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On the new hyperbolic wave solutions to Wu-Zhang system models

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Abstract
In this study, some solitary wave solutions of the Wu-Zhang system are analyzed using the modified expansion function method and the sine–Gordon expansion method. Solitary wave solutions of this non-linear mathematical model consisting of hyperbolic and trigonometric function structures are get. Two and three dimensional, density graphics of the solitary solutions of the mathematical models are drawn by choosing the appropriate parameters. It was seen that all solution functions found as a result provide the mathematical model. In this study, Wolfram Mathematica software program was used for all mathematical calculations.

Keywords The sine–Gordon expansion method (SGEM) · The modified expansion function method (MEFM) · Wu-Zhang system

1 Introduction

Each of the non-linear partial differential equations is a mathematical model that can help to understand and solve problems such as physics, engineering, chemistry, biology. Recently, some approaches have been improved to search analytical solutions of several non-linear mathematical models. Some of those, the extended (G'/G)-expansion method (Kumar et al.

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2012), the B ä cklund transformation method (Lu et al. 2006), the simplified Hirota's method (Darvishi and Najafi 2012), the transformed rational function method (Seadawy 2017), the modified simple equation method (Jawad et al. 2010), the multiple exp-function method (Ma et al. 2010), the extended tanh method (Abdou 2007), the direct algebraic method (Willy et al. 1986), the Jacobi elliptic function method (Shikuo et al. 2001), the homogeneous balance method (Wang et al. 1996), the local fractional Riccati differential equation method (Baskonus and Bulut 2016), the improved Bernoulli sub-equation function method (Baskonus and Bulut 2015), Cornejo-Perez and Rosu method (2005) among others. We used the sine–Gordon expansion method (Chen and Zhenya 2005) and the modified expansion function method (He and Wu 2006) for this purpose. Usually, diverse computational techniques have been improved to obtain solutions for different NLEEs (Gao et al. 2017; Bulut et al. 2018; Bulut et al. 1637; Bulut et al. 2017a; Bulut et al. 2017b; Bulut et al. 2017c; Sulaiman et al. 2018; Zhang 2015; Cattani et al. 2018; Yokus et al. 2018; Inc et al. 2018; Baskonus et al. 2017a, 2017b, 2017c; Zayed and Ibrahim 2012; Hosseini et al. 2018; Hafez et al. 2014; Khan and Akbar 2013; Esen and Yağmurlu 2016; Karaagac et al. 1019).

In this study, it is planned to get the solitary solutions of Wu-Zhang system using MEFM (Baskonus et al. 2016) and the SGEM (Chen and Yan 2005).

The Wu-Zhang system (Jafari et al. 2015) is given as follows:

$$u_t + uu_x + v_x = 0, \tag{1}$$

$$v_t + (uv)_x + \frac{1}{3}u_{xxx} = 0. \tag{2}$$

In the equation system v is the height of the water and u is the surface speed of water.

2 Methods

2.1 Overview of the MEFM

The general form of the non-linear partial differential equation (NPDE) is as fallows:

$$P(u, u_x, u_t, u_{xxx}, v_x, v_t, uv_x, vu_x, \dots) = 0, \tag{3}$$

where $u = u(x, t)$ is unknown function, P is a polynomial in $u(x, t)$ and its derivatives.

Step 1 The travelling wave transformation is as follows:

$$u(x, t) = u(\eta), \quad \eta = r(x - ct). \tag{4}$$

If Eq. (4) is substituted in Eq. (3), the general form of non-linear ordinary differential equation (NODE) is obtained:

$$N(u, u^2, v, u'', \dots) = 0. \tag{5}$$

Step 2 We suppose the following wave equation to be the solution to Eq. (5):

$$u(\eta)=\frac{\sum_{j=0}^{n}A_j[e^{-\vartheta(\eta)}]^j}{\sum_{i=0}^{m}B_i[e^{-\vartheta(\eta)}]^i}=\frac{A_0+A_1e^{-\vartheta}+\ldots+A_ne^{-n\vartheta}}{B_0+B_1e^{-\vartheta}+\ldots+B_me^{-m\vartheta}}, \tag{6}$$

where $A_j, B_i, (0\le j\le n\,,\ 0\le i\le m)$ are constants.

$$\vartheta'(\eta)=e^{-\vartheta(\eta)}+ke^{\vartheta(\eta)}+\lambda. \tag{7}$$

Equation (7) has the following families of solutions (Naher and Abdullah 2014):
Family 1: When, $k\neq 0$ and $\lambda^2-4k>0$,

$$\vartheta(\eta)=ln(\frac{-\sqrt{\lambda^2-4k}}{2k}tanh(\frac{\sqrt{\lambda^2-4k}}{2}(\eta+E))-\frac{\lambda}{2k}). \tag{8}$$

Family 2: When, $k\neq 0$ and $\lambda^2-4k<0$,

$$\vartheta(\eta)=ln(\frac{\sqrt{-\lambda^2+4k}}{2k}tan(\frac{\sqrt{-\lambda^2+4k}}{2}(\eta+E))-\frac{\lambda}{2k}). \tag{9}$$

Family 3: When, $k=0$, $\lambda\neq 0$ and $\lambda^2-4k>0$,

$$\vartheta(\eta)=-ln(\frac{\lambda}{e^{\lambda(\eta+E)}-1}). \tag{10}$$

Family 4: When, $k\neq 0$, $\lambda\neq 0$ and $\lambda^2-4k=0$,

$$\vartheta(\eta)=ln(-\frac{2\lambda(\eta+E)+4}{\lambda^2(\eta+E)}). \tag{11}$$

Family 5: When, $k=0$, $\lambda=0$ and $\lambda^2-4k=0$,

$$\vartheta(\eta)=ln(\eta+E), \tag{12}$$

where $A_j, B_i, (0\le j\le n\,,\ 0\le i\le m), E,\ \lambda,\ k$ are coefficients and m, n positive integers got utilizing the balancing principle.

