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Gradient Estimates for Parabolic Equations in Generalized Weighted Morrey Spaces

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Abstract We consider the Cauchy–Dirichlet problem for linear divergence form parabolic operators in bounded Reifenberg flat domain. The coefficients supposed to be only measurable in one of the space variables and small BMO with respect to the others. We obtain Calderón–Zygmund type estimate for the gradient of the solution in generalized weighted Morrey spaces with Muckenhoupt weight.

Keywords Generalized weighted Morrey spaces, parabolic equations, Cauchy–Dirichlet problem, measurable coefficients, BMO, gradient estimates

MR(2010) Subject Classification 35K20, 35B65, 35J15, 35K10, 35R05, 42B25, 46E30

1 Introduction

In [19], Morrey studied some integral inequalities in connection with Hölder regularity of solutions of nonlinear elliptic and parabolic operators. The classical *Morrey spaces* $L^{q,\lambda}$, usually attributed to him, were formulated in terms of function spaces by Campanato, Brudnyi and Peetre in the 1960s. They introduced notations similar to those used in the definition below (cf. [27]).

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A real valued function $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ with $q \in (1,\infty)$, $\lambda \in (0,n)$ provided the following norm is finite

$$\|f\|_{L^{q,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda} \int_{\mathcal{B}_r(x)} |f(y)|^q \, dy \right)^{1/q},$$

where the supremo is taken over all balls $\mathcal{B}_r(x) \subset \mathbb{R}^n$. The main result connected with these spaces is the following celebrated lemma: *Let* $|Df| \in L^{q,n-\lambda}$ *even locally, with* $n - \lambda < q$. *Then* u *is Hölder continuous of exponent* $\alpha = 1 - \frac{n-\lambda}{q}$. In [10], Chiarenza and Frasca showed boundedness of *the Hardy–Littlewood maximal operator* $\mathcal{M}$ and the *Calderón–Zygmund integral operator* in $L^{q,\lambda}(\mathbb{R}^n)$. This allows them to study the regularity of the solutions of the Dirichlet problem for linear elliptic PDEs with VMO coefficients.

In [18], Mizuhara extended the Morrey concept taking a weight function $\phi(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ in the definition of the norm while in [21], Nakai extended the results of Chiarenza and Frasca from $L^{q,\lambda}$ to $L^{q,\phi}$ imposing integral and doubling conditions on ϕ. These results allow to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $L^{q,\phi}$ (see [24] and the references therein). A further development of the generalized Morrey spaces can be found in the works of Guliyev at al., where the spaces $M^{q,\varphi}$ have been introduced under different conditions on φ (see [1, 12] and the references therein). These give the functional analysis tools to obtain regularity type results in $M^{q,\varphi}$ for various linear boundary value problems, see [14, 15].

Recently, Komori and Shirai [17] defined the weighted Morrey spaces and studied the boundedness of some classical operators in $L^{q,\kappa}(w)$, $q > 1$ where w satisfies the Muckenhoupt condition A_q. In the present work, we consider generalized weighted Morrey spaces $M^{q,\varphi}(w)$ that can be seen as an extension of both $M^{q,\varphi}$ and $L^{q,\kappa}(w)$, see [13].

We call *weight* a positive measurable function w defined on $\mathbb{R}^{n+1}$. Suppose that w satisfies *Muckenhoupt condition* of parabolic type or *parabolic* A_q*-condition.* Precisely, $w \in A_q$ for $q \in (1,\infty)$ if

$$\sup_{\mathcal{I}} \left(\frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} w(x,t)\, dxdt \right) \left(\frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} w(x,t)^{-\frac{1}{q-1}}\, dxdt \right)^{q-1} = [w]_q < \infty \tag{1.1}$$

for all parabolic cylinders $\mathcal{I} \subset \mathbb{R}^{n+1}$. Then $w(\mathcal{I})$ means the weighted measure of $\mathcal{I}$, that is,

$$w(\mathcal{I}) = \int_{\mathcal{I}} w(x,t)\, dxdt\,.$$

This measure satisfies *strong and reverse doubling properties*: for each $\mathcal{I}$ and each measurable subset $\mathcal{A} \subset \mathcal{I}$, there exist constants $c_1 > 0$ and $\tau_1 \in (0,1)$ such that

$$\frac{1}{[w]_q} \left(\frac{|\mathcal{A}|}{|\mathcal{I}|} \right)^q \le \frac{w(\mathcal{A})}{w(\mathcal{I})} \le c_1 \left(\frac{|\mathcal{A}|}{|\mathcal{I}|} \right)^{\tau_1}, \tag{1.2}$$

where c_1 and τ_1 depend on n and q but not on $\mathcal{I}$ and $\mathcal{A}$.

The function $f \in L^1_{\text{loc}}$ belongs to the *weighted Lebesgue space* L^q_w, $q > 1$ if

$$\|f\|^q_{L^q_w} = \int_{\mathbb{R}^{n+1}} |f(x,t)|^q w(x,t)\, dxdt < \infty\,.$$

The notorious result of Muckenhoupt [20] states that (1.1) is a necessary and sufficient condition in order that the *maximal inequality* in L^q_w to hold

$$\|f\|_{L^q_w} \le \|\mathcal{M}f\|_{L^q_w} \le c(n,q)\|f\|_{L^q_w}, \tag{1.3}$$

where $\mathcal{M}f(x,t) = \sup_{(x,t)\in\mathcal{I}} \fint_{\mathcal{I}} |f|$ is the *maximal function* of f and the supremum is taken over all cylinders that contain (x,t).

Consider the *Cauchy–Dirichlet problem*

$$\begin{cases} u_t - D_\alpha(a^{\alpha\beta}(x,t)D_\beta u) = D_\alpha f^\alpha(x,t), & \text{in } Q, \\ u(x,t) = 0, & \text{on } \partial Q, \end{cases} \tag{1.4}$$

where $Q = \Omega \times (0,T]$ is a cylinder in $\mathbb{R}^n \times \mathbb{R}^+$, $\partial Q = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\})$ stands for the parabolic boundary of Q and the summation convention over the repeated lower and upper indexes, running from 1 to n, is adopted. We are going to obtain *Calderón–Zygmund type* estimate for Du in the *generalized weighted Morrey spaces*: denote by $\mathbf{a} = \{a^{\alpha\beta}\}^n_{\alpha,\beta=1}$ the coefficient matrix and by $\mathbf{F} = \big(f^1,\ldots,f^n\big)$ the right-hand side. We are going to prove that

$$\mathbf{F} \in M^{p,\varphi}_w(Q) \text{ implies } |Du| \in M^{p,\varphi}_w(Q), \quad p > 2,$$

with $w \in A_{\frac{p}{2}}$ and φ as in (2.7).

