu)} \|g\|_{L(\log L), B}. \quad (10)$$

We refer, for instance, to [14] and [19] for details on this space and properties. For a given ball B , we define the following local maximal function:

$$M_{B, \nu} f(x) = \sup_{B \supseteq B' \ni x} (m_\nu(B'))^{-1} \int_{B'} |f(y)| dm_\nu(y),$$

where the supremum is taken over all balls B' such that $x \in B' \subseteq B$.

For a function b defined on \mathbb{R} , we let, for any $x \in \mathbb{R}$,

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, for any $x \in \mathbb{R}$, $b^+(x) - b^-(x) = b(x)$.

Lemma 8. *Let $b \in L_{\text{loc}}^1(\mathbb{R}, dm_\nu)$. Then the following statements are equivalent:*

1. $b \in BMO(\mathbb{R}, dm_\nu)$ and $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.
2. There exists $s \in [1, \infty)$ such that

$$\sup_B \frac{\|(b - M_{B,\nu}(b))\chi_B\|_{L_s(\mathbb{R}, dm_\nu)}}{\|\chi_B\|_{L_s(\mathbb{R}, dm_\nu)}} \leq C. \quad (11)$$

3. For all $s \in [1, \infty)$ we have (11).

Proof. Since the proof is similar to the corresponding one in [8], we omit it here. ◀

Lemma 9. *Let $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$. Then*

$$M_\nu(M_\nu f)(x) \approx \sup_{B \ni x} \|f\chi_B\|_{L(1+\log^+ L), \nu}. \quad (12)$$

Proof. Let B be a ball in \mathbb{R} . We are going to use weak type estimates (see [34], for instance): there exists a positive constant $c > 1$ such that every $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ and for every $t > (1/m_\nu(B)) \int_B |f(x)| dm_\nu(x)$ we have

$$\begin{aligned} \frac{1}{ct} \int_{\{x \in B: |f(x)| > t\}} |f(x)| dm_\nu(x) &\leq m_\nu(\{x \in B : M_\nu(f\chi_B)(x) > t\}) \\ &\leq \frac{c}{t} \int_{\{x \in B: |f(x)| > t/2\}} |f(x)| dm_\nu(x). \end{aligned}$$

Then

$$\begin{aligned} \int_B M_\nu(f\chi_B)(x) dm_\nu(x) &= \int_0^\infty m_\nu(\{x \in B : M_\nu(f\chi_B)(x) > \lambda\}) d\lambda \\ &= \int_0^{|f|_B} m_\nu(\{x \in B : M_\nu(f\chi_B)(x) > \lambda\}) d\lambda \\ &+ \int_{|f|_B}^\infty m_\nu(\{x \in B : M_\nu(f\chi_B)(x) > \lambda\}) d\lambda \\ &= m_\nu(B) |f|_B + \int_{|f|_B}^\infty m_\nu(\{x \in B : M_\nu(f\chi_B)(x) > \lambda\}) d\lambda \end{aligned}$$

$$\begin{aligned}
&\geq m_\nu(B) |f|_B + \frac{1}{c} \int_{|f|_B}^{\infty} \left(\int_{\{x \in B: |f(x)| > \lambda\}} |f(x)| dm_\nu(x) \right) \frac{d\lambda}{\lambda} \\
&= m_\nu(B) |f|_B + \frac{1}{c} \int_{\{x \in B: |f(x)| > |f|_B\}} \left(\int_{|f|_B}^{|f(x)|} \frac{d\lambda}{\lambda} \right) |f(x)| dm_\nu(x) \\
&= m_\nu(B) |f|_B + \frac{1}{c} \int_{\{x \in B: |f(x)| > |f|_B\}} |f(x)| \log \frac{|f(x)|}{|f|_B} dm_\nu(x) \\
&\geq \frac{1}{c} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B} \right) dm_\nu(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_B M_\nu(f \chi_B)(x) dm_\nu(x) &= \int_0^\infty m_\nu(\{x \in B : M(f \chi_B)(x) > \lambda\}) d\lambda \\
&\approx \int_0^\infty m_\nu(\{x \in B : M_\nu(f \chi_B)(x) > 2\lambda\}) d\lambda \\
&= \int_0^{|f|_B} m_\nu(\{x \in B : M_\nu(f \chi_B)(x) > 2\lambda\}) d\lambda \\
&+ \int_{|f|_B}^\infty m_\nu(\{x \in B : M_\nu(f \chi_B)(x) > 2\lambda\}) d\lambda \\
&\leq m_\nu(B) |f|_B + c \int_{|f|_B}^\infty \left(\int_{\{x \in B: |f(x)| > \lambda\}} |f(x)| dm_\nu(x) \right) \frac{d\lambda}{\lambda} \\
&= m_\nu(B) |f|_B + c \int_{\{x \in B: |f(x)| > |f|_B\}} |f(x)| \log \frac{|f(x)|}{|f|_B} dm_\nu(x) \\
&\leq c \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B} \right) dm_\nu(x).
\end{aligned}$$