Step 3: Substituting Eq. (6) and its derivatives along with Eq. (7) into Eq. (5), an equation containing the polynomial is obtained. All coefficients are collected by collecting a series of algebraic equations of $e^{-\vartheta(\eta)}$ have the same rank and make each sum equal to zero. To obtain new solutions for (3), the system of equations is solved with the help of Wolfram Mathematica program and the values of $A_j, B_i, (0\le j\le n\,,\ 0\le i\le m), E, \lambda, k$ coefficients are found. Considering the obtained coefficient values and Eqs. (8–12), the solution functions that provide the Eq. (1) are obtained by replacing them in the Eq. (6).

2.2 Overview of the SGEM

Here, we give the analysis of the sine–Gordon equation (Abdelrahman et al. 2015)

$$u_{xx}-u_{tt}=v^2\sin(u), \tag{13}$$

where $u=u(x,t)$ and $v\in\mathbb{R}-\{0\}$.

The travelling wave transformation $u = u(x, t) = u(\eta),\ \eta = r(x - ct)$ on Eq. (13), gives the following non-linear ordinary differential equation (NODE):

$$u'' = \frac{v^2}{r^2\left(1 - c^2\right)} \sin(u), \tag{14}$$

where $u = u(\eta)$ and η stands for the width and k stands for the velocity of the travelling wave respectively. Equation (14) can be simplified in the following forms:

$$\left[\left(\frac{u}{2}\right)'\right]^2 = \frac{v^2}{r^2\left(1 - c^2\right)} \sin^2\left(\frac{u}{2}\right) + q, \tag{15}$$

where q is the integration constant.

Substituting $q = 0,\ w(\eta) = \frac{u}{2}$ and $a^2 = \frac{v^2}{r^2\left(1-c^2\right)}$ into Eq. (15), yields

$$\left(w'\right)^2 = a^2 \sin^2(w(\eta)), \tag{16}$$

substituting $a = 1$ into Eq. (16), yields

$$\left(w'\right)^2 = \sin^2(w(\eta)), \tag{17}$$

From Eq. (17), we have the following four significant equations:

$$\sin(w(\eta)) = \sec h(\eta) \text{ or } \cos(w(\eta)) = -\tanh(\eta), \tag{18}$$

$$\sin(w(\eta)) = -icsch(\eta) \text{ or } \cos(w(\eta)) = -\coth(\eta). \tag{19}$$

The solution of any non-linear partial differential equation (NPDE) is considered to be of the situations:

$$u(\eta) = \sum_{i=1}^{n} \tanh^{i-1}(\eta)\left[B_i \sec h(\eta) + A_i \tanh(\eta)\right] + A_0, \tag{20}$$

$$u(\eta) = \sum_{i=1}^{n} \coth^{i-1}(\eta)\left[B_i \cot h(\eta) + iA_i \mathrm{csch}(\eta)\right] + A_0. \tag{21}$$

According to Eqs. (18) and (19), one can rewrite Eq. (20) as

$$u(w) = \sum_{i=1}^{n} \cos^{i-1}(w)\left[B_i \sin(w) + A_i \cos(w)\right] + A_0. \tag{22}$$

The value of n is defined by the balancing procedure of the highest order non-linear term and the highest order derivative. Substitution Eq. (21) and its possible derivatives into the NODE, gives an equation in different power of trigonometric functions "

$\sin^i(w)\cos^j(w),\ (0 \le i \le n,\ 0 \le j \le n)$". The coefficients of trigonometric functions of the similar sequence are summed and each sum is set equal to zero to obtain some algebraical equations. This set of algebraical equations is solved for the values of the corresponding coefficients. Then the values of these coefficients are included in the Eq. (20a) and (20b) to get the solutions of the given NPDE .

3 Applications

In this section, using two mathematical methods, we are going to obtain the solitary solutions of the Wu-Zhang system.

Let's think the following travelling wave transformation:

$$u = u(\eta),\ v = v(\eta)\,,\ \ \eta = r(x - ct) \tag{23}$$

If the derivative terms required in Eqs. (1) and (2) are obtained from the wave transformation and are replaced in their place, respectively, the following equations are obtained.

$$-cr\,u' + r\,u\,u' + r\,v' = 0, \tag{24}$$

and

$$-cr\,v' + r\,v\,u' + r\,u\,v' + \frac{1}{3}\,r^3\,u''' = 0. \tag{25}$$

Integrating Eq. (23), we get,

$$-cu + \frac{1}{2}u^2 + v = 0. \tag{26}$$

Simplifying Eq. (25), we have,

$$v' = cu' - uu'. \tag{27}$$

Substituting Eq. (26) into Eq. (24), we get,

$$2\,r^2\,u'' + 9\,c\,u^2 - 3\,u^3 - 6\,c^2\,u = 0. \tag{28}$$

3.1 Application of the MEFM

In this section, the MEFM is used to obtain solitary solutions of the Wu-Zhang system (Figs. 1, 2, 3, 4, 5 and 6).

Balancing the highest power non-linear term and the highest derivative in Eq. (27), $n = m + 1$ gives the relationship.

Assume that $m = 1$. Then we have $n = 2$.

For m and n parameters, Eq. (6) is found as follows;

$$u(\zeta) = \frac{A_0 + A_1 e^{-\vartheta} + A_2 e^{-2\vartheta}}{B_0 + B_1 e^{-\vartheta}}. \tag{29}$$

When the derivative term required in Eq. (27) is obtained from the expression (28), then polynomials equation of $e^{-\vartheta}$ is get. The algebraic equation system consisting of the coefficients of $e^{-\vartheta}$ is solved using mathematica program to get the following conditions:

Case-1

$$A_0 = cB_0 - \frac{\sqrt{c^2\lambda^2\left(\lambda^2 - 4k\right)B_0^2}}{\lambda^2 - 4k},$$

