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Original article

The Hardy–Littlewood–Sobolev theorem for Riesz potential generated by Gegenbauer operator

Elman J. Ibrahimov[a,*], Ali Akbulut[b]

[a] *Institute of Mathematics and Mechanics, Baku, Azerbaijan*
[b] *Ahi Evran University, Department of Mathematics, Kirsehir, Turkey*

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Abstract

In this paper we introduced and studied the maximal function (G-maximal function) and the Riesz potential (G-Riesz potential) generated by Gegenbauer differential operator

$$G_\lambda = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}.$$

The $L_{p,\lambda}$ boundedness of the G-maximal operator is obtained. Hardy–Littlewood–Sobolev theorem of G-Riesz potential on $L_{p,\lambda}$ spaces is established.

Keywords: G-Riesz potential; G-maximal function; G-BMO space

0. Introduction

The Hardy–Littlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 (see [1]) for 2π-periodical functions, and later it was extended to the Euclidean spaces, some weighted measure spaces (see [2–4]), symmetric spaces (see [5,6]), various Lie groups [7], for the Jacobi-type hypergroups [8,9], for Chebli–Trimeche hypergroups [10], for the one-dimensional Bessel–Kingman hypergroups [11–13], for the n-dimensional Bessel–Kingman hypergroups ($n \geq 1$) [14–18], and for Laguerre hypergroup [19–22]. The structure of the paper is as follows. In Section 1 we present some definitions, notation and auxiliary results. In Section 2 the $L_{p,\lambda}$ boundedness of the G-maximal operator is proved. In Section 3 we introduce definition of G-Riesz potential. In Section 4 it is proved for the Sobolev type theorem.

* Corresponding author.
E-mail addresses: elmanibrahimov@yahoo.com (E.J. Ibrahimov), aakbulut@ahievran.edu.tr (A. Akbulut).
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1. Definitions, notation and auxiliary results

Let $H(x,r) = (x-r, x+r) \cap [0,\infty)$, $r \in (0,\infty)$, $x \in [0,\infty)$. For all measurable sets $E \subset [0,\infty)$, $\mu E \equiv |E|_\lambda = \int_E sh^{2\lambda}\, t dt$. For $1 \le p \le \infty$ let $L_p([0,\infty), G) \equiv L_{p,\lambda}[0,\infty)$ be the space of functions measurable on $[0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \left(\int_0^\infty |f(ch\ t)|^p sh^{2\lambda}\ t dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|f\|_{\infty,\lambda} = \operatorname*{ess\,sup}_{t\in[0,\infty)} |f(ch\ t)|, \quad p = \infty.$$

Analogy by [9] we define Gegenbauer maximal functions as follows:

$$M_G f(ch\ x) = \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\ t} |f(ch\ x)|\, d\mu_\lambda(t),$$
$$M_\mu f(ch\ x) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(ch\ t)|\, d\mu_\lambda(t), \ d\mu_\lambda(t) = sh^{2\lambda}\ t dt,$$
$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt, \quad |H(x,r)|_\lambda = \int_{H(x,r)} sh^{2\lambda}\ t dt, \quad 0 < \lambda < \frac{1}{2},$$

where

$$H(x,r) = \begin{cases} (0,\ x+r), & x < r, \\ (x-r, x+r), & x > r. \end{cases}$$

Here (see [22])

$$A^\lambda_{ch\ t} f(ch\ x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi f(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi)(\sin\ \varphi)^{2\lambda-1} d\varphi$$

denote the generalized shift operator, associated with the Gegenbauer differential operator

$$G = \left(x^2-1\right)^{1/2-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+1/2} \frac{d}{dx}, \quad x \in (1,\ \infty).$$

Further we will need some auxiliary assertions.

Lemma 1.1. *For $0 < \lambda < 1/2$ the following correlations are true:*

$$|H(0,r)|_\lambda \sim \begin{cases} \left(sh\ \dfrac{r}{2}\right)^{2\lambda+1}, & 0 < r < 2, \\ \left(ch\ \dfrac{r}{2}\right)^{4\lambda}, & 2 \le r < \infty, \end{cases}$$

where c denotes a positive constant.

Here $f \sim g$ denotes that $c_{1,\lambda} g \le f \le c_{2,\lambda} g$ for some positive constants $c_{1,\lambda}$ and $c_{2,\lambda}$ depending on λ.