We suppose that the coefficients are *only measurable* in one spatial variable, i.e., x_1 and possesses *small mean oscillation* (*small* BMO) in the remaining variables (x',t). This *partially* BMO assumption on the coefficients is quite general and allows *arbitrary* discontinuity in one spatial direction which is often related to problems of linear laminates, while the behavior with respect to the other directions, including the time, are controlled in terms of small BMO, such as small multipliers of the Heaviside step function for instance. It is clear that the cases of continuous, VMO or small BMO principal coefficients with respect to *all variables* are particular cases of the situation considered here. Regarding the underlying domain Ω, we suppose that its *non-smooth boundary* is *Reifenberg flat* (cf. Reifenberg [23]). It means that $\partial\Omega$ is well approximated by hyperplanes at each point and at each scale. This kind of minimal regularity of the boundary ensures the validity in Ω of some natural properties of geometric and functional analysis such as $W^{1,q}$-extension, non tangential accessibility property, measure density condition, the Poincaré inequality and so on. We refer the reader to the works of Kenig and Toro [16, 26] and the references therein for further details. In particular, a domain which is sufficiently flat in the sense of Reifenberg is also Jones flat. Moreover, domains with C^1-smooth or Lipschitz continuous boundaries with small Lipschitz constant belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains fractal boundaries domains, such as the von Koch snowflake with a small angle β of the age, for instance $\sin\beta < 1/8$.

The boundary problems and the corresponding regularity theory developed here are related to important variational problems arising in modeling of deformations in composite materials as fiber-reinforced media or, more generally, in the mechanics of membranes and films of simple non-homogeneous materials which form a linear laminated medium. In particular, a highly twinned elastic or ferroelectric crystal is a typical situation where a laminate appears. The equilibrium equations of such a linear laminate usually have only bounded and measurable coefficients in the direction of the stratification. The non-smoothness of the underlying Reifenberg flat domain, instead, is related to models of real-world systems over media with fractal geometry such as blood vessels, the internal structure of lungs, bacteria growth, graphs of stock market data, clouds, semiconductor devices, etc.

The problem considered here was firstly studied in the Lebesgue spaces by Byun and Wang (see [3, 4, 9]). Later, their results have been extended to higher order operators and in various weighted spaces in [5–7, 11, 22] by Byun et al. and in generalized Morrey spaces under various assumptions on the weight function, see [8, 15]. The presented result is a weighted version of the last two cited papers.

The paper is organized as follows. In Section 2 we define the space $M^{q,\varphi}_w$ and prove the maximal inequality in it. The Section 3 presents the problem and the main result. In the following, we obtain a gradient estimate. The technical approach is based on the Vitali-type covering lemma and estimates of the upper level sets of the maximal function of the gradient, see [3] for more details. The technique employed for parabolic equations could be extended without essential difficulties, to elliptic and parabolic systems.

Throughout the paper, the letter c will denote a universal constant that can be explicitly computed in terms of known quantities such as n, L, ν, p, φ, and $[w]_p$. The exact value of c may vary from one occurrence to another.

2 Parabolic Generalized Weighted Morrey Spaces

In the following, we use the domains:

- *parabolic cylinders* centered in $(y,\tau)\in\mathbb{R}^{n+1}$ and of radius $r>0$:

$$\mathcal{I}\equiv\mathcal{I}_r(y,\tau)=\{(x,t)\in\mathbb{R}^{n+1}:\ |x-y|<r,|t-\tau|<r^2\}$$

with Lebesgue measure $|\mathcal{I}_r|=cr^{n+2}$. For each $(y,\tau)\in Q$, we write

$$Q_r(y,\tau)=\mathcal{I}_r(y,\tau)\cap Q,\quad 2\mathcal{I}_r=\mathcal{I}_{2r}(y,\tau),\quad\text{and}\quad \complement(2\mathcal{I}_r)=\mathbb{R}^{n+1}\setminus 2\mathcal{I}_r.$$

- *cylinders* centered in $(y,\tau)=(y_1,y',\tau)$:

$$\mathcal{C}\equiv\mathcal{C}_r(y,\tau)=\{(x_1,x',t)\in\mathbb{R}^{n+1}:|x_1-y_1|<r,\ |x'-y'|<r,|t-\tau|<r^2\}$$

with $|\mathcal{C}_r|=c(n)r^{n+2}$.

- *elliptic cubes* centered in $y=(y_1,y')\in\mathbb{R}^n$:

$$\mathcal{C}'_r(y)=\{(x_1,x')\in\mathbb{R}^n:\ |x_1-y_1|<r,|x'-y'|<r\}$$

with $|\mathcal{C}'_r|=c(n)r^n$.

Definition 2.1 *Let Q be a cylinder in $\mathbb{R}^{n+1}$. A function $f\in L^q_w(Q)$, $w\in A_q$, $q>1$, belongs to the generalized weighted Morrey space $M^{q,\varphi}_w(Q)$ if the following norm is finite*

$$\|f\|^q_{M^{q,\varphi}_w(Q)}=\sup_{\substack{(y,\tau)\in Q\\ r>0}}\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q w(\mathcal{I}_r(y,\tau))}\int_{Q_r(y,\tau)}|f(x,t)|^q w(x,t)dxdt, \tag{2.1}$$

where φ is a measurable non-negative function defined on $Q\times\mathbb{R}_+$.

If $w\equiv 1$, then $M^{q,\varphi}_w(Q)\equiv M^{q,\phi}(Q)$ with $\phi(\mathcal{I}_r(y,\tau))=\varphi(\mathcal{I}_r(y,\tau))^q r^{n+2}$.

If $\varphi\equiv r^{(\lambda-n-2)/q}$ and $w\equiv 1$, then $M^{q,\varphi}_w(Q)\equiv L^{q,\lambda}(Q)$, $\lambda\in(0,n+2)$.

If $\varphi \equiv w^{-1/q}$, then $M_w^{q,\varphi}(Q) \equiv L_w^q(Q)$.

In [15], we proved maximal inequality in $M^{q,\varphi}$ under quite general condition on φ. In fact, we consider a couple of measurable non-negative weights (φ_1, φ_2) for which the next result holds.

