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Preservice middle school mathematics teachers' strategy repertoire in proportional problem solving

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ABSTRACT
This study examined 91 preservice mathematics teachers' strategy repertoire that they referred to when solving one direct and one inverse proportion missing-value word problems. When encouraged to provide multiple solutions, the preservice teachers exhibited the ability to solve the two problems using more than one strategy. However, they used a significantly greater number of strategies for solving the direct than for the inverse proportion problem. The most frequently used strategies for the two problems were the cross-multiplication and across-multiplication, respectively, and many of the preservice teachers also used these two strategies as their first strategy. On the other hand, the number of strategies applied by the preservice teachers did not differ significantly according to their first choice of strategies. A key finding of this study was that the preservice teachers possessing less common strategies in their repertoire had a significantly larger strategy repertoire than those who had more common strategies.

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Background

In elementary and advanced school mathematics, proportional reasoning plays a key role in students' mathematical development (Kilpatrick et al., 2001). Proportional reasoning is also essential in learning science (Cramer & Post, 1993) and STEM concepts (Lobato et al., 2010). The importance of developing students' proportional reasoning is stated in national mathematics curricula of countries, in mathematical standards of well-known organisations (e.g. National Council of Teachers of Mathematics [NCTM] and Australian Curriculum, Assessment and Reporting Authority [ACARA]), and in large-scale international assessment studies (e.g. the Trends in International Mathematics and Science Study [TIMSS] and Programme for International Student Assessment [PISA]).

Proportional reasoning consists of "the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities" (Lamon, 2007, pp. 637–638). Three concepts are essential for developing students' proportional reasoning: ratio, proportion, and (direct and inverse) proportional relationships. A ratio is defined as a multiplicative comparison of two quantities (Lobato

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et al., 2010), whereas a proportion is defined as the statement of equality of two ratios, which is usually presented by the mathematical notation $a/b = c/d$ (Riehl & Steinthorsdottir, 2019). A directly proportional relationship exists between two quantities if "the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor" (Lobato et al., 2010, p. 11). It is modelled by the equation $y = k^*x$, in which y and x are quantities compared and k refers to the constant of proportionality. If the ratio formed by the values of a quantity (i.e. within ratio) is equal to the inverse of the ratio formed by the values of the second quantity (i.e. $a/b = 1/[c/d]$), then an inversely proportional relationship exists between these two quantities. An inversely proportional relationship is modelled by the equation $y^*x = k$.

For students to solve direct and inverse proportion problems successfully, their proportional reasoning ability should be developed. The most common instructional strategy in schools for developing students' proportional reasoning is to provide them with proportion problems (Lesh et al., 1988). Similarly, students' proportional reasoning has been assessed by their ability to solve these problems (Misailidou & Williams, 2003; Riehl & Steinthorsdottir, 2019). The literature contains a variety of strategies that students apply when solving proportion problems. While some students solve them using only one single strategy, others apply multiple strategies. As stated by Brakoniecki et al. (2021), proportional reasoning ripe with multiple solution strategies because effective mathematics teaching practices emphasise the importance of promoting student reasoning and sense making through analysing multiple solution strategies. Also more generally, literature reports that encouraging students to apply multiple solution strategies is an effective method, which has generally been used as a teaching technique in high-performing countries, for improving mathematical knowledge and performance, creativity, critical thinking, and problem-solving competences (Große & Renkl, 2006; Guberman & Leikin, 2013; Leikin & Levav-Waynberg, 2008; Schukajlow et al., 2015; Schukajlow & Krug, 2014; Silver et al., 2005).

According to Carpenter et al. (1993), if given the opportunity, children as young as kindergarten can invent strategies to solve various problems. Hence, the instruction should build upon students' existing knowledge on informal and intuitive strategies (Carpenter & Fennema, 1992). To assist students in developing multiple solutions, teachers themselves should know various strategies including informal ones that students who are at the beginning of mastering a certain topic might engage in and know how to apply these strategies in solving specific problems (Ball et al., 2008; Copur-Gencturk & Doleck, 2021). However, teachers often solve proportional reasoning problems using formal strategies (Fisher, 1988). On the other hand, using informal strategies, especially in proportional reasoning tasks, assists teachers improve students' reasoning skills and understand their mathematical competence (Ruchti & Bennett, 2013). Therefore, teachers should be given "more opportunities in teacher education and professional development programs to learn to solve problems by using both formal and informal strategies" (Copur-Gencturk & Doleck, 2021, p. 6).

Notwithstanding the above pleas that – in order for them to improve students' learning – teachers should know and be able to use various strategies to solve problems, research on teachers' strategy repertoire is much scarcer compared to the large body of research on students' proportional reasoning (Lobato et al., 2011). It is important to note, however, that existing research generally identifies and classifies in-service teachers' and preservice teachers' (PSTs) solution strategies to proportion problems, but lacks diagnostic information on their knowledge of multiple strategies. Hence, in the present study, we investigate PSTs' strategy repertoire to obtain diagnostic information about their knowledge of multiple strategies. Investigating PSTs' strategy repertoire can shed light on the *mathematical knowledge* that they will need for teaching (MKT) (Ball et al., 2008) ratio and proportion concepts in middle school.

Purpose of the study and research questions

This study investigates the strategy repertoire that PSTs rely on when solving direct and inverse proportion missing-value word problems, documents the characteristics of their strategy uses, and examines the relationship between the presence of a certain strategy in PSTs' repertoire and the total number of strategies they have in their repertoire. In literature, it is more common to report particular strategies used by PSTs or in-service teachers to solve proportion problems (e.g. Arican, 2018; Livy & Vale, 2011) and their interpretation of students' strategies (e.g. Hines & McMahon, 2005; Jacobson et al., 2018) than to explore their entire repertoire of strategies. In addition, PSTs may tend to use a single strategy that they believe is best for solving proportion problems assigned to them (Arican, 2018). Hence, to get an appropriate view on the content and structure of PSTs' strategy repertoire, they were encouraged to provide multiple solutions to one direct and one inverse missing-value proportion problem.