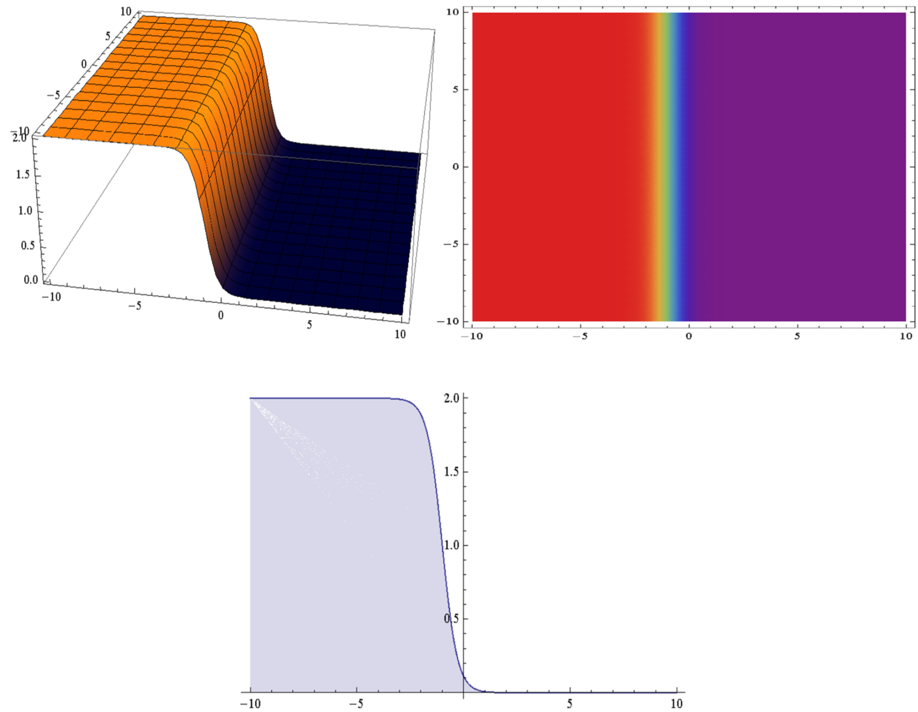

Fig. 1 The three dimensional, density graphic and two dimensional graphic surfaces of Eq. (29) in $k=1$, $\lambda=3$, $E=0.75$, $c=1$, $B0=0.35$ and $t=1$

$$A_1 = \frac{-\lambda\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}B_1 + B_0\left(-2\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2} + c\lambda\left(\lambda^2-4k\right)B_1\right)}{\lambda\left(\lambda^2-4k\right)B_0},$$

$$A_2 = -\frac{2c^2\lambda B_0 B_1}{\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}},\ s = \frac{\sqrt{3}c}{\sqrt{\lambda^2-4k}}.$$

The following solutions are obtained after these coefficients are put in Eq. (28), Family 1: When $k \neq 0$, $\lambda^2 - 4k > 0$, solution of Eq. (1),

$$u_1(x,t) = \frac{c\left(\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}(\lambda+\tau) - c\lambda B_0\left(\lambda^2-4k+\lambda\tau\right)\right)}{\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}(\lambda+\tau)}, \tag{30}$$

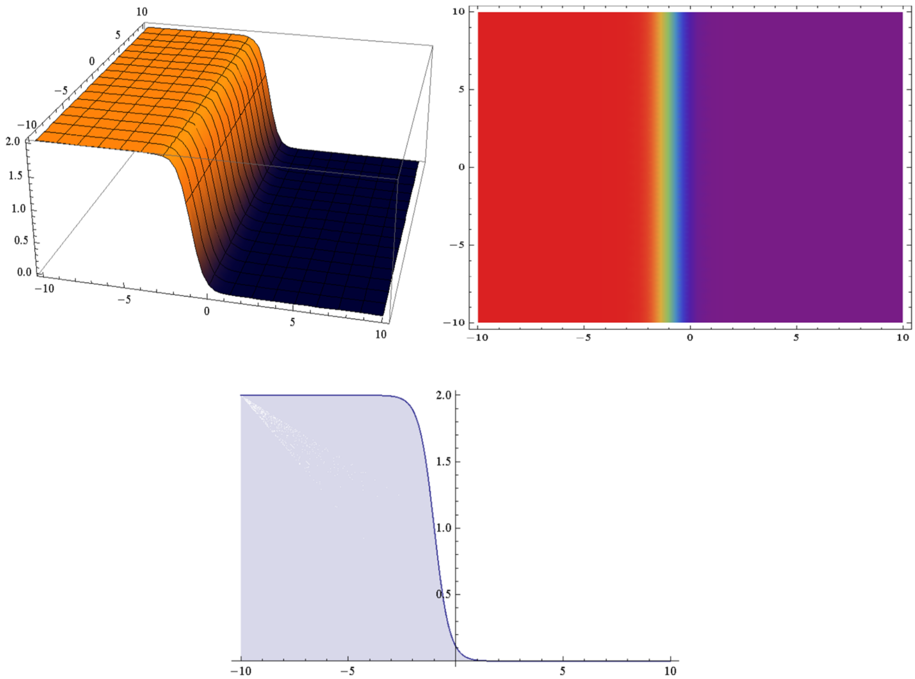

Fig. 2 The three dimensional, density and two dimensional graphic surfaces of Eq. (30) in $k=1$, $\lambda=3$, $c=1$, $B_0=0.35$, $E=0.75$, and $t=1$

$$v_1(x,t) = \frac{\left(4\,c^2\,k\right)}{\left(\left(1 + Cosh\left[\sqrt{3}\,c\,\eta + E\sqrt{\lambda^2 - 4k}\right]\right)(\lambda + \tau\,)^2\right)}.$$

where $\tau = \left(\sqrt{\lambda^2 - 4k}Tanh\left[\frac{1}{2}\sqrt{3}\,c\,\eta\left(E + \sqrt{\lambda^2 - 4k}\right)\right]\right)$.

Family 2: When $k \neq 0$, $\lambda^2 - 4k < 0$,

$$u_2(x,t) = \frac{c}{c\lambda B_0\kappa}(\lambda + \kappa\,\gamma)^{-1}\left(c\lambda B_0\kappa(\lambda + \kappa\,\gamma)\ - c\lambda B_0\left(\lambda^2 - 4k + \lambda\kappa\,\gamma\right)\right), \tag{31}$$

$$v_2(x,t) = 4k\,c^2\left(\left(1 + Cosh\left[\sqrt{3}\,c\,\eta + E\kappa\right]\right)\left(\lambda + \kappa\,Tanh\left[\frac{1}{2}\left(\sqrt{3}\,c\,\eta + E\kappa\right)\right]\right)^{-2},$$

where $\left(\kappa = \sqrt{\lambda^2 - 4k},\ \gamma = Tanh\left[\frac{\kappa}{2}(E + \eta)\right]\right)$.