Theorem 2.2 *Suppose that, for any fixed $(y,\tau) \in \mathbb{R}^{n+1}$ and any $r > 0$, there exists $\kappa > 0$ such that*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(\mathcal{I}_\sigma(y,\tau))\sigma^{\frac{n+2}{q}}}{s^{\frac{n+2}{q}}} \leq \kappa\, \varphi_2(\mathcal{I}_r(y,\tau)) \,. \tag{2.2}$$

Then for any $q > 1$, $\mathcal{M}$ is bounded from M^{q,φ_1} to M^{q,φ_2} and

$$\|\mathcal{M}f\|_{M^{q,\varphi_2}(\mathbb{R}^{n+1})} \leq c(n,q,\kappa)\|f\|_{M^{q,\varphi_1}(\mathbb{R}^{n+1})} \,.$$

We need analogous result in $M_w^{q,\varphi}$. Let $L^{\infty,v}(0,\infty)$ be the space of all functions $g(\zeta)$, $\zeta > 0$ with finite norm

$$\|g\|_{L^{\infty,v}(0,\infty)} = \sup_{\zeta>0} v(\zeta)g(\zeta)$$

and $L^\infty(0,\infty) \equiv L^{\infty,1}(0,\infty)$. Denote by $\mathfrak{M}(0,\infty)$ the set of all measurable functions on $(0,\infty)$, by $\mathfrak{M}^+(0,\infty)$ the subset of all nonnegative functions on $(0,\infty)$, by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all non-decreasing functions in $\mathfrak{M}^+(0,\infty)$, and by $\mathbb{A}$ the subset

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{\zeta\to 0+} \varphi(\zeta) = 0 \right\}.$$

Let ν be a continuous and non-negative function on $(0,\infty)$. We define the *sup-operator* $\overline{S}_\nu$ acting on $\mathfrak{M}(0,\infty)$ by

$$(\overline{S}_\nu g)(\eta) := \|\nu\, g\|_{L^\infty(\eta,\infty)}, \quad \eta \in (0,\infty) \,.$$

This operator turns to be bounded on $\mathbb{A}$, as it is proved in [2].

Theorem 2.3 ([2]) *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L^\infty(\eta,\infty)} < \infty$ for any $\eta > 0$ and let ν be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_\nu$ is bounded from $L^{\infty,v_1}(0,\infty)$ to $L^{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\|v_2\overline{S}_\nu(\|v_1\|^{-1}_{L^\infty(\cdot,\infty)})\|_{L^\infty(0,\infty)} < \infty. \tag{2.3}$$

The following lemma give some estimates of the maximal function.

Lemma 2.4 *Let $w \in A_q$, $q > 1$. Then the inequality*

$$\|\mathcal{M}f\|_{L_w^q(\mathcal{I}_r)} \leq c\Big(\|f\|_{L_w^q(\mathcal{I}_{2r})} + w(\mathcal{I}_r)^{\frac{1}{q}} \cdot \sup_{s>2r} s^{-n-2}\|f\|_{L^1(\mathcal{I}_s)}\Big) \tag{2.4}$$

holds for all $f \in L^q_{w,\mathrm{loc}}(\mathbb{R}^{n+1})$.

Proof Fix a point $(y,\tau) \in \mathbb{R}^{n+1}$ and write $\mathcal{I}_r \equiv \mathcal{I}_r(y,\tau)$ and $\mathcal{I}_s \equiv \mathcal{I}_s(y,\tau)$. Consider the decomposition

$$f = f_1 + f_2 = f\chi_{2\mathcal{I}_r} + f\chi_{\complement(2\mathcal{I}_r)} \,.$$

Then

$$\|\mathcal{M}f\|_{L_w^q(\mathcal{I}_r)} \leq \|\mathcal{M}f_1\|_{L_w^q(\mathcal{I}_r)} + \|\mathcal{M}f_2\|_{L_w^q(\mathcal{I}_r)}.$$

Because of (1.3), we have

$$\|\mathcal{M}f_1\|_{L^q_w(\mathcal{I}_r)} \le \|\mathcal{M}f_1\|_{L^q_w(\mathbb{R}^{n+1})} \le c\|f_1\|_{L^q_w(\mathbb{R}^{n+1})} = c\|f\|_{L^q_w(\mathcal{I}_{2r})}.$$

As in [15, Lemma 3.2], we get that for all $(x,t)\in\mathcal{I}_r$ holds

$$\mathcal{M}f_2(x,t) \le c \sup_{s>2r} \frac{1}{|\mathcal{I}_s|}\int_{\mathcal{I}_s} |f(z,\zeta)|\,dzd\zeta, \tag{2.5}$$

where the right-hand side does not depend on (x,t) anymore. Hence

$$\|\mathcal{M}f_2\|_{L^q_w(\mathcal{I}_r)} \le c \sup_{s>2r} \frac{1}{|\mathcal{I}_s|}\int_{\mathcal{I}_s} |f(z,\zeta)|\,dzd\zeta \left(\int_{\mathcal{I}_r} w(x,t)\,dxdt\right)^{\frac{1}{q}}.$$

Unifying the both estimates, we get

$$\|\mathcal{M}f\|_{L^q_w(\mathcal{I}_r)} \le c\left(\|f\|_{L^q_w(2\mathcal{I}_r)} + w(\mathcal{I}_r)^{\frac{1}{q}}\cdot \sup_{s>2r}\frac{1}{|\mathcal{I}_s|}\int_{\mathcal{I}_s}|f(z,\zeta)|\,dzd\zeta\right). \qquad \square$$

Lemma 2.5 *Let $w\in A_q, q>1$. Then the inequality*

$$\|\mathcal{M}f\|_{L^q_w(\mathcal{I}_r)} \le cw(\mathcal{I}_r)^{\frac{1}{q}}\cdot \sup_{s>2r} w(\mathcal{I}_s)^{-\frac{1}{q}}\,\|f\|_{L^q_w(\mathcal{I}_s)} \tag{2.6}$$

holds for all $f\in L^q_{w,\mathrm{loc}}(\mathbb{R}^{n+1})$.

Proof Denote by

$$\mathcal{A}_1 := w(\mathcal{I}_r)^{\frac{1}{q}}\cdot\sup_{s>2r}\frac{1}{|\mathcal{I}_s|}\int_{\mathcal{I}_s}|f(z,\zeta)|\,dzd\zeta\,,$$
$$\mathcal{A}_2 := \|f\|_{L^q_w(2\mathcal{I}_r)}\,.$$

Applying Hölder's inequality and (1.1), we get

$$\mathcal{A}_1 \le [w]_q^{\frac{1}{q}}w(\mathcal{I}_r)^{\frac{1}{q}}\cdot\sup_{s>2r}\frac{1}{w(\mathcal{I}_s)^{\frac{1}{q}}}\left(\int_{\mathcal{I}_s}|f(z,\zeta)|^q w(z,\zeta)\,dzd\zeta\right)^{\frac{1}{q}}.$$