The following information was collected from PSTs separately for the direct and inverse proportion problem: the distribution and number of strategies being part of the strategy repertoire task, the most and least used strategies, the strategies used as the first or second one, and the average number of strategies calculated for applicants of each strategy category. Using the collected information, we aimed to answer the following research questions:

1. What is the distribution and number of strategies used by PSTs in solving the direct and inverse proportion problem?
2. What are the most and least common strategies used by PSTs in solving the two problems?
3. Which strategy do PSTs use as their first strategy and which strategies follow the first strategy?
4. Does the number of strategies applied by PSTs differ significantly according to their first strategies?
5. Do the means of the number of strategies calculated for each strategy category differ significantly?

Literature on strategies used in solving proportion problems

Researchers have made various attempts to identify and classify students', PSTs', and in-service teachers' strategies to solve proportion problems. Classifications of strategies have usually been done by considering one or more of the following solution characteristics: The level of sophistication of proportional reasoning (i.e. intuitive, additive, pre-proportional, proportional, qualitative, quantitative, etc.) applied in problem solving (e.g.

Bart et al., 1994; Carney et al., 2015; Karplus et al., 1983; Lamon, 1993; Thornton & Fuller, 1981), correctness of solutions (e.g. Cabero-Fayos et al., 2020; Fisher, 1988), and external representations used in solving problems (e.g. Arican, 2018).

The literature on proportional reasoning mostly consists of research that focused on students' strategies and only very few of them (e.g. Arican, 2018; Cabero-Fayos et al., 2020; Fisher, 1988; Joshua & Lee, 2022; Lo, 2004; Riley, 2010) investigated in-service and PSTs' strategies. Fisher's (1988) article was one of the first to report on in-service teachers' strategies. Fisher (1988) determined nine strategies of secondary teachers in which she classified the first five strategies as incorrect and the remaining four as correct strategies (pp. 161–162): No Answer; Intuitive (guessing the answer or answering the question by just relying on feelings or intuition); Additive (making additive comparisons); Proportion Attempt (express some understanding of a proportional relationship was involved but cannot show this relationship); Incorrect Other (an incorrect strategy that cannot be placed in categories 1–4); Proportion Formula (demonstrating the equality of two ratios or formulating an equation that expresses the equality of two products followed by an explicit statement indicating the inverse relationship); Proportional Reasoning (using a correct proportion strategy other than the proportion formula); Algebra (using an algebraic equation other than the proportion formula to solve a problem); and Correct Other (a correct strategy that cannot be placed in categories 6–8).

In a recent study, Arican (2018) used Fisher's (1988) strategy categories to identify strategies presented by eight PSTs, but followed a slightly different approach. Unlike Fisher (1988), Arican (2018) did not consider correctness of strategies in his classification process and also reported PSTs' external representations as strategy categories. Arican (2018) identified 10 different strategies: additive, proportion formula, algebra, unit ratio, ratio table, double number line, strip diagram, unit conversion, double counting, and visual.

In the current study, following Fisher (1988) and Arican (2018) studies, we developed a classification of all strategies that can be considered as correct including the strategies that would – according to some – not necessarily refer to true proportional reasoning but that would still lead to a correct solution. As a result, we developed two closely related strategy frameworks, one for direct proportion problems (Table 1) and one for inverse proportion problems (Table 2) to guide researchers in categorising PSTs' responses to missing-value proportion problems. Tables 1 and 2 present formal and operational definitions of the direct and inverse proportion strategies, respectively and provide additional considerations for coding purposes. We determined eight and seven direct and inverse proportion strategies, respectively. Except for the cross-multiplication, which is also called rule of three, (e.g. Arican, 2019; Boston et al., 2003) and build-up strategies (e.g. Kaput & West, 1994; Lamon, 2007), the remaining direct proportion strategies were reported by either Fisher (1988) or Arican (2018). Similarly, except for the across-multiplication (e.g. Arican, 2019) and unit product strategies, the remaining inverse proportion strategies were reported by either Fisher (1988) or Arican (2018). We have included the unit product strategy in Table 2 as it entails an important understanding of inversely proportional relationships which is not generally reported in the literature.

Table 1. Formal and operational definitions of solution strategies that can be used for solving direct proportion problems and additional considerations for coding purposes