Proof. Let first $0 < r < 2$, then

$$\begin{aligned} |H(0,r)|_\lambda &= \int_0^r sh^{2\lambda}\ t dt = \int_0^r (sh\ t)^{2\lambda-1} d(ch\ t) = \int_0^r (ch^2\ t-1)^{\lambda-\frac{1}{2}} d(ch\ t) \\ &= \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} (t+1)^{\lambda-\frac{1}{2}} dt \ge (ch\ r+1)^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} dt \\ &\ge (ch1+1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}} \Big|_1^{ch\ r} = \frac{2(ch\ r-1)^{\lambda+\frac{1}{2}}}{(2\lambda+1)(1+ch1)^{\frac{1}{2}-\lambda}} \\ &= \frac{2^{\lambda+\frac{3}{2}}}{(2\lambda+1)(1+ch1)^{\frac{1}{2}-\lambda}} \left(sh\ \frac{r}{2}\right)^{2\lambda+1}. \end{aligned} \tag{1.1}$$

On the other hand,

$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt = \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}}(t+1)^{\lambda-\frac{1}{2}} dt \le 2^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} dt$$
$$= \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}(t-1)^{\lambda+\frac{1}{2}}\Big|_1^{ch\ r} = \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}(ch\ r-1)^{\lambda+\frac{1}{2}} = \frac{2^{2\lambda+1}}{2\lambda+1}\left(sh\ \frac{r}{2}\right)^{2\lambda+1}.$$

Let now $2 \le r < \infty$. Then

$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt = \int_0^r (sh\ t)^{2\lambda-1} d(ch\ t) = \int_0^r (ch^2\ t-1)^{\lambda-\frac{1}{2}} d(ch\ t)$$
$$= \int_1^{ch\ r} \frac{(t-1)^{\lambda-\frac{1}{2}}}{(t+1)^{\frac{1}{2}-\lambda}} dt \ge (ch\ r+1)^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} dt$$
$$= (ch\ r+1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}}\Bigg|_1^{ch\ r} = \frac{2}{2\lambda+1}\frac{(ch\ r-1)^{\lambda+\frac{1}{2}}}{(ch\ r+1)^{\frac{1}{2}-\lambda}}$$
$$= \frac{2}{2\lambda+1}\frac{\left(2sh^2\ \frac{r}{2}\right)^{\lambda+\frac{1}{2}}}{\left(2ch^2\ \frac{r}{2}\right)^{\frac{1}{2}-\lambda}} = \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}}\frac{\left(2sh\ \frac{r}{2}\right)^{2\lambda+1}}{\left(ch\ \frac{r}{2}\right)^{1-2\lambda}}$$
$$\ge \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}}\left(ch\ \frac{r}{2}\right)^{4\lambda} \Leftrightarrow \left(2sh\ \frac{r}{2}\right)^{2\lambda+1}$$
$$\ge \left(ch\ \frac{r}{2}\right)^{2\lambda+1} \Leftrightarrow 2sh\ \frac{r}{2} \ge ch\ \frac{r}{2} \Leftrightarrow 2\frac{e^{r/2}-e^{-r/2}}{2}$$
$$\ge \frac{e^{r/2}+e^{-r/2}}{2} \Leftrightarrow 2(e^r-1) \ge e^r+1 \Leftrightarrow 2e^r \ge 3,$$

that takes place for $r \ge 2$.

Thus,

$$|H(0,r)|_\lambda \ge \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}}\left(ch\ \frac{r}{2}\right)^{4\lambda}.$$

Let us obtain an upper bound for $|H(0,r)|_\lambda$.

$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt = \int_0^r \left(2sh\ \frac{t}{2} ch\ \frac{t}{2}\right)^{2\lambda} dt$$
$$= 2^{2\lambda+1}\int_0^r \left(sh\ \frac{t}{2}\right)^{2\lambda}\left(ch\ \frac{t}{2}\right)^{2\lambda-1} d\left(sh\ \frac{t}{2}\right) \le 2^{2\lambda+1}\int_0^r \left(sh\ \frac{t}{2}\right)^{4\lambda-1} d\left(sh\ \frac{t}{2}\right)$$
$$= \frac{2^{2\lambda+1}}{4\lambda}\left(sh\ \frac{t}{2}\right)^{4\lambda}\Bigg|_0^r = \frac{4^\lambda}{2\lambda}\left(sh\ \frac{r}{2}\right)^{4\lambda} \le \frac{4^\lambda}{2\lambda}\left(ch\ \frac{r}{2}\right)^{4\lambda}. \tag{1.2}$$