On the other hand,

$$\begin{aligned} & w(\mathcal{I}_r)^{\frac{1}{q}}\cdot\sup_{s>2r}\frac{1}{w(\mathcal{I}_s)^{\frac{1}{q}}}\left(\int_{\mathcal{I}_s}|f(z,\zeta)|^q w(z,\zeta)\,dzd\zeta\right)^{\frac{1}{q}} \\ &\ge\ [w]_q^{-\frac{1}{q}}w(\mathcal{I}_r)^{\frac{1}{q}}\left(\int_{\mathcal{I}_{2r}}|f(z,\zeta)|^q w(z,\zeta)\,dzd\zeta\right)^{\frac{1}{q}}\cdot\sup_{s>2r}\frac{1}{w(\mathcal{I}_s)^{\frac{1}{q}}} = c\mathcal{A}_2\,. \end{aligned}$$

Hence by Lemma 2.4 we get (2.6) (see [15]). $\square$

Theorem 2.6 (Maximal inequality) *Let $w\in A_q, q>1$ and φ satisfy*

$$\sup_{r<s<\infty}\frac{\operatorname*{ess\,inf}_{s<\sigma<\infty}\varphi(\mathcal{I}_\sigma(y,\tau))w\big(\mathcal{I}_\sigma(y,\tau)\big)^{\frac{1}{q}}}{w\big(\mathcal{I}_s(y,\tau)\big)^{\frac{1}{q}}} \le \kappa\,\varphi(\mathcal{I}_r(y,\tau)) \tag{2.7}$$

with κ independent of r and (y,τ). Then

$$\|f\|_{M^{q,\varphi}_w(\mathbb{R}^{n+1})} \le \|\mathcal{M}f\|_{M^{q,\varphi}_w(\mathbb{R}^{n+1})} \le c(n,q,\kappa)\|f\|_{M^{q,\varphi}_w(\mathbb{R}^{n+1})}, \quad \forall f\in M^{q,\varphi}_w(\mathbb{R}^{n+1})\,.$$

The proof follows by Lemma 2.5 as in [15, Theorem 3.4].

Impose in addition a kind of monotonicity condition on φ, precisely

$$\varphi(\mathcal{I}_r(y,\tau))^q|\mathcal{I}_r(y,\tau)| \le \varphi(\mathcal{I}_s(z,\zeta))^q|\mathcal{I}_s(z,\zeta)|, \quad \text{for all } \ \mathcal{I}_r(y,\tau)\subset\mathcal{I}_s(z,\zeta)\,. \tag{2.8}$$

This implies that, for a given $Q = \Omega \times (0,T] \subset \mathbb{R}^{n+1}$, there holds

$$\sup_{\substack{(y,\tau)\in Q\\ r>0}} \frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi(\mathcal{I}_r(y,\tau))^q |\mathcal{I}_r(y,\tau)|^\alpha} \le \varkappa, \quad \forall \alpha > 0 \tag{2.9}$$

with $\varkappa = \varkappa(n, q, \kappa, \varphi, Q)$ (see [8]).

3 Statement of the Problem and Main Result

Consider the Cauchy–Dirichlet problem

$$\begin{cases} u_t - D_\alpha(a^{\alpha\beta}(x,t)D_\beta u) = D_\alpha f^\alpha(x,t), & \text{in } Q,\\ u(x,t) = 0, & \text{on } \partial Q. \end{cases} \tag{3.1}$$

Suppose that the coefficients are uniformly bounded and uniformly elliptic, that is, there exist positive constants Λ and ν such that

$$\begin{cases} \|\mathbf{a}\|_{L^\infty(Q)} \le \Lambda,\\ a^{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \ge \nu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ for a.a. } (x,t) \in Q. \end{cases} \tag{3.2}$$

For each $\mathcal{C}_r(y,\tau)$ and for a fixed $x_1 \in (y_1 - r, y_1 + r)$, we consider the x_1-slice of $\mathcal{C}_r(y,\tau)$, that is,

$$\mathcal{C}_r^{x_1}(y,\tau) = \{(x',t) \in \mathbb{R}^{n-1}\times\mathbb{R} : |x'-y'| < r, |t-\tau| < r^2\}$$

and the integral average with respect to (x',t)

$$\overline{\mathbf{a}}_{\mathcal{C}_r(y,\tau)}(x_1) = \frac{1}{|\mathcal{C}_r^{x_1}(y,\tau)|}\int_{\mathcal{C}_r^{x_1}(y,\tau)} \mathbf{a}(x_1,x',t)\,dx'dt.$$

Definition 3.1 ([8]) *The couple* $(\mathbf{a},\Omega)$ *is* (δ,R)*-vanishing of co-dimension* 1*, if*

- *For every point* $(y,\tau) \in Q$ *and for every number* $r \in (0, \frac{1}{3}R]$ *such that*

$$\operatorname{dist}(y,\partial\Omega) > \sqrt{2}r, \tag{3.3}$$

there exists a coordinate system centered in $(y,\tau) \equiv (0,0)$*, which variables we still denote by* (x,t) *and in which*

$$\frac{1}{|\mathcal{C}_r(0,0)|}\int_{\mathcal{C}_r(0,0)} |\mathbf{a}(x,t) - \overline{\mathbf{a}}_{\mathcal{C}_r(0,0)}(x_1)|^2\,dxdt \le \delta^2. \tag{3.4}$$

- *For any point* $(y,\tau) \in Q$ *and for every number* $r \in (0, \frac{1}{3}R]$ *such that*

$$\operatorname{dist}(y,\partial\Omega) = \operatorname{dist}(y,x_0) \le \sqrt{2}r$$

for some $x_0 \in \partial\Omega$*, there exists a coordinate system centered in* $(x_0,\tau) \equiv (0,0)$*, which variables we still denote by* (x,t) *such that*

$$\Omega \cap \{x \in \mathcal{C}'_{3r}(0) : x_1 > 3r\delta\} \subset \Omega \cap \mathcal{C}'_{3r}(0) \subset \Omega \cap \{x \in \mathcal{C}'_{3r}(0) : x_1 > -3r\delta\} \tag{3.5}$$

and

$$\frac{1}{|\mathcal{C}_{3r}(0,0)|}\int_{\mathcal{C}_{3r}(0,0)} |\mathbf{a}(x,t) - \overline{\mathbf{a}}_{\mathcal{C}_{3r}(0,0)}(x_1)|^2\,dxdt \le \delta^2. \tag{3.6}$$

Because of the scaling invariance property of the Reifenberg domains (see [7, Lemma 5.2]), we can take for simplicity $R = 1$ while $\delta > 0$ is invariant under such scaling argument. The condition (3.3) means that away from the boundary the coefficients are *small* BMO in all variables except x_1. In that variable, they are *only measurable* and may have arbitrary jumps in the direction x_1. In addition, the domain Ω is (δ, R)-*Reifenberg flat* satisfying (3.5) (see Reifenberg [23]) and the coefficients have a small oscillation along the flat direction x' of the boundary and are only measurable along the normal direction x_1. The number $\sqrt{2}r$ in (3.3) is selected for convenience since we need to take the size of the cylinders in (3.4) such that there is enough room to have the rotation of $\mathcal{C}_r(y,\tau)$ in any spatial direction.