Strategies	Formal definition	Operational definition	Additional considerations for coding
Build-Up	An intensive unit is formed using two external quantities and the desired unit is obtained by duplication, iteration, or partitioning (Kaput & West, 1994; Lamon, 2007)	First, the given information is written down, and the desired answer is obtained by iterating this given information as many times as required. For instance, if three pencils cost 2 dollars, subjects iterate this information three times to calculate the price of nine pencils.	Students at an early age generally use build up strategy to solve proportion problems. This strategy cannot be treated as a proportional reasoning strategy because it does not consider the invariance of the ratio formed by the corresponding values of two quantities.
Cross-Multiplication	Cross-multiplication strategy is a direct result of the ratios formed by the values of two separate quantities being equivalent. If Q1 and Q2 are two quantities, then a/b = c/d and a/c = b/d in which both yields ad = bc. *Q1* *Q2* a c b d	First, subjects enter the values of two quantities side-by-side. Next, they check if the values of quantities increase or decrease simultaneously. Finally, they cross-multiply the values at the corners.	Subjects who use the cross-multiplication strategy generally do not state two ratios are being equivalent, and they do not necessarily understand multiplicative relationships formed within and between values of quantities. Hence, this strategy is usually considered as a rote computation technique.
Unit Ratio	A unit ratio is determined between quantities and this ratio is multiplied by the corresponding value to calculate the missing-value. Mathematically speaking, a/b = c/x means that x = b* (c/a) in which c/a = (c/a)/1 (i.e. unit ratio). Multiplying c/a by *b* yields the missing-value.	Subjects infer a unit ratio relationship between two quantities. This relationship is presented by inserting the term *per* between two corresponding values. For instance, if three pencils cost 2 dollars, then 2/3 dollars per one pencil, which is the unit ratio. To calculate the price of nine pencils, 2/3 should be multiplied by 9 that yields 6 dollars.	Like the cross-multiplication strategy, subjects who use the unit ratio strategy also do not state equivalence of two ratios. However, in the unit ratio strategy, subjects infer a unit ratio relationship between quantities and use this relationship to calculate the missing-value. This inference does not occur in the cross-multiplication strategy.
Ratio Table	The factor of change existing either within values (a*k1 = b) or between values (a*k2 = c) are determined and the corresponding values are multiplied with these two factors to calculate the missing-value (c*k1 = x or b*k2 = x).	The values of quantities are usually entered side by side, not necessarily separated by table lines, and the factor of change is determined either within values or between values. In some forms, subjects may display values in pictorial or graphical forms. Hence, one should check that if they determined the factor of change and multiplied within or between values to calculate the missing-value.	Subjects usually do not immediately recognise that ratios formed by corresponding values of two quantities are all equal to a constant number. However, some subjects may recognise this constant ratio relationship after making some exploration. Ratio table strategy differs from the cross-multiplication and unit ratio strategies in that it involves multiplication within (scalar approach) and between (functional approach) values.
Proportion Formula	In this strategy, subjects either form a proportion showing the equality of two ratios (e.g. a/b = c/d) or write an equation expressing the	Subjects form a proportion by equating two ratios and calculate the missing-value either by cross-multiplying the values in the proportion or simplifying/expanding the	The proportion formula strategy involves setting up a proportion or creating an equation, both of which require recognising an equivalence relation

(*Continued*)

Table 1. Continued.

Strategies	Formal definition	Operational definition	Additional considerations for coding
	equality of two products (e.g. a*d = b*c) (Fisher, 1988).	values in the numerator and denominator by division/multiplication. Some subjects can calculate the missing-value from the equation that states the equality of two products.	(usually between numbers). Recognition of this equivalence relationship does not appear in the cross-multiplication strategy. However, in the ratio table strategy, some subjects may recognise this equivalence relationship after exploring numerical relationships between the values.
Algebra	Algebra strategy involves solving proportion problems by setting up an algebraic equation other than the proportion formula.	Subjects may obtain an algebraic equation that represents the multiplicative relationship between corresponding values of two quantities. The algebraic equation is a general expression of relationship between quantities.	Algebra strategy can be confused with the ratio table and proportion formula strategies. While the algebra strategy involves deriving a generic algebraic expression (e.g. Distance = Velocity * Time) that reflects the multiplicative relationship between quantities, both the ratio table and proportion formula strategies begin with concrete values.
Graphical	Subjects can present relationships between quantities graphically and deduce their answers using these graphical representations.	Subjects may draw a direct proportion graph and use it to solve the given problem.	Subjects may calculate the answer by calculating the slope or solve it by paying attention to the area below the graph.
Other	Strategies that cannot be placed in any of above categories are placed in here.		Drawings, pictorial solutions that are not presented in graphical forms, and intuitive strategies (Fisher, 1988) can be placed in this category.

Methods

Participants

This study included 91 PSTs, 51 third year and 40 fourth year, from a university located in the central Turkey. PSTs attended to a four-year-long middle school mathematics programme of the university. In the fall semester of the 2020–2021 academic year, PSTs were contacted during two online mathematics education courses (one for the third year and one for the fourth year PSTs). Participation in the study was on a voluntary basis. Except for six third-year PSTs and one fourth-year PST, all PSTs participated in the study. In each course, PSTs were given the strategy repertoire task in an online environment as an extracurricular activity. The university programme, including the two courses, did not include instruction on ratio, proportion, and proportional relationships. Hence, PSTs solved the direct and inverse proportion problem using their previously acquired knowledge on these concepts during middle and high school. However, we should

Table 2. Formal and operational definitions of solution strategies that can be used for solving inverse proportion problems and additional considerations for coding purposes.