Family 3: When $k = 0$, $\lambda \neq 0$ and $\lambda^2 - 4k > 0$,

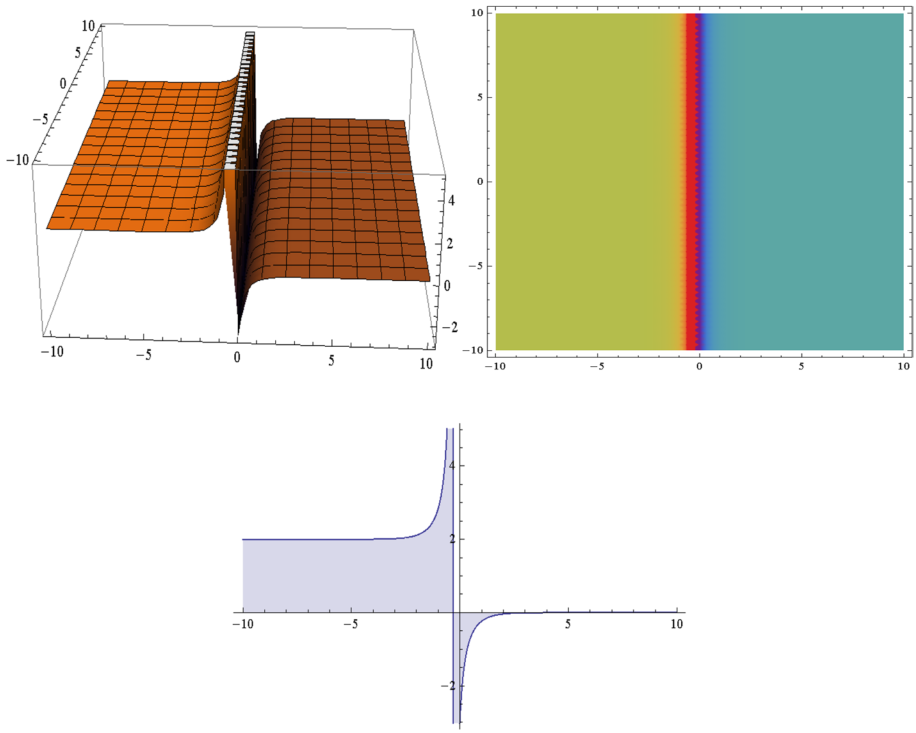

Fig. 3 The three dimensional, density and two dimensional graphic surfaces of Eq. (31) in $k=1$, $\lambda=3$, $c=1$, $E=0.75$, $B_0=0.35$ and $t=1$

$$u_3(x,t) = c - \frac{Coth\left[\frac{1}{2}\lambda\left(E + \frac{\sqrt{3}c\eta}{\sqrt{\lambda^2}}\right)\right]\sqrt{c^2\lambda^4 B_0^2}}{\lambda^2 B_0}, \tag{32}$$

$$v_3(x,t) = -\frac{c^2}{-1 + Cosh\left[\lambda\left(E + \frac{\sqrt{3}c\eta}{\sqrt{\lambda^2}}\right)\right]}.$$

According to the cases Family 4 and Family 5, the solution can not be get. Because, the solution function u is calculated as undefined due to the term $\lambda^2 - 4k = 0$.

Case-2:

$$A_0 = -\sqrt{\frac{2}{3}}\sqrt{s^2\left(\lambda^2 - 2k\right)B_0^2 + \sqrt{s^4\lambda^2\left(\lambda^2 - 4k\right)B_0^4}},$$

$$A_1 = \frac{\sqrt{s^2\left(\lambda^2 - 2k\right)B_0^2 + \sqrt{s^4\lambda^2\left(\lambda^2 - 4k\right)B_0^4}}\left(\sqrt{s^4\lambda^2\left(\lambda^2 - 4k\right)B_0^4} - s^2\lambda B_0\left(\lambda B_0 + 2kB_1\right)\right)}{\sqrt{6}s^2\lambda k B_0^2},$$

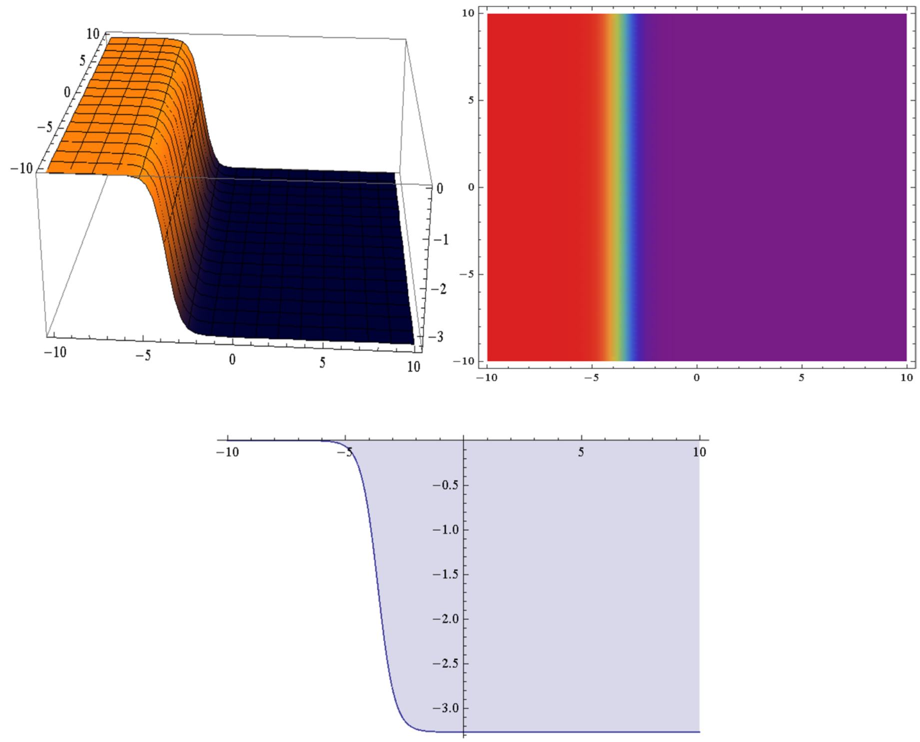

Fig. 4 The three dimensional, density and two dimensional graphic surfaces of Eq. (32)in $k=1$, c = 1, $\lambda=3$, $E=0.75$, $s=1$ and $B_0=0.35$, $t=1$

$$A_2=\frac{\left(-s^2\lambda^2B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}\right)\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}B_1}{\sqrt{6}s^2\lambda\,k\,B_0^3},$$

$$c=\frac{\left(s^2\left(\lambda^2-4k\right)B_0^2-\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}\right)\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}}{2\sqrt{6}s^2\lambda\,k\,B_0^3}.$$