Recall that under a *weak solution* of (3.1), we mean a function

$$u \in C^0(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$$

that satisfies

$$\int_Q u\phi_t \, dxdt - \int_Q a^{\alpha\beta} D_\beta u D_\alpha \phi \, dxdt = \int_Q f^\alpha D_\alpha \phi \, dxdt$$

for all $\phi \in C_0^\infty(Q)$ with $\phi(x,T) \equiv 0$. Moreover, the following L^2-estimate holds

$$\int_Q |Du(x,t)|^2 \, dxdt \le c \int_Q |\mathbf{F}(x,t)|^2 \, dxdt, \tag{3.7}$$

where the constant c depends only on n, Λ, ν and T.

We suppose that $\mathbf{F} \in M_w^{p,\varphi}(Q)$ with $w \in A_{\frac{p}{2}}$, $p \in (2,\infty)$ and φ satisfies (2.7). This implies $\mathbf{F} \in L_w^p(Q)$. Precisely, choose $(y,\tau) \in Q$, then

$$\sup_{(z,\zeta)\in Q} \{|y-z| + \sqrt{|\tau-\zeta|}\} < \operatorname{diam} Q\,.$$

Hence there exist $r^* < \operatorname{diam} Q$ and $d > 0$, such that $Q \subset \mathcal{I}_{r^*}(y,\tau) \subset \mathcal{I}_{2d}(0,0)$ and

$$\begin{aligned}\|\mathbf{F}\|^2_{L_w^p(Q)} = \||\mathbf{F}|^2\|_{L_w^{\frac{p}{2}}(Q)} &\le \varphi(\mathcal{I}_{r^*}(y,\tau)) w\big(\mathcal{I}_{r^*}(y,\tau)\big)^{\frac{2}{p}} \||\mathbf{F}|^2\|_{M_w^{\frac{p}{2},\varphi}(Q)} \\ &\le c\varphi(\mathcal{I}_{2d}(0,0)) w\big(\mathcal{I}_{2d}(0,0)\big)^{\frac{2}{p}} \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}(Q)}\,.\end{aligned}$$

By the Hölder inequality and (1.1), we get

$$\begin{aligned}\|\mathbf{F}\|^2_{L^2(Q)} &= \int_Q |\mathbf{F}(x,t)|^2 w(x,t)^{\frac{2}{p}} w(x,t)^{-\frac{2}{p}} \, dxdt \\ &\le \left(\int_Q \big(|\mathbf{F}(x,t)|^2\big)^{\frac{p}{2}} w(x,t)\, dxdt\right)^{\frac{2}{p}} \left(\int_Q w(x,t)^{-\frac{2}{p-2}} \, dxdt\right)^{\frac{p-2}{p}} \\ &\le |Q| [w]_{\frac{p}{2}}^{\frac{2}{p}} \left(\frac{w(\mathcal{I}_{r^*}(y,\tau))}{w(Q)}\right)^{\frac{2}{p}} \varphi(\mathcal{I}_{r^*}(y,\tau)) \times \\ &\quad \times \frac{1}{\varphi(\mathcal{I}_{r^*}(y,\tau))} \left(\frac{1}{w(\mathcal{I}_{r^*}(y,\tau))} \int_{\mathcal{I}_{r^*}(y,\tau)} \big(|\mathbf{F}(x,t)|^2\big)^{\frac{p}{2}} w(x,t)\, dxdt\right)^{\frac{2}{p}}.\end{aligned}$$

Because of the doubling property (1.2) of w, we get

$$\left(\frac{w(\mathcal{I}_{r^*}(y,\tau))}{w(Q)}\right)^{\frac{2}{p}} \le [w]_{\frac{p}{2}}^{\frac{2}{p}} \frac{|\mathcal{I}_{r^*}(y,\tau)|}{|Q|}\,.$$

Hence, applying first (1.2) to w and then (2.8) to φ, we get

$$\|\mathbf{F}\|^2_{L^2(Q)} \le c\,\varphi(\mathcal{I}_{2d}(0,0))\,w\big(\mathcal{I}_{2d}(0,0)\big)^{\frac{2}{p}}\,\||\mathbf{F}|^2\|_{M_w^{\frac{p}{2},\varphi}(Q)}\,. \tag{3.8}$$

The last estimate ensures the *existence of unique weak solution* u of (3.1) according to [3, 9]. Then our goal is to show that the gradient of this solution possesses the same regularity as the right-hand side of the equation (3.1).

Theorem 3.2 *Let $p \in (2,\infty)$, $w \in A_{\frac{p}{2}}$ and φ satisfy (2.7). Then there exists a small positive constant $\delta = \delta(n, L, \nu, p, \kappa, \varphi, [w], Q)$ such that if the couple $(\mathbf{a},\Omega)$ is (δ, R)-vanishing of codimension 1 and $\mathbf{F} \in M_w^{p,\varphi}(Q)$, then $Du \in M_w^{p,\varphi}(Q)$ and the following estimate holds:*

$$\|Du\|_{M_w^{p,\varphi}(Q)} \le c\|\mathbf{F}\|_{M_w^{p,\varphi}(Q)}$$

with a constant c depending on known quantities.

4 Gradient Estimate in $M_w^{p,\varphi}(Q)$

In what follows, we establish a suitable version of the Vitali covering lemma. We use it to derive a power decay estimate of the upper level sets for the Hardy–Littlewood maximal function of the spatial gradient of the weak solution. The regularity estimate in the main result then follows by the standard procedure of summation over the level sets.

Fix a point $(y_0,\tau_0) \in Q$, take $\mathcal{I}_r(y_0,\tau_0)$ and consider $Q_r = \mathcal{I}_r(y_0,\tau_0) \cap Q$. Let u be a weak solution of (3.1). Then we define the sets

$$\begin{aligned} \mathfrak{C} &= \{(x,t) \in Q_r \colon \mathcal{M}(|Du|^2) > \lambda_1^2\}, \\ \mathfrak{D} &= \{(x,t) \in Q_r \colon \mathcal{M}(|Du|^2) > 1\} \cup \{\mathcal{M}(|\mathbf{F}|^2) > \delta^2\} \end{aligned} \tag{4.1}$$

with $\lambda_1 > 1$ and $\delta > 0$. It is easy to see that $\mathfrak{C} \subset \mathfrak{D} \subset Q_r$. For each $(y,\tau) \in \mathfrak{C}$ consider $\mathcal{C}_\rho(y,\tau)$ and define the following auxiliary function

$$\Theta(\rho) = \frac{w(\mathfrak{C} \cap \mathcal{C}_\rho(y,\tau))}{w(\mathcal{C}_\rho(y,\tau))}, \quad \rho > 0, w \in A_q, q \in (1,\infty)\,.$$

Because of (1.2) $\Theta(0) = \lim_{\rho\to 0_+} \Theta(\rho) = 1$ and $\lim_{\rho\to+\infty} \Theta(\rho) = 0$. We start with some preliminary lemma taking $R = 1$ because of the invariance property of the Reifenberg domain [9] (see also [3–8]).