Strategies	Formal definition	Operational definition	Additional considerations for coding
Across-multiplication	Mathematically, in the inverse proportion, we have a/b = 1/(c/d) that gives us a/b = d/c, which yields ac = bd. *Q1* *Q2* a c b d	Like the cross-multiplication strategy, the values of two quantities are entered side-by-side, and corresponding values at the sides are across multiplied. The desired value is calculated from the equality of products.	Subjects generally do not recognise that the product of corresponding values are equal and do not recognise *reciprocal* multiplicative relationships formed between values of quantities (i.e. *a* is *1/b* times as large as *ab* or *b* is *1/a* times as large as *ab*).
Ratio Table	The values of two separate quantities are entered into a table, not necessarily separated by lines, and multiplicative relationships within values (a*k = b and c*(1/k) = d) are considered when calculating the missing-value.	In the inverse proportion, when we multiply the value of a quantity by a number, then we need to divide the value of the corresponding quantity by that number. Multiplying between values does not work in this strategy. In some forms, rather than using a table from, subjects may display values in pictorial or graphical forms.	Subjects usually do not immediately recognise that products formed by corresponding values of two quantities are all equal to a constant number. However, some subjects may recognise this constant product relationship between numbers after making some exploration.
Proportion Formula	In this strategy, subjects either form an inverse proportion showing the equality of two ratios (e.g. a/b = 1/[c/d]) or write an equation expressing the equality of two products (e.g. a*c = b*d) (Fisher, 1988).	Subjects usually use this strategy after forming an inverse proportion using the corresponding values of two quantities. However, some subjects may recognise the equality of two products without forming an inverse proportion.	Proportion formula strategy necessitates recognising an equivalence relation between numbers. Subjects do not recognise this relation in the across-multiplication strategy. Some subjects may recognise this relationship later in the ratio table strategy after exploring numeric multiplicative relationships.
Algebra	Subjects solve the problem by setting up an algebraic equation other than the proportion formula.	Subjects may obtain an algebraic equation that represents the multiplicative relationship between corresponding values of two quantities. The algebraic equation is a general expression of relationship between quantities.	Algebra strategy can be confused with the ratio table and proportion formula strategies. Algebra strategy involves obtaining a generic algebraic expression (e.g. Distance = Velocity * Time). However, both the ratio table and proportion formula strategies begin with concrete values.
Unit Product	Mathematically speaking, x*y = k is the inverse proportion formula and to calculate the unit product, the subject assigns x = 1 that yields 1 * (x*y) = k.	This strategy requires the calculation of the product per one unit (e.g. three workers decorate some cakes in 12 min, then one worker can do the same job in 36 min) and dividing this product by the corresponding value to calculate the missing-value.	This strategy is similar to the unit ratio strategy. The only difference is that rather than calculating a unit ratio, unit product strategy entails calculating the product per one unit.

(*Continued*)

Table 2. Continued.

Strategies	Formal definition	Operational definition	Additional considerations for coding
Graphical	Subjects can present relationships between quantities graphically and deduce their answers using these representations.	Subjects may draw an inverse proportion graph and use it to solve given problems.	Subjects may calculate the answer by recognising numerical relationships between values or by paying attention to the features of the graph.
Other	Strategies that cannot be placed in any of above categories are placed in here.		Drawings, pictorial solutions that are not presented in graphical forms, and intuitive strategies (Fisher, 1988) can be placed in this category.

note that the instruction mostly relies on the cross-multiplication and across-multiplication algorithms and ratio table strategy (Arican, 2019). Furthermore, the graphs of directly proportional relationships are taught in eighth grade as a special case of linear equations, but neither in middle school (Turkish Ministry of National Education [TMNE], 2018a) nor in high school programmes (TMNE, 2018b) instruction is provided on graphs of inversely proportional relationships.

The data collection instrument

PSTs' responses were collected by means of a strategy repertoire task that included two missing-value word problems, one with a directly and one with an inversely proportional relationship (see Appendix). The problems were written as missing-value problems because classroom instruction often focuses on missing-value problems in teaching proportions. Hence, PSTs have more experience in solving missing-value problems as compared to comparison (e.g. Lamon, 2007) and qualitative reasoning problems (e.g. Cramer et al., 1993). The two problems included real-life contexts that were supposed to be familiar to PSTs: The direct proportion problem involved mixing water and lemon juice to make lemonade whereas the inverse proportional problem was about decorating cakes in a certain amount of time. We used integer ratios in both problems since the purpose of the study was to reveal as many strategies of PSTs as possible, not to determine their difficulty. Consistent with our decision to use missing-value problems, researchers (e.g. Van Dooren et al., 2009; Fernández et al., 2011, 2012; Jiang et al., 2017) reported that, compared to missing-value proportion problems with non-integer ratios, students perform better on missing-value proportion problems when integer ratios are involved.

To better understand the depth of PSTs' repertoire, their solutions to these two problems were collected in two stages. In stage 1, we just asked PSTs to solve the given problem without giving any indication on which strategy to use. In stage 2, we explicitly asked them to solve the same problem applying as many different strategies as they could think of, using a form containing the following titles: Strategy A, Strategy B, Strategy C, and Other strategies. The "Other strategies" title was preceded by the following statement: "If you can still come up with other strategies than the ones you used before, you can write them in here". Stage 1 was immediately followed by Stage 2, and PSTs solved the Lemonade problem first and solved the Bakery problem next without having a break between the two problems.

Data coding and analysis

PSTs' responses to the two problems were separately coded by one author and one trained graduate student, following the strategy frameworks in Tables 1 and 2. We provided examples of PSTs' solutions in the Appendix as an illustration of the coding process. We emphasise that this coding system does not consider the correctness of PSTs' solutions. Hence, also the strategies used in solutions that led to incorrect answers to the two proportional problems were categorised by giving a code from Table 1 or Table 2. However, we should note that there were only a few incorrect solutions that occurred because the inversely proportional relationship was confused with the directly proportional relationship. The two coders coded all the data and recorded the name of strategies used by each PST in stage 1 and stage 2 of the task in two separate Excel files. The intercoder reliability between the coders was 89.3%, suggesting a nearly perfect agreement between them (Miles & Huberman, 1994). The remaining disagreed codes were examined by the three authors and decisions were made on the strategy categories that these codes belonged.

The final dataset included descriptive information on the number of strategies used by PSTs, order of these strategies (i.e. first used, second used, and so on), and application status of specific strategies (e.g. a code 1 was entered if a specific strategy was used, otherwise 0 was entered). This final dataset was analysed using the SPSS software. In the following section, we present findings on the strategies that PSTs used to solve the two problems.