Therefore, for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ we get

$$M_\nu(M_\nu f)(x) \approx \sup_{B \ni x} m_\nu(B)^{-1} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B} \right) dm_\nu(x). \quad (13)$$

Since

$$1 \leq \frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B} \right) dm_\nu(x),$$

we have

$$|f|_B \leq \|f \chi_B\|_{L(1+\log^+ L), \nu}.$$

Using the inequality $\log^+(ab) \leq \log^+ a + \log^+ b$ with $a, b > 0$, we get

$$\begin{aligned}
& \frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B}\right) dm_\nu(x) \\
&= \frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \left(\frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L), \nu}} \frac{\|f\chi_B\|_{L(1+\log^+ L), \nu}}{|f|_B}\right)\right) dm_\nu(x) \\
&= \frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L), \nu}}\right) dm_\nu(x) \\
&+ \frac{1}{m_\nu(B)} \int_B |f(x)| \log^+ \frac{\|f\chi_B\|_{L(1+\log^+ L), \nu}}{|f|_B} dm_\nu(x) \\
&\leq \|f\chi_B\|_{L(1+\log^+ L), \nu} + |f|_B \log^+ \frac{\|f\chi_B\|_{L(1+\log^+ L), \nu}}{|f|_B}.
\end{aligned}$$

Since $\frac{\|f\chi_B\|_{L(1+\log^+ L), \nu}}{|f|_B} \geq 1$ and $\log t \leq t$ when $t \geq 1$, we get

$$\frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B}\right) dm_\nu(x) \leq 2\|f\chi_B\|_{L(1+\log^+ L), \nu}. \quad (14)$$

On the other hand, since

$$\|f\chi_B\|_{L(1+\log^+ L), \nu} = \frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{\|f\chi_B\|_{L(1+\log^+ L), \nu}}\right) dm_\nu(x),$$

on using

$$|f|_B \leq \|f\chi_B\|_{L(1+\log^+ L), \nu},$$

we get

$$\|f\chi_B\|_{L(1+\log^+ L), \nu} \lesssim \frac{1}{m_\nu(B)} \int_B |f(x)| \left(1 + \log^+ \frac{|f(x)|}{|f|_B}\right) dm_\nu(x). \quad (15)$$

Therefore, from (13), (14) and (15) we have (12). ◀

For proving our main results, we need the following estimate.

Lemma 10. [15, Lemma 1] *If $b \in BMO(\mathbb{R}, dm_\nu)$, then for any $q \in (0, 1)$, there exists a positive constant C such that*

$$M_q^\sharp(M_{b, \nu} f)(x) \leq C \|b\|_{BMO(\mathbb{R}, dm_\nu)} M_\nu(M_\nu f)(x) \quad (16)$$

for every $x \in \mathbb{R}$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\nu}$ on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$, when b belongs to the space $BMO(\mathbb{R}, dm_\nu)$.

Theorem 2. *Let $1 < p < \infty$, $0 \leq \lambda < 2\nu + 2$ and $0 \leq \mu < 2\nu + 2$. The following assertions are equivalent:*

- (i) $b \in BMO(\mathbb{R}, dm_\nu)$.
- (ii) The operator $M_{b,\nu}$ is bounded on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Proof. (i) \Rightarrow (ii). Suppose that $b \in BMO(\mathbb{R}, dm_\nu)$. Combining Theorem 1 and Lemma 10, we get

$$\begin{aligned} \|M_{b,\nu}f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} &\lesssim \|M_q^\sharp(M_{b,\nu}f)\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \\ &\lesssim \|b\|_{BMO(\mathbb{R}, dm_\nu)} \|M_\nu(M_\nu f)\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \\ &\lesssim \|b\|_{BMO(\mathbb{R}, dm_\nu)} \|M_\nu f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \\ &\lesssim \|b\|_{BMO(\mathbb{R}, dm_\nu)} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)}. \end{aligned}$$