Substituting these coefficients into Eq. (28), the following solutions are found out: Family 1: When $k\neq0$, $\lambda^2-4k>0$, solution of Eq. (1),

$$u_4(x,t)=-\frac{\sqrt{\frac{2}{3}}\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}\left(\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}+s^2\lambda\sqrt{\lambda^2-4k}B_0^2\,\omega\right)}{s^2\lambda B_0^3\left(\lambda+\sqrt{\lambda^2-4k}\,\omega\right)}, \tag{33}$$

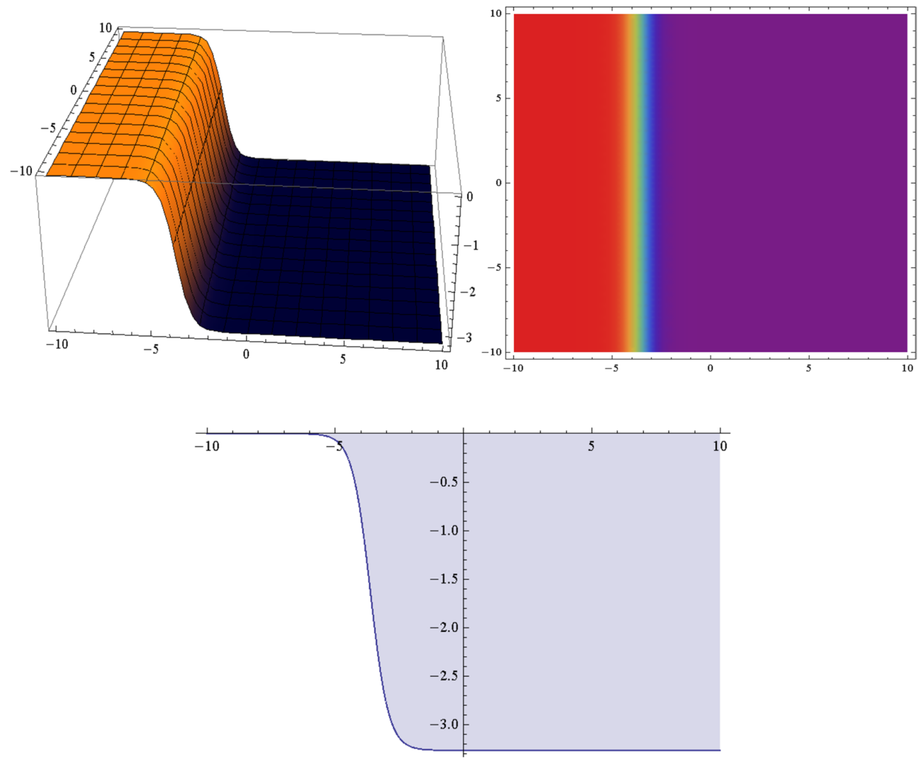

Fig. 5 The three dimensional, density and two dimensional graphic surfaces of Eq. (33) in $k=1$, $c=1$, $\lambda=3$, $s=1$, $B_0=0.35$, $E=0.75$ and $t=1$

$$\text{where,}\left(\omega=\left(Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{t\left(-s^2\left(\lambda^2-4k\right)B_0^2+\kappa s\right)\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\kappa s}}{2\sqrt{6}skB_0^3}\right)\right]\right)\right).$$

$$v_4(x,t)=\frac{\left(2s^2\left(\lambda^2-4k\right)kSech\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{\tau}{2\sqrt{6}\,skB_0^3}\right)\right]^2\right)}{\left(3\left(\lambda+\sqrt{\lambda^2-4k}\,Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{1}{\left(2\sqrt{6}skB_0^3\right)\tau}\right)\right]\right)^2\right)},$$

where, $\tau=\left(t\left(-s^2\left(\lambda^2-4k\right)B_0^2+\kappa\right)\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\kappa},\ \kappa=s\lambda B_0^2\sqrt{\left(\lambda^2-4k\right)}\right)$.

Family 2: When $k\neq 0$, $\lambda^2-4k<0$,

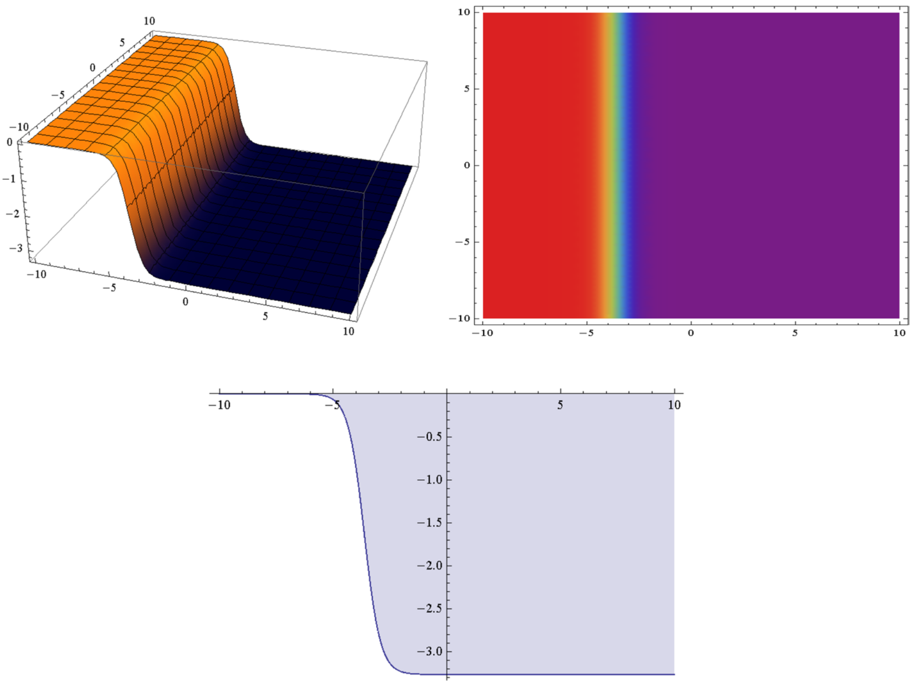

Fig. 6 The three dimensional, density and two dimensional graphic surfaces of Eq. (34)in $k=1$, $c=1$, $\lambda=3$, $s=1$, $B_0=0.35$, $E=0.75$ and $t=1$