Results

PSTs' solutions to the direct proportion problem were classified into one of the eight strategy categories presented in Table 1. Table 3 shows that all PSTs used at least one strategy from Table 1. The mean number of strategies used by PSTs was 2.90 (*SD*: 1.00). The cross-multiplication strategy was the most used strategy; it was used 68 times (25.7%). The second and third most used strategies were the ratio table (23.9%) and proportion formula strategies (23.5%). These three strategies formed about 73% of the total number of strategies used in solving the direct proportion problem. The

Table 3. The distribution and frequencies of PSTs' strategies in the direct proportion problem.

Attempt number	Cross-Multip.	Ratio table	Proportion formula	Unit ratio	Build-up	Graphical	Other	Algebra	Total
1	46	22	19	3	1	0	0	0	91
2	15	18	22	9	11	2	0	0	77
3	5	16	9	8	5	6	2	2	53
4	2	4	9	4	2	2	6	2	31
5	0	3	3	1	0	0	0	1	8
6	0	0	0	0	1	1	0	1	3
7	0	0	0	0	0	1	0	0	1
Total	68	63	62	25	20	12	8	6	264

Note. PSTs' strategies were entered in the table based on their number of total appearances.

Table 4. The distribution and frequencies of PSTs' strategies in the inverse proportion problem.

Attempt number	Across-Multip.	Ratio table	Proportion formula	Unit product	Algebra	Other	Graphical	Total
1	60	20	4	2	2	2	0	90
2	11	24	7	10	3	2	0	57
3	3	12	8	6	3	4	3	39
4	0	1	3	4	2	1	2	13
5	0	1	1	1	0	0	1	4
Total	74	58	23	23	10	9	6	203

Note. PSTs' strategies were entered in the table based on their number of total appearances.

remaining five strategies were not often used: While the unit ratio and build up strategies were rarely used, the graphical, other, and algebra strategies were almost never used. The algebra strategy was the least frequently used of these five strategies.

Regarding PSTs' order of strategy uses, approximately half of them used the cross-multiplication as their first strategy. The ratio table (24%) and proportion formula (21%) strategies were the following in line to be used as the first strategy, respectively. Furthermore, the proportion formula and ratio table were the most frequent strategies in the second and third solution attempts of PSTs, respectively. In addition, we analysed whether the number of strategies applied by PSTs differed based on their first strategy. An ANOVA analysis showed that there was no significant relationship between PSTs' first strategy and the number of strategies they applied, $F(3, 87) = 1.54$ and $p = .21$.

PSTs' solutions to the inverse proportion problem were classified into one of the seven strategy categories presented in Table 2. Table 4 shows that, except one PST who did not provide a solution, all PSTs used at least one strategy from Table 2. The mean number of strategies used by PSTs was 2.23 (*SD*: .857). The Paired *T*-test results showed that PSTs applied significantly more strategies in solving the direct proportion problem than the inverse proportion problem, $t(90) = 6.32$, $p < .001$. The across-multiplication strategy was the most used strategy that constituted 36% of the total strategies, and the ratio table was the second most used strategy (28.5%). The across-multiplication and ratio table strategies formed approximately 65% of the total number of strategies used in solving the inverse proportion problem. On the other hand, PSTs rarely used the proportion formula and constant product strategies, and they almost never applied graphical, other, and algebra strategies with graphical being the least used strategy.

Regarding PSTs' order of in which the strategies were used, two thirds of them applied the across-multiplication as their first strategy. The ratio table strategy followed the across-multiplication strategy that was used by 20 PSTs (22%) in their first solution attempts. Moreover, the ratio table was the most frequent strategy in PSTs' second and third solution attempts. As in direct proportion problem, there was no significant relationship between PSTs' first strategy and the number of strategies they applied, $F(4, 85) = .604$ and $p = .66$.

We also investigated, both for the direct and the inverse proportional problem, if the application of a specific strategy (i.e. whether a strategy was part of a PST's repertoire) was related to the number of strategies used by that PST (i.e. the size of that PST's repertoire). To have enough sample observation in each strategy category involved in this analysis, for both problems, we grouped graphical, algebra, and other strategies together and re-named them as "atypical strategies" since they were not used very often. The mean of the number of strategies used by applicants and nonapplicants of each strategy and

Table 5. The mean (and standard deviations) of PSTs' strategies for applicants and nonapplicants of each strategy.

	App. Status	Cross-Multip.	Across-Multip.	Ratio table	Proportion formula	Unit ratio	Unit product	Build up	Atyp.
DP	1	*n* = 68 3.06 (1.04)		*n* = 63 3.11 (.98)	*n* = 62 3.00 (.99)	*n* = 25 3.52 (.87)		*n* = 20 3.55 (1.09)	*n* = 20 4.05 (.78)
	0	*n* = 23 2.43 (.66)		*n* = 28 2.43 (.87)	*n* = 29 2.69 (1.00)	*n* = 66 2.67 (.95)		*n* = 71 2.72 (.89)	*n* = 71 2.58 (.82)
IP	1		*n* = 74 2.38 (.82)	*n* = 58 2.40 (.77)	*n* = 23 2.83 (.83)		*n* = 23 2.87 (.81)		*n* = 21 2.81 (.92)
	0		*n* = 17 1.59 (.71)	*n* = 33 1.94 (.93)	*n* = 68 2.03 (.77)		*n* = 68 2.01 (.76)		*n* = 70 2.06 (.75)

Note. DP: Direct Proportion; and IP: Inverse Proportion; Standard deviations are presented in between parentheses; 1 and 0 indicate PSTs' application status of a strategy; and *n* represents the number of PSTs.

standard deviations are presented in Table 5. For instance, the number 3.06 on the top of the first column in this table indicates the mean number of strategies used by PSTs who had a cross-multiplication strategy in their repertoire. In the direct proportion problem, the presence of strategies that were less common (i.e. unit ratio, build up, and atypical) in the repertoire was associated with a larger total number of strategies as compared to the presence of more common strategies (i.e. cross-multiplication, ratio table, and proportion formula). For example, PSTs having atypical strategies in their repertoire applied an average of 4.05 strategies, while those having cross-multiplication in their repertoire applied an average of 3.06 strategies. We obtained a similar finding for the inverse proportion problem: The presence of strategies that were less common (i.e. proportion formula, unit product, and atypical) in the repertoire was associated with a larger total number of strategies than the presence of more common strategies (i.e. cross-multiplication and ratio table) in the repertoire. For instance, PSTs having the unit product strategy in their repertoire applied an average of 2.87 strategies, while those having across-multiplication in their repertoire only applied an average of 2.38 strategies.