Combining (1.1)–(1.2), we obtain assertion of Lemma 1.1. □

Lemma 1.2. *Let* $0 < \lambda < 1/2$ *and* $x \in [0,\infty)$, $r \in (0,\infty)$. *Then the following estimates are reasonable for* $0 < r < 2$

$$|H(x,r)|_\lambda \le c_\lambda \begin{cases} \left(sh\ \frac{r}{2}\right)^{2\lambda+1}, & 0 \le x < r, \\ sh\ \frac{r}{2} ch^{2\lambda}\ x, & r \le x < \infty. \end{cases} \tag{a}$$

For $2 \le r < \infty$.

$$|H(x,r)|_\lambda \le c_\lambda \begin{cases} ch^{2\lambda}\ r, & 0 \le x < r, \\ ch^{2\lambda}\ x ch^{2\lambda}\ r, & r \le x < \infty. \end{cases} \tag{b}$$

Here and further $c_\lambda, c_{\alpha,\lambda}, c_{\alpha,\lambda,p}$ will denote some constants, depending only on subscribed indexes and generally speaking different in different formulas.

Proof. First we consider the case $0 < r < 1$ and $x \in [0, \infty)$.

Let $0 \le t \le 2$, Then we have

$$t \le sh\ t \le e \cdot t. \tag{1.3}$$

We prove left-hand part of this estimate. We consider the function $f(t) = sh\ t - t$. Since, $f'(t) = ch\ t - 1 \ge 0$, then $f(t)$ increases on $[0, \infty)$, and that takes the smallest value for $t = 0$, $f(0) = 0$, consequently $f(t) \ge 0$ is equivalent to $sh\ t \ge t$.

We prove right-hand part of estimate (1.3).

$$\frac{e^t - e^{-t}}{2} \le e \cdot t \quad \Leftrightarrow \quad e^{2t} - 1 \le 2 \cdot e^{1+t} \cdot t \quad \Leftrightarrow \quad e^{2t} \le 2 \cdot e^{1+t} \cdot t + 1.$$

We consider the function $f(t) = 2 \cdot e^{1+t} \cdot t + 1 - e^{2t}$.

$$f'(t) = 2 \cdot e^{1+t} + 2 \cdot e^{1+t} \cdot t - 2e^{2t} = 2e^t \left(e + t \cdot e - e^t\right) \ge e(t+1) - e^t \ge 0 \quad \text{as, } t \le 2.$$

Thus, the estimate (1.3) is proved.

Hence it follows that for $0 \le x < r < 2$

$$|H(x,r)|_\lambda = \int_0^{x+r} sh^{2\lambda}\ t dt \le e^{2\lambda} \int_0^{2r} t^{2\lambda} dt = \frac{(2e)^{2\lambda}}{2\lambda+1} \cdot r^{2\lambda+1} \le c_\lambda \left(sh\ \frac{r}{2}\right)^{2\lambda+1}. \tag{1.4}$$

For $r \le x < 2$

$$|H(x,r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda}\ t dt \le e^{2\lambda} \int_{x-r}^{x+r} t^{2\lambda} dt \le 2e^{2\lambda} \cdot r \cdot (x+r)^{2\lambda} \le 2e^{2\lambda} \cdot r \cdot (2x)^{2\lambda} \le c_\lambda\, sh\ \frac{r}{2} ch^{2\lambda}\ x.$$

Let now $0 < r < 2 \le x < \infty$, then we have

$$|H(x,r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda}\ t dt \le 2r \cdot sh^{2\lambda}(x+r) = 2r(sh\ xch\ r + ch\ xsh\ r)^{2\lambda}$$
$$\le 2r(sh\ xch\ 1 + ch\ xsh\ 1)^{2\lambda} \le 2r(2ch\ xch\ 1)^{2\lambda} \le c_\lambda\, sh\ \frac{r}{2} ch^{2\lambda}\ x. \tag{1.5}$$

Now we consider the case, $2 \le r < \infty$, $x \in [0, \infty)$.