(ii) \Rightarrow (i). Assume that $M_{b,\nu}$ is bounded on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. Let $B = B(x, r)$ be a fixed ball. Consider $f = \chi_B$. It is easy to calculate that

$$\begin{aligned} \|\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} &\approx \sup_{y \in \mathbb{R}^n, t > 0} \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{B(y,t)} \chi_B(z) dz \right)^{\frac{1}{p}} \\ &= \sup_{y \in \mathbb{R}^n, t > 0} \left(|B(y,t) \cap B| [t]_1^{-\lambda} [1/t]_1^\mu \right)^{\frac{1}{p}} \\ &= \sup_{B(y,t) \subseteq B} \left(|B(y,t)| [t]_1^{-\lambda} [1/t]_1^\mu \right)^{\frac{1}{p}} = r^{\frac{2\nu+2}{p}} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}}. \quad (17) \end{aligned}$$

On the other hand, since

$$M_{b,\nu}(\chi_B)(x) \gtrsim \frac{1}{|B|} \int_B |b(z) - b_B| dz \quad \text{for all } x \in B,$$

we have

$$\begin{aligned} \|M_{b,\nu}(\chi_B)\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} &\approx \sup_{B(y,t)} \left([t]_1^{-\lambda} [1/t]_1^\mu \int_{B(y,t)} |M_{b,\nu}(\chi_B)(z)|^p dz \right)^{\frac{1}{p}} \\ &\gtrsim r^{\frac{n}{p}} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \frac{1}{|B|} \int_B |b(z) - b_B| dz. \quad (18) \end{aligned}$$

Since by assumption

$$\|M_{b,\nu}(\chi_B)\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \lesssim \|\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)},$$

by (17) and (18) we get

$$\frac{1}{|B|} \int_B |b(z) - b_B| dz \lesssim 1.$$

◀

From Theorem 2 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3. *Let $1 < p < \infty$ and $0 \leq \lambda < 2\nu + 2$. The following assertions are equivalent:*

- (i) $b \in BMO(\mathbb{R}, dm_\nu)$.
- (ii) The operator $M_{b,\nu}$ is bounded on $L_{p,\lambda}(\mathbb{R}, dm_\nu)$.

Corollary 4. *Let $1 < p < \infty$ and $0 \leq \lambda < 2\nu + 2$. The following assertions are equivalent:*

- (i) $b \in BMO(\mathbb{R}, dm_\nu)$.
- (ii) The operator $M_{b,\nu}$ is bounded on $\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$.

4. Commutator of maximal operator $[b, M_\nu]$ in total Morrey spaces $L_{p,\lambda,\mu}$

In this section, we find necessary and sufficient conditions for the commutator of the maximal operator $[b, M_\nu]$ to be bounded on the spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

For a function b defined on \mathbb{R} , we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\nu]$ and $M_{b,\nu}$ are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ and $x \in \mathbb{R}$ the following inequality is valid:

$$\begin{aligned} |[b, M_\nu]f(x)| &= |b(x)M_\nu f(x) - M_\nu(bf)(x)| \\ &= |M_\nu(b(x)f)(x) - M_\nu(bf)(x)| \leq M_\nu(|b(x) - b|f)(x) = M_{b,\nu}f(x). \end{aligned}$$

If b is any locally integrable function on \mathbb{R} , then

$$|[b, M_\nu]f(x)| \leq M_{b,\nu}f(x) + 2b^-(x)M_\nu f(x), \quad x \in \mathbb{R}, \quad (19)$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}, dm_\nu)$ (see, for example, [1, 7, 9, 30]).

Obviously, the operators $M_{b,\nu}$ and $[b, M_\nu]$ are essentially different from each other because $M_{b,\nu}$ is positive and sublinear while $[b, M_\nu]$ is neither positive nor sublinear.

Applying Theorem 2, we obtain the following result.

Theorem 3. *Let $1 < p < \infty$, $0 \leq \lambda \leq 2\nu + 2$ and $0 \leq \mu \leq 2\nu + 2$. Suppose that b is a real valued locally integrable function in \mathbb{R} . Then the following assertions are equivalent:*

- (i) $b \in BMO(\mathbb{R}, dm_\nu)$ such that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.
- (ii) The operator $[b, M_\nu]$ is bounded on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Proof. (i) \Rightarrow (ii). Suppose that $b \in BMO(\mathbb{R}, dm_\nu)$. Combining Theorems 1 - 2 and the inequality (19), we get

$$\begin{aligned} \|[b, M_\nu]f\|_{L_{p,\lambda,\mu}} &\leq \|M_{b,\nu}f + 2b^- M_\nu f\|_{L_{p,\lambda,\mu}} \\ &\leq \|M_{b,\nu}f\|_{L_{p,\lambda,\mu}} + \|b^-\|_{L_\infty} \|M_\nu f\|_{L_{p,\lambda,\mu}} \\ &\lesssim (\|b\|_* + \|b^-\|_{L_\infty}) \|f\|_{L_{p,\lambda,\mu}}. \end{aligned}$$

(ii) \Rightarrow (i). Assume that $[b, M_\nu]$ is bounded on $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. Let $B = B(x, r)$ be a fixed ball.