Using the information presented in Table 5, we conducted a *T*-test analysis to determine if there were significant differences among the mean of the number of strategies used, depending on whether or not a particular strategy was in the repertoire of a participant. For the direct proportion problem, Table 6 shows that the presence of an atypical

Table 6. The *T*-test and *p*-values for the comparison of mean number of strategies PSTs used in the direct proportion problem for each specific strategy.

	Ratio table	Proportion formula	Unit ratio	Build up	Atypical
Cross-multiplication	$t = -.283$ $p = .777$	$t = .336$ $p = .737$	$t = -1.971$ $p = .052$	$t = -1.832$ $p = .070$	$t = -3.937$ $p = .000$**
Ratio table		$t = .624$ $p = .534$	$t = -1.825$ $p = .072$	$t = -1.703$ $p = .092$	$t = -3.909$ $p = .000$**
Proportion formula			$t = -2.292$ $p = .024$*	$t = -2.108$ $p = .038$*	$t = -4.324$ $p = .000$**
Unit ratio				$t = -.103$ $p = .919$	$t = -2.125$ $p = .039$*
Build up					$t = -1.668$ $p = .103$

*The difference in means is significant at a level of .05; **The difference in means is significant at a level of .01.

Table 7. The *T*-test and *p*-values for the comparison of mean number of strategies PSTs used in the inverse proportion problem for each specific strategy.

	Ratio table	Proportion formula	Unit product	Atypical
Across-multiplication	$t = -.143$ $p = .887$	$t = -2.292$ $p = .024$*	$t = -2.510$ $p = .014$*	$t = -2.064$ $p = .042$*
Ratio table		$t = -2.217$ $p = .030$*	$t = -2.441$ $p = .017$*	$t = -1.984$ $p = .051$
Proportion formula			$t = -.165$ $p = .869$	$t = .076$ $p = .940$
Unit product				$t = .230$ $p = .819$

*The difference in means is significant at a level of .05.

strategy in the repertoire was associated with significantly more strategies than the presence of cross-multiplication, ratio table, proportion formula, and unit ratio strategies in the repertoire (4.05 strategies versus 3.06, 3.11, 3.00, and 3.52 strategies, respectively). Moreover, the presence of unit ratio and build up strategies in one's repertoire was associated with significantly more strategies than the presence of proportion formula strategy in the repertoire (3.52 and 3.55 strategies versus 3.00 strategies). For the inverse proportion problem, except for one comparison (atypical [2.81 strategies] versus ratio table [2.40 strategies]), the presence of less common strategies in the repertoire was associated with significantly more strategies than the presence of more common strategies (Table 7).

Discussion and conclusions

The purpose of this study was to examine PSTs' strategy repertoire by examining the distribution of their solution strategies for two missing-value problems, one direct and one inverse proportion. In the direct proportion problem, all PSTs used at least one strategy from Table 1. Similarly, except for one PST who did not provide any solution, PSTs used at least one strategy from Table 2. Compared to the inverse proportion problem, PSTs, on average, used a significantly greater number of strategies to solve the direct proportion problem. This result might be related to the fact that there are more strategies in Table 1 for direct proportion problems than in Table 2 for inverse proportion problems. Moreover, the inverse proportion problem included an extra piece of information (i.e. 30 cakes) that was in fact not needed for the solution. This extra information may have made the inverse proportion problem even more challenging, since PSTs had to ignore it to determine the proportional relationship between quantities. On the other hand, in agreement with Berk et al. (2009), average number of strategies used by PSTs in solving the two problems suggested that, when prompted, they were able to solve direct and inverse proportion problems using more than a single strategy. According to various researchers (e.g. Ball et al., 2008; Baumert et al., 2010; Copur-Gencturk & Doleck, 2021; Guberman & Leikin, 2013; Schukajlow et al., 2015), mastery of multiple solutions is part of PSTs' domain-specific professional knowledge. Thus, the current study was effective in revealing the potential of this domain-specific professional knowledge base of PSTs.

In agreement with the literature (e.g. Arican, 2019; Fisher, 1988), the cross-multiplication and across-multiplication strategies were the most used strategies for the direct and inverse proportion problem, respectively. PSTs might have used these two strategies more than the remaining strategies because of the great importance attached to rule memorisation and rote calculations in proportion instruction (Izsák & Jacobson, 2013). Although the cross-multiplication and across-multiplication algorithms are derived from direct proportion and inverse proportion, respectively, many PSTs apply these two algorithms without realising this detail (Arican, 2018). Hence, to develop PSTs' knowledge and experience on using less common proportional reasoning strategies, the instruction of the cross-multiplication and across-multiplication algorithms should be delayed in teaching proportions (Shield & Dole, 2013).