Let $0 \le x < 2 \le r$. As in the proof of the estimate (1.2), we obtain

$$|H(x,r)|_\lambda = \int_0^{x+r} sh^{2\lambda}\ t dt = \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{t}{2} \Big|_0^{x+r} = \frac{4^\lambda}{2\lambda} sh^{4\lambda}\ \frac{x+r}{2}$$
$$= \frac{4^\lambda}{2\lambda}\left(sh\ \frac{x}{2} ch\ \frac{r}{2} + ch \frac{x}{2} sh\ \frac{r}{2}\right)^{4\lambda} \le c_\lambda \left(sh\ \frac{1}{2}\, ch\ \frac{r}{2} + ch\ \frac{1}{2}\, sh\ \frac{r}{2}\right)^{4\lambda} \le c_\lambda ch^{4\lambda}\ \frac{r}{2}. \tag{1.6}$$

Let now $2 \le r \le x < \infty$, then

$$|H(x,r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda}\ t dt \le \frac{4^\lambda}{2\lambda}\ sh^{4\lambda}\ \frac{t}{2}\Big|_{x-r}^{x+r} = \frac{4^\lambda}{2\lambda}\left(sh^{4\lambda}\ \frac{x+r}{2} - sh^{4\lambda}\ \frac{x-r}{2}\right)$$
$$\le \frac{4^\lambda}{2\lambda} sh^{4\lambda}\ \frac{x+r}{2} \le c_\lambda ch^{4\lambda}\ \frac{x}{2} ch^{4\lambda}\ \frac{r}{2} \le c_\lambda ch^{2\lambda}\ xch^{2\lambda}\ r. \tag{1.7}$$

From (1.6) and (1.7) it follows that at $2 \le r < \infty$ and $0 \le x < \infty$

$$|H(x,r)|_\lambda \le c_\lambda ch^{2\lambda}\ r, \quad 0 \le x < r, \tag{1.8}$$
$$|H(x,r)|_\lambda \le ch^{2\lambda}\ xch^{2\lambda}\ r, \quad r \le x < \infty. \tag{1.9}$$

Assertion of Lemma 1.2 follows from (1.4)–(1.5), (1.8) and (1.9). $\square$

2. $L_{p,\lambda}$-boundedness of the G-maximal operator

Theorem 2.1. *For $0 \le x < \infty$ and $0 < r < \infty$ the following inequality is valid*

$$M_G f(ch\ x) \le c_\lambda M_\mu f(ch\ x),$$

where c_λ is a positive constant.

Proof. Consider the integral

$$\begin{aligned} I(x,r) &= \int_0^r A^\lambda_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ t dt \\ &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_0^\pi |f(ch\ x \cdot ch\ t - sh\ x \cdot sh\ t \cos\ \varphi)|\, (\sin\ \varphi)^{2\lambda-1} d\varphi \right\} sh^{2\lambda}\ tdt. \end{aligned}$$

Making the substitution
$z = ch\ x \cdot ch\ t - sh\ x \cdot sh\ t \cos\ \varphi$, we get that

$$\cos\ \varphi = \frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t}, \quad \varphi =$$
$$mboxarccos \frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t},$$
$$\begin{aligned} d\varphi &= \frac{dz}{\sqrt{1 - (\frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t})^2} sh\ x \cdot sh\ t} \\ &= (sh^2\ x \cdot sh^2\ t - ch^2\ x \cdot ch^2\ t + 2 \cdot z \cdot ch\ x \cdot ch\ t - z^2)^{-\frac{1}{2}} dz. \end{aligned}$$

Since,

$$\begin{aligned} sh^2\ x \cdot sh^2\ t - ch^2\ x \cdot ch^2\ t &= (ch^2\ x - 1) sh^2\ t - ch^2\ x \cdot ch^2\ t = ch^2\ x \cdot sh^2\ t - sh^2\ t - ch^2\ x \cdot ch^2\ t \\ &= -sh^2\ t + ch^2\ x (sh^2\ t - ch^2\ t) = -sh^2\ t - ch^2\ x, \end{aligned}$$

that

$$d\varphi = (2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2)^{-\frac{1}{2}} dz$$

and

$$(\sin\ \varphi)^{2\lambda-1} = (2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2)^{\lambda-\frac{1}{2}} (sh\ x \cdot sh\ t)^{1-2\lambda}.$$

Then $I(x,r)$ makes a list of form

$$\begin{aligned} I(x,r) &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \\ &\times \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)|\, (2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2)^{\lambda-1} (sh\ x)^{1-2\lambda} dz \right\} sh\ tdt. \end{aligned} \tag{2.1}$$