Since

$$M_\nu(b\chi_B)\chi_B = M_{B,\nu}(b) \quad \text{and} \quad M_\nu(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned} |M_{B,\nu}(b) - b\chi_B| &= |M_\nu(b\chi_B)\chi_B - bM_\nu(\chi_B)\chi_B| \\ &\leq |M_\nu(b\chi_B) - bM_\nu(\chi_B)| = |[b, M_\nu]\chi_B|. \end{aligned}$$

Hence

$$\|M_{B,\nu}(b) - b\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \leq \|[b, M_\nu]\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)}.$$

Thus from (17) we get

$$\begin{aligned} \frac{1}{|B|} \int_B |b - M_{B,\nu}(b)| &\leq \left(\frac{1}{|B|} \int_B |b - M_{B,\nu}(b)|^p \right)^{\frac{1}{p}} \\ &\leq |B|^{-\frac{1}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|b\chi_B - M_{B,\nu}(b)\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \\ &\lesssim r^{-\frac{2\nu+2}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|[b, M_\nu]\chi_B\|_{L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)} \\ &\lesssim r^{-\frac{2\nu+2}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|\chi_B\|_{L_{p,\lambda,\mu}} \approx 1. \end{aligned}$$

Denote

$$E := \{x \in B : b(x) \leq b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B| dt = \int_F |b(t) - b_B| dt,$$

in view of the inequality $b(x) \leq b_B \leq M_{B,\nu}(b)$, $x \in E$, we get

$$\begin{aligned} \frac{1}{|B|} \int_B |b - b_B| &= \frac{2}{|B|} \int_E |b - b_B| \\ &\leq \frac{2}{|B|} \int_E |b - M_{B,\nu}(b)| \leq \frac{2}{|B|} \int_B |b - M_{B,\nu}(b)| \lesssim 1. \end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}, dm_\nu)$.

In order to show that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$, note that $M_{B,\nu}(b) \geq |b|$. Hence

$$0 \leq b^- = |b| - b^+ \leq M_{B,\nu}(b) - b^+ + b^- = M_{B,\nu}(b) - b.$$

Thus

$$(b^-)_B \leq c,$$

and by the Lebesgue differentiation theorem we get

$$b^-(x) = \lim_{m_\nu B \rightarrow 0} \int_B b^-(y) dm_\nu(y) \leq c \quad \text{for a.e. } x \in \mathbb{R}.$$

◀

From Theorem 3 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 5. *Let $1 < p < \infty$ and $0 \leq \lambda < 2\nu + 2$. Suppose that b is a real valued locally integrable function in \mathbb{R} . Then the following assertions are equivalent:*

- (i) $b \in BMO(\mathbb{R}, dm_\nu)$ such that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.
- (ii) The operator $[b, M_\nu]$ is bounded on $L_{p,\lambda}(\mathbb{R}, dm_\nu)$.

Corollary 6. *Let $1 < p < \infty$ and $0 \leq \lambda < 2\nu + 2$. Suppose that b is a real valued locally integrable function in \mathbb{R} . Then the following assertions are equivalent:*

- (i) $b \in BMO(\mathbb{R}, dm_\nu)$ such that $b^- \in L_\infty(\mathbb{R}, dm_\nu)$.
- (ii) The operator $[b, M_\nu]$ is bounded on $\tilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$.

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References

- [1] G.A. Abasova, M.N. Omarova, *Commutator of anisotropic maximal function with BMO functions on total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics, **43(1)**, 2022, 3-15.
- [2] C. Abdelkefi, M. Sifi, *Dunkl translation and uncentered maximal operator on the real line*, JIPAM. J. Inequal. Pure Appl. Math. **8(3)**, 2007, Article 73, 11 p.
- [3] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in \mathbb{R}^n* , Doctor's degree dissertation, Moscow, Mat. Inst. Steklov, 1994, 1-329 (in Russian).
- [4] V.S. Guliyev, Y.Y. Mammadov, *Some estimations for Riesz potentials in terms maximal and fractional maximal functions associated with the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math. Mech., **27(7)**, 2007, 71-76.
- [5] V.S. Guliyev, Y.Y. Mammadov, *On fractional maximal function and fractional integral associated with the Dunkl operator on the real line*, J. Math. Anal. Appl., **353(1)**, 2009, 449-459.
- [6] V.S. Guliyev, Y.Y. Mammadov, F.A. Muslumova, *Boundedness characterization of maximal commutators on Orlicz spaces in the Dunkl setting*, Journal of Mathematical Study **53(1)**, 2019, 1-21.
- [7] V.S. Guliyev, *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal., **16(4)**, 2022, 1509-1524.
- [8] V.S. Guliyev, *Some characterizations of BMO spaces via commutators in Orlicz spaces on stratified Lie groups*, Results Math., **77 (1)**, 2022, 18 p.
- [9] V.S. Guliyev, *Corrigendum to : Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal., **19(1)**, 2025, 305-306.
- [10] V.S. Guliyev, *Characterization of commutators of the maximal function in total Morrey spaces on stratified Lie groups*, Anal. Math. Physics, **15 (2)**, 2025, article number : 42, 1-25.
- [11] C.F. Dunkl, *Differential-difference operators associated with reflections groups*, Trans. Amer. Math. Soc., **311**, 1989, 167-183.