The algebra and graphical strategies were the least used strategies in solving the two problems, respectively. Similar to this result, using Fisher's (1988) strategy framework, Cabero-Fayos et al. (2020) reported that none of the participating Spanish PSTs used the algebra strategy to solve given missing-value proportion problems. Regarding graphical strategies, Lo (2004) reported PSTs' difficulty in providing pictures in explaining their solutions to proportion problems. Similarly, Arican and Kiymaz (2022) reported Turkish PSTs' difficulties in providing formal textbook definitions, formulas, and graphs of directly and inversely proportional relationships. As stated in the Methods section, in Turkey, the graphs of directly proportional relationships are taught in eighth grade as a special case of linear equations, but no instruction is provided on the graphs of inversely proportional relationships in middle and high school programmes (TMNE, 2018a, 2018b). However, using algebraic and graphical strategies could have benefit PSTs to develop deeper understanding of proportional relationships (Ruchti & Bennett, 2013; Shield & Dole, 2013).

Ratio tables are frequently used in middle school mathematics textbooks as an instruction technique in teaching ratios and proportions. Hence, after, respectively, the cross-multiplication and across-multiplication strategy, the ratio table was the second most used strategy in both problems. Similarly, Arican (2018) reported ratio table to be the most commonly used strategy among eight PSTs attending middle and secondary mathematics programmes in the United States. Furthermore, although many PSTs used the proportion formula for solving the direct proportion problem, they rarely did so for solving the inverse proportion problem. This scarcity suggested PSTs' difficulties in forming a proportion from corresponding values of two inverse quantities, which requires equating the first ratio to the multiplicative inverse of the second ratio. Finally, PSTs' solutions suggested a balance between strategies relying on algorithmic approaches (i.e. cross-multiplication, across-multiplication, proportion formula, and algebra) and conceptual approaches (i.e. ratio table, unit ratio, unit product, build up, and graphical). This balance appears to be a result of ratio and proportion instruction in Turkey mostly relying on the cross-multiplication and across-multiplication algorithms and proportion formula strategy and ratio table and unit ratio strategies (TMNE, 2018a, 2018b).

PSTs used more common strategies first, and then less common strategies to solve the two problems. Hence, many PSTs used the cross-multiplication and across-multiplication strategies as their first solution strategy for both problems. On the other hand, in both problems, we did not detect a significant relationship between the first strategy of PSTs and the number of strategies they used. One key finding was that PSTs who (also) had the less common strategies (i.e. unit ratio, build up, and atypical strategies for the direct proportion problem and proportion formula, unit product, and atypical strategies for the inverse proportion problem) in their repertoire had significantly more strategies in their repertoire than PSTs who only used more common strategies. Hence, PSTs who had the less common strategies in their repertoire also had the more common ones in their repertoire, whereas the opposite was not always true: Some PSTs only used the more common ones. Thus, in agreement with Silver et al. (2005), we speculate that limitations in mathematical content knowledge of PSTs were an obstacle to their use of less common strategies in solving the two proportion problems.

Limitations and suggestions for future studies

The study included a relatively small sample of PSTs, and the educational background of PSTs may have influenced their strategy use. Hence, it would be beneficial to conduct future studies with a larger sample and in PSTs with different educational backgrounds. Furthermore, the current study only collected PSTs' solutions to two missing-value proportion problems. Similarly, future studies should include a variety of problems with different contexts and levels of numerical complexity, with a view to get a better understanding of the impact of the problem context and the numbers being used in problems on PSTs' strategy repertoire. Finally, this study only included PSTs and not in-service teachers. Therefore, we suggest conducting a study with in-service teachers to obtain information on their professional knowledge base about strategies for solving proportion problems and to make recommendations for what they should do to improve their teaching.

Implications for teaching and teacher education

Encouraging students to generate multiple solutions to given problems is an important instruction technique in improving their mathematical knowledge (Schukajlow & Krug, 2014). To assist students in developing multiple solutions, teachers should know multiple solution strategies and know how to apply them in solving specific problems, and so they should be given "more opportunities in teacher education and professional development programs to learn to solve problems by using both formal and informal strategies" (Copur-Gencturk & Doleck, 2021, p. 6). As stated in the Methods section, the university programme that PSTs attended did not include any type of course specifically designed for providing instruction on ratio, proportion, and proportional relationships. Yet, based on previous experiences with ratios and proportions, which usually relied on rule memorisation and rote computations (Arican, 2018), PSTs showed some capacity to use different strategies in solving the two proportion problems when prompted. As stated by Burgos and Godino (2020), teachers should be able to use different solution strategies beyond rules and rote computations to provide students with a deeper understanding of proportional situations. However, Hines and McMahon (2005) reported PSTs' lack of familiarity with middle school students' proportional reasoning strategies. Therefore, teacher education programmes should provide opportunities for PSTs to improve their professional knowledge bases on strategies used to solve proportion problems. Finally, also discussed by Shield and Dole (2013), PSTs' over-reliance on the rote computations and rule memorisation suggested limitations in mathematics textbooks on promoting their proportional reasoning.

Thus, mathematics textbooks should be designed in such a way as to develop students' and PSTs' proportional reasoning.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Data availability statement

The datasets used and/or analysed during the current study are available from authors on a reasonable request.

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Appendix

1. Strategy Repertoire Task

Dear students. Two problems are presented in below. Follow the steps in the booklet.

Stage 1a:

Please solve the given problem. Show clearly in your written work how you solve it.

Beyza decided to sell lemonade in front of their house. Yesterday she mixed 6 L of water and 2 L of lemon juice to make this lemonade. Today, Beyza used 18 L of water, please calculate how many litres of lemon juice she should include in the mixture to make lemonade with the exact same taste?

Solution:

Stage 2a:

You may use other strategies to solve the problem from the previous page. Please try to solve the problem in as many different ways as possible.

Strategy A:

Strategy B:

Strategy C:

Other strategies:

If you can still come up with other strategies than the ones you used before, you can write all of them here.