Transform expansion

$$\begin{aligned} &2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t - sh^2\ t (ch^2\ x - sh^2\ x) - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t - sh^2\ t \cdot ch^2\ t + sh^2\ t \cdot sh^2\ x - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2\ t \cdot sh^2\ x - (ch^2\ t - 1) ch^2\ x - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2\ x \cdot (ch^2\ t - 1) - ch^2\ t \cdot ch^2\ x - z^2 (ch^2\ x - sh^2\ x) \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2\ x \cdot ch^2\ t - sh^2\ x - ch^2\ t \cdot ch^2\ x - z^2 \cdot ch^2\ x - z^2 \cdot sh^2\ x \end{aligned}$$

$$= 2z \cdot ch\ x \cdot ch\ t - sh^2\ x - ch^2\ t - z^2 ch^2\ x - z^2 sh^2\ x = (z^2-1) sh^2\ x - (ch\ t - z \cdot ch\ x)^2$$

$$= \left(z^2-1\right) sh^2\ x \left[1 - \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right)^2\right]. \tag{2.2}$$

Taking into account (2.1) and (2.2) we get

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)| (z^2-1)^{\lambda-1} \left[1 - \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right)^2\right]^{\lambda-1} dz \right\} \frac{sh\ t}{sh\ x} dt. \tag{2.3}$$

Note that

$$\frac{sh\ t}{sh\ x} = (z^2-1)^{\frac{1}{2}} \frac{\partial}{\partial t} \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right),$$

rewrite (2.3) of form

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)| (z^2-1)^{\lambda-\frac{1}{2}} \right.$$

$$\left. \times \left[1 - \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right)^2\right]^{\lambda-1} \frac{\partial}{\partial t} \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right) \right\} dz dt. \tag{2.4}$$

Since $ch\ (x-t) \le z \le ch\ (x+t)$, then we obtain

$$\begin{cases} ch\ (x-r) \le z \le ch\ x \\ x - \operatorname{arc}chz \le t \le r \end{cases} \quad \text{and} \quad \begin{cases} ch\ x \le z \le ch\ (x+r) \\ \operatorname{arc}chz - x \le t \le r. \end{cases}$$

That is why changing the order of integration in (2.4), we get

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \left(\int_{ch\ (x-r)}^{ch\ x} dz \int_{x-\operatorname{arc}chz}^{r} dt + \int_{ch\ x}^{ch\ (x+r)} dz \int_{\operatorname{arc}chz - x}^{r} dt \right). \tag{2.5}$$

Consider the integral

$$A(x,z,r) \equiv A(x,r) = \int_{x-\operatorname{arc}chz}^{r} \left[1 - \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right)^2\right]^{\lambda-1} \frac{\partial}{\partial t} \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right) dt.$$

Putting $u = \frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}$, we get

$$A(x,z,r) \equiv A(x,r) = \int_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} (1-u^2)^{\lambda-1} du. \tag{2.6}$$

On the even power of $ch\ t$

$$B(x,r) = \int_{\operatorname{arc}chz - x}^{r} \left[1 - \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right)^2\right]^{\lambda-1} \frac{\partial}{\partial t} \left(\frac{ch\ t - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right) dt$$

$$= \int_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} (1-u^2)^{\lambda-1} du. \tag{2.7}$$

Taking into account (2.6) and (2.7) in (2.5), we have

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_{ch\ (x-r)}^{ch\ (x+r)} |f(z)| (z^2-1)^{\lambda-\frac{1}{2}} \int_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} (1-u^2)^{\lambda-1} du dz. \tag{2.8}$$

Since $ch\ (x-r) \le z \le ch\ (x+r)$, then

$$\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x} \ge \frac{ch\ r - ch\ x\cdot ch\ (x+r)}{sh\ x\cdot sh(x+r)} = \frac{2ch\ r - 2ch\ x\cdot ch\ (x+r)}{2sh\ x\cdot sh(x+r)}$$
$$= \frac{2ch\ r - ch(2x+r) - ch\ r}{ch(2x+r) - ch\ r} = \frac{ch\ r - ch(2x+r)}{ch(2x+r)-ch\ r} = -1. \tag{2.9}$$

On the other hand for $ch\ (x-r) \le z \le ch\ (x+r)$,

$$\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x} \le \frac{ch\ r - ch\ x\cdot ch\ (x-r)}{sh\ x\,|sh(x-r)|} = \frac{2ch\ r - 2ch\ x\cdot ch\ (x-r)}{2sh\ x\cdot sh(r-x)}$$
$$= \frac{2ch\ r - ch(2x-r) - ch\ r}{ch\ r - ch(2x-r)} = \frac{ch\ r - ch(2x-r)}{ch\ r - ch(2x-r)} = 1. \tag{2.10}$$