$$u_5(x,t) = -\frac{\sqrt{\frac{2}{3}}\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}\left(\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}+ s^2\lambda\sqrt{\lambda^2-4k}B_0^2\,\omega\right)}{s^2\lambda B_0^3\left(\lambda+\sqrt{\lambda^2-4k}\,\omega\right)}, \tag{34}$$

where, $$\omega = \left(Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{1}{\frac{\left(2\sqrt{6}skB_0^3\right)t\left(-s^2\left(\lambda^2-4k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}\right)}{\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}}}\right)\right]\right).$$

$$v_5(x,t)=\frac{\left(2s^2\left(\lambda^2-4k\right)kSech\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{\tau}{2\sqrt{6}skB_0^3}\right)\right]^2\right)}{\left(3\left(\lambda+\sqrt{\lambda^2-4k}Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{1}{\left(2\sqrt{6}skB_0^3\right)\tau}\right)\right]\right)^2\right)},$$

where, $\tau=\left(t\left(-s^2\left(\lambda^2-4k\right)B_0^2+\sqrt{s^4\lambda^2(\lambda^2-4k)B_0^4}\right)\sqrt{s^2(\lambda^2-2k)B_0^2+\sqrt{s^4\lambda^2(\lambda^2-4k)B_0^4}}\right).$

Family 3: When $k=0$, $\lambda\neq 0$ and $\lambda^2-4k>0$,

$$u_6(x,t)=-\frac{\sqrt{\frac{2}{3}}\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}\left(\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}+s^2\lambda\sqrt{\lambda^2-4k}B_0^2\,\omega\right)}{s^2\lambda B_0^3\left(\lambda+\sqrt{\lambda^2-4k}\,\omega\right)},\tag{35}$$

where,
$$\omega=\left(Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{1}{\left(2\sqrt{6}skB_0^3\right)t\left(-s^2\left(\lambda^2-4k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}\right)\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}}\right)\right]\right).$$

$$v_6(x,t)=\frac{\left(2s^2\left(\lambda^2-4k\right)\mu Sech\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{\tau}{2\sqrt{6}skB_0^3}\right)\right]^2\right)}{\left(3\left(\lambda+\sqrt{\lambda^2-4k}\,Tanh\left[\frac{1}{2}\sqrt{\lambda^2-4k}\left(E+sx+\frac{1}{\left(2\sqrt{6}skB_0^3\right)\tau}\right)\right]\right)^2\right)},$$

where, $\tau=\left(t\left(-s^2\left(\lambda^2-4k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}\right)\sqrt{s^2\left(\lambda^2-2k\right)B_0^2+\sqrt{s^4\lambda^2\left(\lambda^2-4k\right)B_0^4}}\right).$

3.2 Application of the SGEM

In this section, solitary solutions are obtained by applying the SGEM method to the Wu-Zhang system.

Balancing the highest power non-linear term and the highest derivative in Eq. (26), gives $n=1$.

Equation (21) is written as follows according to n value,

$$u(w)=B_1\sin(w)+A_1\cos(w)+A_0.\tag{36}$$

Substituting Eq. (35) and its second derivative along with Eq. (17) into Eq. (27), gives an equation in power of trigonometric functions.We collect a set of algebraical equations by equating the summations of the coefficients of the trigonometric functions with the same power to zero. The set of algebraical equations is simplified to obtained the values of the