Stage 1b:

Here is another problem. Please solve it. Show clearly in your written work how you solve it.

In a bakery, four workers can decorate 30 cakes in 8 h. Please calculate how many hours is needed for 16 workers to decorate these many cakes?

Solution:

Stage 2b:

You may use other strategies to solve the problem from the previous page. Please try to solve the problem in as many different ways as possible.

Strategy A:

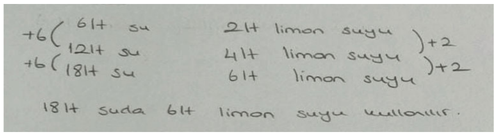
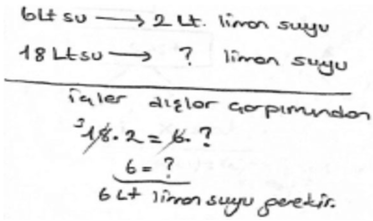
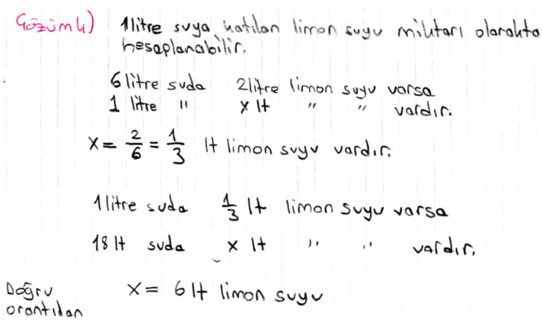
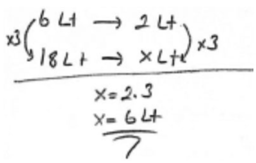
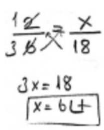
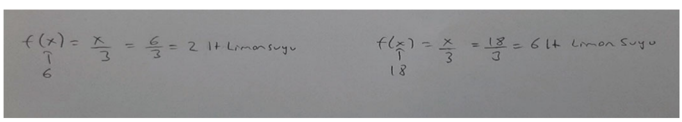
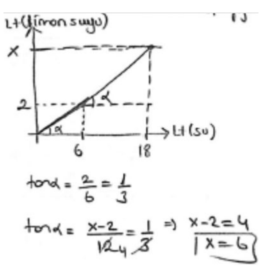
Strategy B:

Strategy C:

Other strategies:

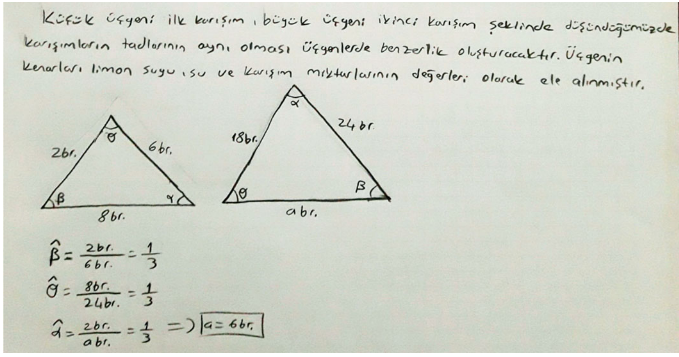
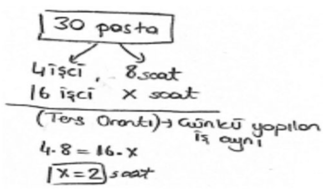
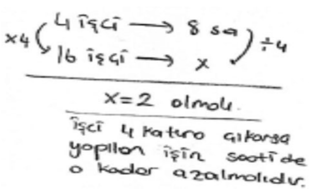
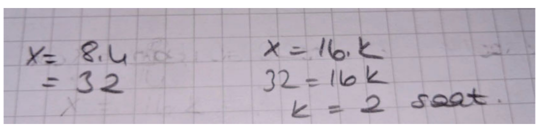
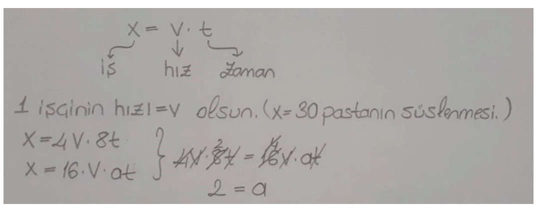
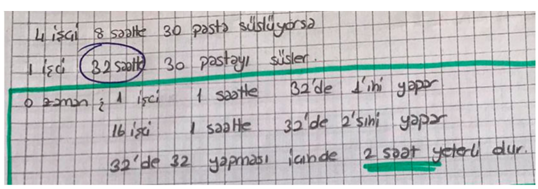
If you can still come up with other strategies than the ones you used before, you can write all of them here.

2. Strategy Examples from the Preservice Teachers' Responses

Strategy	Sample solutions for the inverse proportion problem
Build-Up	
Cross-Multiplication	
Unit Ratio	
Ratio Table	
Proportion Formula	
Algebra	
Graphical	

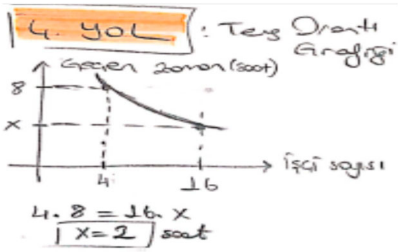
(*Continued*)

Continued.

Strategy	Sample solutions for the inverse proportion problem
Other	
Strategy	Sample solutions for the inverse proportion problem
Across-Multiplication	
Ratio Table	
Proportion Formula	
Algebra	
Unit Product	

(*Continued*)

Continued.

Strategy	Sample solutions for the inverse proportion problem
Graphical	
Other