Lemma 4.1 *Let $(\mathbf{a},\Omega)$ be $(\delta,1)$-vanishing of codimension 1 and $\mathfrak{C}$, $\mathfrak{D}$ and $\Theta(\rho)$ be as above. Suppose that, for any $(y,\tau) \in \mathfrak{C}$, there exists $\varepsilon \in (0,1)$ such that $\Theta(1) < \varepsilon$. Then $\Theta(\rho) \ge \varepsilon$ implies $Q_r \cap \mathcal{C}_\rho(y,\tau) \subset \mathfrak{D}$ and*

$$w(\mathfrak{C}) \le \varepsilon[w]_q^2\left(\frac{10\sqrt{3}}{1-\delta}\right)^{(n+2)q} w(\mathfrak{D})\,. \tag{4.2}$$

Proof The implication follows by (1.2) and [7, Lemma 5.3]. Since $\Theta(1) < \varepsilon$, there exists $\rho_{(y,\tau)} \in (0,1)$ such that $\Theta(\rho_{(y,\tau)}) = \varepsilon$ and $\Theta(\rho) < \varepsilon$ for all $\rho > \rho_{(y,\tau)}$ and $(y,\tau) \in \mathfrak{C}$. Consider the family of cylinders $\{\mathcal{C}_{\rho_{(y,\tau)}}(y,\tau)\}_{(y,\tau)\in\mathfrak{C}}$ which is an open covering of $\mathfrak{C}$. By the Vitali lemma (cf. [25, Lemma I.3.1]), there exists a disjoint sub-collection $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\ge1}$ with $\rho_i = \rho_{(y_i,\tau_i)} \in (0,1)$, $(y_i,\tau_i) \in \mathfrak{C}$ such that $\Theta(\rho_i) = \varepsilon$,

$$\sum_{i\ge1} |\mathcal{C}_{\rho_i}(y_i,\tau_i)| \ge c|\mathfrak{C}| \quad \text{and} \quad \mathfrak{C} \subset \bigcup_{i\ge1} \mathcal{C}_{5\rho_i}(y_i,\tau_i)$$

with a positive constant $c = c(n)$. Since $\Theta(5\rho_i) < \varepsilon$, by (1.2), we have

$$w(\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)) < \varepsilon w(\mathcal{C}_{5\rho_i}(y_i,\tau_i)) \le \varepsilon [w]_q 5^{(n+2)q} w(\mathcal{C}_{\rho_i}(y_i,\tau_i))\,.$$

Furthermore, making use of the bound (see [5, 7])

$$\sup_{\substack{(y,\tau)\in Q_r\\ 0<\rho<1}} \frac{|\mathcal{C}_\rho(y,\tau)|}{|Q_r \cap \mathcal{C}_\rho(y,\tau)|} \le \left(\frac{2\sqrt{2}}{1-\delta}\right)^{n+2},$$

we get by (1.2)

$$w(\mathcal{C}_{\rho_i}(y_i,\tau_i)) \le [w]_q \left(\frac{2\sqrt{2}}{1-\delta}\right)^{q(n+2)} w(Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)).$$

Now we have

$$\begin{aligned} w(\mathfrak{C}) &= w\left(\bigcup_{i\ge 1} \left(\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)\right)\right) \le \sum_{i\ge 1} w(\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)) \\ &< \varepsilon \sum_{i\ge 1} w(\mathcal{C}_{5\rho_i}(y_i,\tau_i)) \le \varepsilon [w]_q^2 5^{q(n+2)} \sum_{i\ge 1} w(\mathcal{C}_{\rho_i}(y_i,\tau_i)) \\ &\le \varepsilon [w]_q^2 \left(\frac{10\sqrt{2}}{1-\delta}\right)^{q(n+2)} \sum_{i\ge 1} w(Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i))\,. \end{aligned}$$

Having in mind that $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\ge 1}$ are mutually disjoint, $\Theta(\rho_i) = \varepsilon$ and (4.2), we get

$$\begin{aligned} w(\mathfrak{C}) &\le \varepsilon [w]_q^2 \left(\frac{10\sqrt{2}}{1-\delta}\right)^{q(n+2)} w\left(\bigcup_{i\ge 1} Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)\right) \\ &\le \varepsilon [w]_q^2 \left(\frac{10\sqrt{2}}{1-\delta}\right)^{q(n+2)} w(\mathfrak{D}). \end{aligned}$$

□

Lemma 4.2 *Let $\Theta(1) < \varepsilon$. Then for each $k \in \mathbb{N}$ holds*

$$\begin{aligned} w\left(\{(x,t) \in Q_r \colon \mathcal{M}(|Du|^2) > \lambda_1^{2k}\}\right) &\le \varepsilon_1^k w\left(\{(x,t)\in Q_r \colon \mathcal{M}(|Du|^2) > 1\}\right) \\ &\quad + \sum_{i=1}^{k} \varepsilon_1^i w(\{(x,t)\in Q_r \colon \mathcal{M}(|\mathbf{F}|^2) > \delta^2 \lambda_1^{2(k-i)}\}), \end{aligned} \tag{4.3}$$

where $\varepsilon_1 = \varepsilon\,[w]_q^2 \big(\frac{10\sqrt{2}}{1-\delta}\big)^{q(n+2)}$.

Proof By Lemma 4.1, we have

$$\begin{aligned} w\left(\{(x,t) \in Q_r \colon \mathcal{M}(|Du|^2) > \lambda_1^{2}\}\right) \le\ & \varepsilon_1 w\left(\{(x,t)\in Q_r \colon \mathcal{M}(|Du|^2) > 1\}\right) \\ & + \varepsilon_1\, w\left(\{(x,t)\in Q_r \colon \mathcal{M}(|\mathbf{F}|^2) > \delta^2\}\right), \end{aligned}$$

where $\varepsilon_1 = \varepsilon\,[w]_q^2 \big(\frac{10\sqrt{2}}{1-\delta}\big)^{q(n+2)}$.