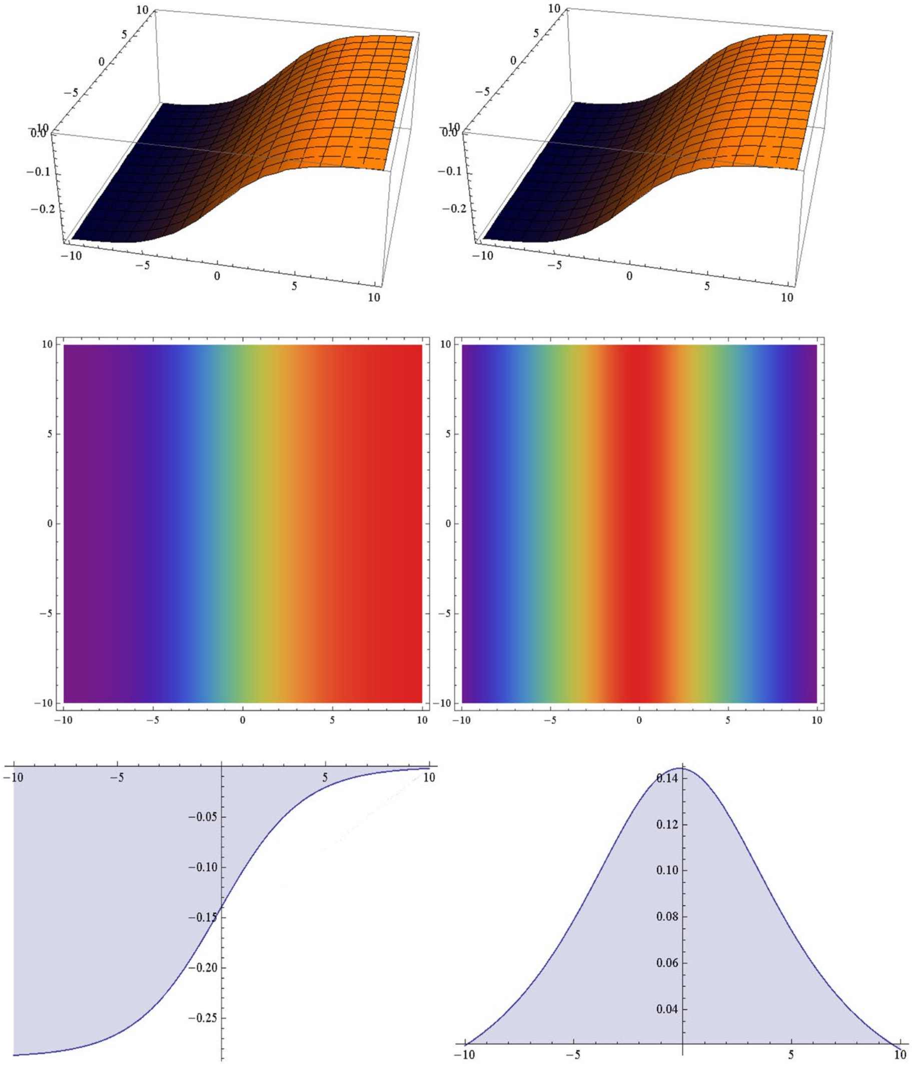

Fig. 7 The three dimensional, density and two dimensional graphics urfaces of the imaginary and real part of the Eq. (36) respectively in $k=1$, $t=1$

parameters included. The values of parameters are then substituted into Eq. (20) and (21) with fixed value of m to get the solutions of Eq. (1) (Figs. 7, 8 and 9).

Case-1: When,

$$A_0 = -\frac{k}{\sqrt{3}},\ A_1 = -\frac{k}{\sqrt{3}},\ B_1 = \frac{ik}{\sqrt{3}},\ c = -\frac{k}{\sqrt{3}},$$

we have the following compound non-topological and topological kink-type soliton:

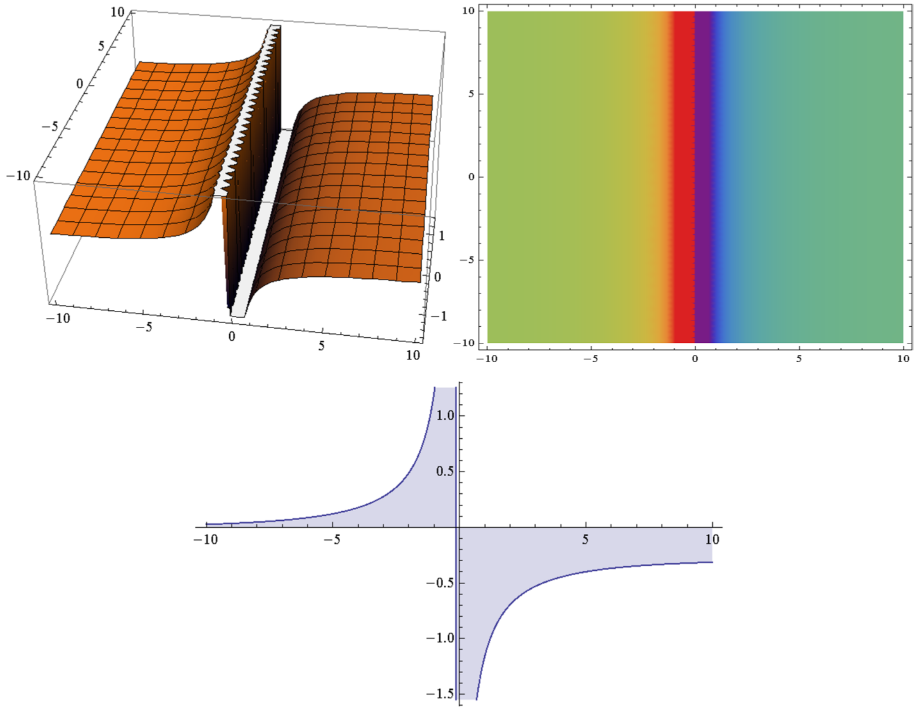

Fig. 8 The three dimensional, density and two dimensional graphic surfaces of the Eq. (37) in $k=1$, $t=1$

$$u_7(x,t) = \frac{k\left(-1 + iSech\left[k\left(x+\frac{tu}{\sqrt{3}}\right)\right] + Tanh\left[k\left(x+\frac{tu}{\sqrt{3}}\right)\right]\right)}{\sqrt{3}}, \tag{37}$$

$$v_7(x,t) = -\frac{k^2}{3 + 3i\,Sinh\left[k\left(x+\frac{tk}{\sqrt{3}}\right)\right]},$$

and the singular soliton,

$$u_8(x,t) = -\frac{k\left(-1 + Coth\left[k\left(x+\frac{tu}{\sqrt{3}}\right)\right] + Csch\left[k\left(x+\frac{tu}{\sqrt{3}}\right)\right]\right)}{\sqrt{3}}, \tag{38}$$

$$v_8(x,t) = -\frac{1}{6}k^2 Csch\left[\frac{1}{2}k\left(x+\frac{tk}{\sqrt{3}}\right)\right]^2.$$

Case-2: When,

$$A_0 = \frac{2k}{\sqrt{3}},\ A_1 = -\frac{2k}{\sqrt{3}},\ B_1 = 0,\ c = \frac{2k}{\sqrt{3}},$$