- [12] C.F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables*, 2nd edn, Encyclopedia of Mathematics and its Applications, 155 (Cambridge University Press, Cambridge, 2014).
- [13] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16**, 1978, 263-270.
- [14] F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14**, 415-426 (1961).
- [15] G. Hu, D. Yang, *Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type*, J. Math. Anal. Appl., **354**, 2009, 249-262.
- [16] L. Kamoun, *Besov-type spaces for the Dunkl operator on the real line*, J. Comput. Appl. Math. **199**, 2007, 56-67.
- [17] M. Kronz, *Some function spaces on spaces of homogeneous type*. *Manuscripta Math*, **106(2)**, 2001, 219-248.
- [18] A.K. Lerner, S. Ombrosi, C. Pérez, R.H. Torres, R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math., **220**, 2009, 1222-1264.
- [19] R. Long, L. Yang, *BMO functions in spaces of homogeneous type*, Sci. Sinica Ser. A **27(7)**, 1984, 695-708.
- [20] Y.Y. Mammadov, *On the boundedness of the maximal operator in Morrey spaces associated with the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math. Mech., **30(1)**, 2010, 147-154.
- [21] Y.Y. Mammadov, *Some embeddings into the modified Morrey spaces associated with the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math. Mech., **29(1)**, 2009, 111-120.
- [22] Y.Y. Mammadov, F.A. Muslimova, Z.V. Safarov, *Fractional maximal commutator on Orlicz spaces for the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math., **40(4)**, 2020, 130-144.
- [23] Y.Y. Mammadov, F.A. Muslumova, *Some embeddings into the total Morrey spaces associated with the Dunkl operator on the real line*, Proceedings of IAM, **13(1)**, 2023, 1-12.

- [24] H. Mejjaoli, Kh. Trimeche, *Harmonic analysis associated with the Dunkl-Bessel Laplace operator and a mean value property*, Fract. Calc. Appl. Anal., **4(4)**, 2001, 443-480.
- [25] M.A. Mourou, *Transmutation operators associated with a Dunkl-type differential-difference operator on the real line and certain of their applications*, Integral Transforms Spec. Funct., **12(1)**, 2001, 77-88.
- [26] F. Muslumova, *On embeddings into the Morrey and modified Morrey spaces in the Dunkl setting*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math., **40(1)**, 2020, 135-145.
- [27] F.A. Muslumova, *Some embeddings into the total Morrey spaces associated with the Dunkl operator on \mathbb{R}^d* , Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math., **43(1)**, 2023, 94-102.
- [28] F.A. Muslumova, *Corrigendum to : "Some embeddings into the total Morrey spaces associated with the Dunkl operator on \mathbb{R}^d "*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **44(1)**, 2024, p. 132.
- [29] F.A. Muslimova, Y.Y. Mammadov, J.N. Abdullayeva, *On embeddings of total Morrey spaces associated with Bessel operators*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Math., **45(1)**, 2025, 54-62.
- [30] M.N. Omarova, *Commutators of parabolic fractional maximal operators on parabolic total Morrey spaces*, Math. Meth. Appl. Sci., **48(11)**, 2025, 11037-11044.
- [31] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, M. Dekker, Inc., New York, 1991.
- [32] F. Soltani, *L_p -Fourier multipliers for the Dunkl operator on the real line*, J. Funct. Anal., **209**, 2004, 16-35.
- [33] F. Soltani, *On the Riesz-Dunkl potentials*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **40(2)**, 2014, 14-21.
- [34] E.M. Stein, *Harmonic analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, **43**, Princeton University Press, Princeton, New Jersey, 1993.
- [35] K. Trimeche, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Integral Transforms Spec. Funct., **13**, 2002, 17-38.

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