The last inequality is exactly (4.3) with $k = 1$. Furthermore, we proceed with the proof by induction, as it is done in [4, Corollary 4.15]. Suppose that (4.3) holds true for some $k \ge 1$. Define the functions $u_1 = \frac{u}{\lambda_1}$ and $\mathbf{F}_1 = \frac{\mathbf{F}}{\lambda_1}$. It is easy to see that u_1 is a weak solution to (3.1) with a right-hand side $\mathbf{F}_1$. Hence Lemma 4.1 holds with sets $\mathfrak{C}$ and $\mathfrak{D}$ corresponding to u_1 as defined in (4.1). According to (4.3), the inductive assumption holds true for u_1 with the same $k \ge 1$. The definition of u_1 ensures the inductive passage from k to $k+1$ for u. Namely,

$$w(\{(x,t)\in Q_r \colon \mathcal{M}(|Du|^2) > \lambda_1^{2(k+1)}\}) = w(\{(x,t)\in Q_r \colon \mathcal{M}(|Du_1|^2) > \lambda_1^{2k}\})$$

$$
\begin{aligned}
&\le \varepsilon_1^k\, w(\{(x,t)\in Q_r\colon \mathcal{M}(|Du_1|^2)>1\})\\
&\quad+\sum_{i=1}^{k}\varepsilon_1^i w(\{(x,t)\in Q_r\colon \mathcal{M}(|\mathbf{F}_1|^2)>\delta^2\lambda_1^{2(k-i)}\})\\
&=\varepsilon_1^k\, w(\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>\lambda_1^2\})\\
&\quad+\sum_{i=1}^{k}\varepsilon_1^i\, w(\{(x,t)\in Q_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k-i)}\lambda_1^2\})\\
&\le \varepsilon_1^{k+1}\, w(\{(x,t)\in Q_r\colon \mathcal{M}(|Du|^2)>1\})\\
&\quad+\sum_{i=1}^{k+1}\varepsilon_1^i\, w(\{(x,t)\in Q_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k+1-i)}\}).
\end{aligned}
$$

Noting that because of the arbitrary choice of the point $(y_0,\tau_0)\in Q$, the above estimates hold locally for any $Q_r=\mathcal{I}_r(y,\tau)\cap Q$ with $(y,\tau)\in Q$.

Lemma 4.3 *Let $h\in L^1(Q)$ be a nonnegative function, $w\in A_q$, $q\in(1,\infty)$, φ satisfy (2.7), and $\theta>0,\lambda>1$ be constants. Then $h\in M_w^{q,\varphi}(Q)$ if and only if*

$$
\mathcal{S}:=\sup_{\substack{(y,\tau)\in Q\\ r>0}}\sum_{k\ge 1}\frac{\lambda^{kq}w(\{(x,t)\in Q_r\colon h(x,t)>\theta\lambda^k\})}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}<\infty\,.
$$

Moreover,

$$
\frac{1}{c}\mathcal{S}\le \|h\|^q_{M_w^{q,\varphi}(Q)}\le c(1+\mathcal{S}),
$$

where $c=c(\theta,\lambda,q,\kappa,\varphi,[w],Q)$.

Proof Consider Q_r as above, then

$$
\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q}\frac{1}{w\big(\mathcal{I}_r(y,\tau)\big)}\int_{Q_r}h(x,t)^q w(x,t)\,dxdt\\
&\quad=\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q}\frac{1}{w\big(\mathcal{I}_r(y,\tau)\big)}\int_{\{(x,t)\in Q_r:h\le\theta\lambda\}}h(x,t)^q w(x,t)\,dxdt\\
&\qquad+\sum_{k\ge1}\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q}\frac{1}{w\big(\mathcal{I}_r(y,\tau)\big)}\int_{\{(x,t)\in Q_r:\theta\lambda^k<h\le\theta\lambda^{k+1}\}}h(x,t)^q w(x,t)\,dxdt\\
&\quad\le(\theta\lambda)^q\frac{w(Q_r)}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\\
&\qquad+\sum_{k\ge1}\frac{(\theta\lambda^{k+1})^q}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}w\big(\{(x,t)\in Q_r:h(x,t)>\theta\lambda^k\}\big)\\
&\quad\le(\theta\lambda)^q\Big[\frac{c_1|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))^q|\mathcal{I}_r(y,\tau)|}\Big(\frac{|Q_r|}{|\mathcal{I}_r(y,\tau)|}\Big)^{\tau_1-1}\\
&\qquad+\sum_{k\ge1}\frac{\lambda^{kq}w\big(\{(x,t)\in Q_r:h(x,t)>\theta\lambda^k\}\big)}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\Big].
\end{aligned}
$$

Taking the supremum over $\mathcal{I}_r(y,\tau)$ and making use of (1.2) and (2.9), we get

$$
\|h\|^q_{M_w^{q,\varphi}(Q)}\le c\,(1+\mathcal{S})\,.
$$

On the other hand

$$\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\int_{Q_r} h(x,t)^q w(x,t)\,dxdt\\
&=\frac{q}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\int_{Q_r}\bigg(\int_0^{h(x,t)}\xi^{q-1}d\xi\bigg)w(x,t)\,dxdt\\
&=\frac{q}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\int_0^{\infty} w\big(\{(x,t)\in Q_r : h(x,t)>\xi\}\big)\xi^{q-1}\,d\xi\\
&\geq\frac{q}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\sum_{k\geq 1} w\big(\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}\big)\int_{\theta\lambda^{k-1}}^{\theta\lambda^k}\xi^{q-1}\,d\xi\\
&=\theta^q(1-\lambda^{-q})\frac{1}{\varphi(\mathcal{I}_r(y,\tau))^q w\big(\mathcal{I}_r(y,\tau)\big)}\sum_{k\geq 1}\lambda^{kq} w\big(\{(x,t)\in Q_r : h(x,t)>\theta\lambda^k\}\big).
\end{aligned}$$

Taking again the supremum over $\mathcal{I}_r(y,\tau)$, we get $\|h\|^q_{M^{q,\varphi}_w(Q)} \geq \frac{1}{c}\mathcal{S}$.

We are in a position now to prove Theorem 3.2.

Proof Recall that $\mathbf{F}\in M^{p,\varphi}_w(Q)$, $w\in A_{\frac{p}{2}}$, $p\in(2,\infty)$ and φ satisfies (2.7). Because of the scaling invariance property of (3.1) under a normalization, we can assume that the norm of $\mathbf{F}$ is small enough. In fact, taking

$$\bar{u}(x,t)=\frac{\delta u(x,t)}{\sqrt{\||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}_w(Q)}}}\qquad\text{and}\qquad \bar{\mathbf{F}}(x,t)=\frac{\delta \mathbf{F}(x,t)}{\sqrt{\||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}_w(Q)}}}$$

instead of u and $\mathbf{F}$ in (3.1), we get $\||\bar{\mathbf{F}}|^2\|_{M^{\frac{p}{2},\varphi}_w(Q)}=\delta^2$. Then we need to prove boundedness of the norm of $D\bar{u}$. Hence, it is enough to get $\|\mathcal{M}(|D\bar{u}|^2)\|_{M^{\frac{p}{2},\varphi}_w(Q)}\leq c$. For this goal, we apply Lemma 4.3 with $h=\mathcal{M}(|D\bar{u}|^2)$, $\lambda=\lambda_1^2$, $\theta=1$ and $q=\frac{p}{2}$.