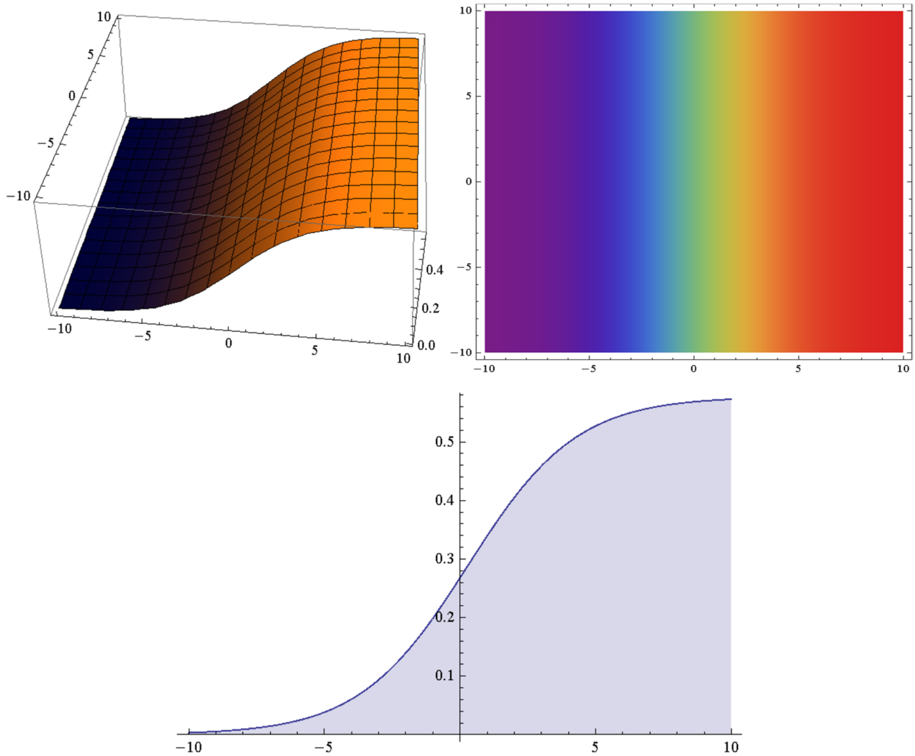

Fig. 9 The three dimensional, density and two dimensional graphic surfaces of Eq. (38) in $k=1$, $t=1$

we get the following topological kink-type soliton:

$$u_9(x,t) = \frac{2k}{\sqrt{3}}\left(1 + Tanh\left[k\left(x - \frac{2tu}{\sqrt{3}}\right)\right]\right),$$
$$v_9(x,t) = \frac{2}{3}k^2 Sech\left[k\left(x - \frac{2t\,k}{\sqrt{3}}\right)\right]^2. \tag{39}$$

4 Results and discussion

The modified exp function method and the Sine–Gordon expansion method have been successfully employed to secure obtain the wave solutions of an important non-linear model; the Wu-Zhang system. Various wave solutions obtained by using these powerful schemes have been reported in this study. On the other hand, when we compare the results obtained by using two methods in this article with the results obtained in Eslami and Rezazadeh (2016), some new travelling wave solutions have been presented to the literature. The hyperbolic function solutions acquired by using the modified expansion function method of Wu-Zhang system are obtained from the following coefficients.

$$A_0 = cB_0 - \frac{\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}}{\lambda^2-4k},$$

$$A_1 = \frac{-\lambda\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}B_1 + B_0\left(-2\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2} + c\lambda\left(\lambda^2-4k\right)B_1\right)}{\lambda\left(\lambda^2-4k\right)B_0},$$

$$A_2 = -\frac{2c^2\lambda B_0 B_1}{\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}},\ s = \frac{\sqrt{3}c}{\sqrt{\lambda^2-4k}}.$$

From these coefficients, diverse solution functions are obtained according to the conditions of Families-1-2-3. For example;

$$u_1(x,t) = \frac{c\left(\begin{array}{l}\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}\left(\lambda+\sqrt{\lambda^2-4k}\,Tanh\left[\frac{1}{2}\sqrt{3}\,c\,\eta\left(E+\sqrt{\lambda^2-4k}\right)\right]\right)- \\ c\lambda B_0\left(\lambda^2-4k+\lambda\sqrt{\lambda^2-4k}\,Tanh\left[\frac{1}{2}\sqrt{3}\,c\,\eta\left(E+\sqrt{\lambda^2-4k}\right)\right]\right)\end{array}\right)}{\sqrt{c^2\lambda^2\left(\lambda^2-4k\right)B_0^2}\left(\lambda+\sqrt{\lambda^2-4k}\,Tanh\left[\frac{1}{2}\sqrt{3}\,c\,\eta\left(E+\sqrt{\lambda^2-4k}\right)\right]\right)},$$

$$v_1(x,t) = \frac{\left(4\,c^2\,k\right)}{\left(\begin{array}{l}\left(1+Cosh\left[\sqrt{3}\,c\,\eta+E\sqrt{\lambda^2-4k}\right]\right) \\ \left(\lambda+\sqrt{\lambda^2-4k}Tanh\left[\frac{1}{2}\left(\sqrt{3}\,c\,\eta+E\sqrt{\lambda^2-4k}\right)\right]\right)^2\end{array}\right)}.$$

Some of the solutions of the Wu-Zhang system according to the coefficients found according to the other method, sine–Gordon expansion method, are as follows.

$$A_0 = -\frac{k}{\sqrt{3}},\ A_1 = -\frac{k}{\sqrt{3}},\ B_1 = \frac{ik}{\sqrt{3}},\ c = -\frac{k}{\sqrt{3}},$$

$$u_7(x,t) = \frac{k\left(-1+iSech\left[k\left(x+\frac{tu}{\sqrt{3}}\right)\right]+Tanh\left[k\left(x+\frac{tu}{\sqrt{3}}\right)\right]\right)}{\sqrt{3}},$$

$$v_7(x,t) = -\frac{k^2}{3+3i\,Sinh\left[k\left(x+\frac{tk}{\sqrt{3}}\right)\right]}.$$

The results acquired successfully in our study are thought to have an important physical meaning for the dynamical system. Periodic features are needed to make an estimate of the prospective behavior of the system.

5 Conclusions

In this article, we construct various wave solutions to the Wu-Zhang system by using the modified expansion function method and the sine–Gordon expansion method. We successfully get topological kink-type, non-topological, singular soliton and trigonometric function solutions. When the graphics of the obtained solutions of the non-linear differential equation are analyzed, they physically conform to the motion pattern of the wave. When the graphs of the solution functions obtained as a result of the methods applied to the equations are physically interpreted, it is observed that the movements intensify according to the characteristic feature of the solution function as time progresses and pigs in certain points. The graphics indicate that the solution functions obtained have periodic features. It is beneficial to obtain functions with such features. Because it is quite easy to physically interpret equations with periodic function properties. It also helps us to easily comment on the motion model within the desired range.

The reported results show that the two methods are very efficient and suitable mathematical tools for solving non-linear partial differential equations.

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Declarations

Conflict of interest None

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