Take $\mathfrak{C}$ corresponding to the solution $\bar{u}$. We note that, for each $(y,\tau)\in\mathfrak{C}$,

$$\begin{aligned}
\frac{w(\mathfrak{C}\cap\mathcal{C}_1(y,\tau))}{w(\mathcal{C}_1(y,\tau))}&\leq cw(\mathfrak{C})=cw\big(\{(x,t)\in Q_r\colon \mathcal{M}(|D\bar{u}|^2)>\lambda_1^2\}\big)\\
&\leq c\int_{Q_r}\mathcal{M}(|D\bar{u}|^2)(x,t)w(x,t)\,dxdt\leq c\int_{Q_r}|D\bar{u}(x,t)|^2w(x,t)\,dxdt\\
&\leq c\int_{Q}|D\bar{u}(x,t)|^2w(x,t)\,dxdt\leq c\int_{Q}|\bar{\mathbf{F}}(x,t)|^2w(x,t)\,dxdt\\
&\leq c\||\bar{\mathbf{F}}|^2\|_{L^{\frac{p}{2},\varphi}_w(Q)}\leq c\delta^2,
\end{aligned}$$

according to (3.8). Taking δ small enough, we get

$$\Theta(1)=\frac{w(\mathfrak{C}\cap\mathcal{C}_1(y,\tau))}{w(\mathcal{C}_1(y,\tau))}\leq c\delta^2<\varepsilon\,.$$

Therefore, Lemma 4.2 gives

$$\begin{aligned}
&\sum_{k\geq 1}\lambda_1^{2k\frac{p}{2}}\frac{w\big(\{(x,t)\in Q_r\colon \mathcal{M}(|D\bar{u}|^2)>\lambda_1^{2k}\}\big)}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}r^{n+2}}\\
&\quad\leq\sum_{k\geq 1}\lambda_1^{kp}\varepsilon_1^k\frac{w\big(\{(x,t)\in Q_r\colon \mathcal{M}(|D\bar{u}|^2)>1\}\big)}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}\,w\big(\mathcal{I}_r(y,\tau)\big)}\\
&\qquad+\sum_{k\geq 1}\sum_{i=1}^{k}\lambda_1^{kp}\varepsilon_1^i\frac{w\big(\{(x,t)\in Q_r\colon \mathcal{M}(|\bar{\mathbf{F}}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big)}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}\,w\big(\mathcal{I}_r(y,\tau)\big)}
\end{aligned}$$

$$\leq \sum_{k\geq 1} \left(\lambda_1^p \varepsilon_1\right)^k \frac{w(Q_r)}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}\, w\big(\mathcal{I}_r(y,\tau)\big)}$$

$$+ \sum_{i\geq 1} \left(\lambda_1^p \varepsilon_1\right)^i \underbrace{\sum_{k\geq i} \lambda_1^{p(k-i)} \frac{w\big(\{(x,t)\in Q_r \colon \mathcal{M}(|\bar{\mathbf{F}}|^2) > \delta^2 \lambda_1^{2(k-i)}\}\big)}{\varphi^{\frac{p}{2}}(\mathcal{I}_r(y,\tau))\, w\big(\mathcal{I}_r(y,\tau)\big)}}_{\mathcal{S}'}$$

$$\leq \kappa_4 \sum_{k\geq 1} \left(\lambda_1^p \varepsilon_1\right)^k + \sum_{i\geq 1} \left(\lambda_1^p \varepsilon_1\right)^i \mathcal{S}',$$

where we have used (3.8) for the last inequality. Note that

$$\mathcal{S}' = \sum_{k\geq i} \lambda_1^{p(k-i)} \frac{w\big(\{(x,t)\in Q_r \colon \mathcal{M}(|\bar{\mathbf{F}}|^2) > \delta^2 \lambda_1^{2(k-i)}\}\big)}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}} w\big(\mathcal{I}_r(y,\tau)\big)}$$

$$= \sum_{k\geq i} \left(\lambda_1^{2(k-i)}\right)^{\frac{p}{2}} \frac{w\big(\{(x,t)\in Q_r \colon \mathcal{M}\big(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\big) > \lambda_1^{2(k-i)}\}\big)}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}} w\big(\mathcal{I}_r(y,\tau)\big)}$$

$$\leq \frac{k_p}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}\, w\big(\mathcal{I}_r(y,\tau)\big)} \left(w(Q_r) + \int_{Q_r} \mathcal{M}\left(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\right)^{\frac{p}{2}} (x,t) w(x,t)\, dxdt \right)$$

$$\leq \frac{k_p}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}\, w\big(\mathcal{I}_r(y,\tau)\big)} \left(w(Q_r) + \int_{Q_r} \left(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\right)^{\frac{p}{2}} (x,t) w(x,t)\, dxdt \right).$$

Taking again the supremum over $\mathcal{I}_r(y,\tau)$ and making use of (3.8), we get

$$\mathcal{S}' \leq c\left(1 + \left\| \left| \frac{\bar{\mathbf{F}}}{\delta} \right|^2 \right\|^{\frac{p}{2}}_{M^{\frac{p}{2},\varphi}(Q)} \right) \leq c\left(1 + \frac{1}{\delta^p} \| |\bar{\mathbf{F}}|^2 \|^{\frac{p}{2}}_{M^{\frac{p}{2},\varphi}(Q)} \right) \leq c.$$

Taking ε, and the corresponding δ, small enough such that $0 < \lambda_1^p \varepsilon_1 < 1$, we get

$$\sum_{k\geq 1} \lambda_1^{2k\frac{p}{2}} \frac{|\{(x,t)\in Q_r \colon \mathcal{M}(|D\bar{u}|^2) > \lambda_1^{2k}\}|}{\varphi(\mathcal{I}_r(y,\tau))^{\frac{p}{2}}\, w\big(\mathcal{I}_r(y,\tau)\big)} \leq c \sum_{k\geq 1} \left(\lambda_1^p \varepsilon_1\right)^k \leq c\,.$$

Taking again the supremum over $(y,\tau)\in Q, r>0$ in the estimates above and making use of Lemma 4.3, we find that $\|\mathcal{M}(|D\bar{u}|^2)\|_{M_w^{\frac{p}{2},\varphi}(Q)} \leq c$. This way, (2.7) and the definition of $\bar{u}$ imply

$$\| |Du|^2 \|_{M_w^{\frac{p}{2},\varphi}(Q)} \leq c \| |\mathbf{F}|^2 \|_{M_w^{\frac{p}{2},\varphi}(Q)}$$

with constant depending on known quantities. □

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