From (2.9) and (2.10) it follows that for $ch\ (x-r) \le z \le ch\ (x+r)$, and $0 < x < r < 2$

$$-1 \le \frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x} \le 1. \tag{2.11}$$

From (2.11) it follows that for $0 < x < r < 2$

$$A(x,r) = \int_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} (1-u^2)^{\lambda-1}du \le \int_{-1}^{1}(1-u^2)^{\lambda-1}du = \frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}. \tag{2.12}$$

But taking into account (2.12) and (2.8), we obtain that for $0 < x < r < 2$

$$I(x,r) \le \int_{ch\ (x-r)}^{ch\ (x+r)} |f(z)|\,(z^2-1)^{\lambda-\frac{1}{2}}dz = \int_{x-r}^{x+r} |f(ch\ t)|\,sh^{2\lambda}\ tdt. \tag{2.13}$$

Now let $2 \le r \le x < \infty$ and $ch\ (x-r) \le z \le ch\ (x+r)$.
Then we have

$$\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x} \le \frac{ch\ x - z\cdot ch\ x}{\sqrt{z^2-1}sh\ x} = \frac{(1-z)ch\ x}{\sqrt{z^2-1}sh\ x} = -\frac{\sqrt{z-1}ch\ x}{\sqrt{z+1}sh\ x} \le 0.$$

From (2.9) it follows that

$$\max(1-u)^{\lambda-1} \le \max_{-1\le u\le 0}(1-u)^{\lambda-1} = \max(2^{\lambda-1},1) = 1,$$
$$-1 \le u \le \frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}.$$

Taking into account this circumstance, for the integral $A(x,r)$ we obtain of (2.6)

$$A(x,r) = \int_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} \left(1-u^2\right)^{\lambda-1} du$$
$$\le \int_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} (1+u)^{\lambda-1}\,du = \frac{1}{\lambda}\,(1+u)^{\lambda}\Big|_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} = \frac{1}{\lambda}\left(1+\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right)^{\lambda}$$
$$= \frac{1}{\lambda}\left(1-\frac{z\cdot ch\ x - ch\ r}{\sqrt{z^2-1}\cdot sh\ x}\right)^{\lambda} \le \frac{1}{\lambda}\left[1-\left(\frac{z\cdot ch\ x - ch\ r}{\sqrt{z^2-1}\cdot sh\ x}\right)^2\right]^{\lambda}. \tag{2.14}$$

We find extremum of the function

$$f(z) = 1-\left(\frac{z\cdot ch\ x - ch\ r}{\sqrt{z^2-1}\cdot sh\ x}\right)^2.$$

$$f'(z) = -2\left(\frac{z\cdot ch\ x - ch\ r}{\sqrt{z^2-1}\cdot sh\ x}\right)\times\frac{(z^2-1)sh\ x\cdot ch\ x - z^2 sh\ x\cdot ch\ x + z\cdot ch\ r\cdot sh\ x}{(z^2-1)^{\frac{3}{2}}sh^2\ x}$$
$$= -2\left(\frac{z\cdot ch\ x - ch\ r}{\sqrt{z^2-1}\cdot sh\ x}\right)\frac{z\cdot ch\ r\cdot sh\ x - ch\ x\cdot sh\ x}{(z^2-1)^{\frac{3}{2}}sh^2\ x} = \frac{2(z\cdot ch\ x - ch\ r)(ch\ x - z\cdot ch\ r)}{(z^2-1)^2 sh^2\ x}.$$

Since $ch\ (x-r)\le z\le ch\ (x+r)$, then the function $f(z)$ for $z = ch\ x/ch\ r$ has a maximum

$$f_{\max}\left(\frac{ch\ x}{ch\ r}\right) = 1-\left(\frac{ch^2\ x - ch^2\ r}{\sqrt{ch^2\ x - ch^2\ r}\cdot sh\ x}\right)^2 = 1-\frac{ch^2\ x - ch^2\ r}{sh^2\ x} = \frac{ch^2\ r - 1}{sh^2\ x} = \left(\frac{sh\ r}{sh\ x}\right)^2.$$

From (2.14) we have

$$A(x,r)\le\frac{1}{\lambda}\left(\frac{sh\ r}{sh\ x}\right)^{2\lambda}. \tag{2.15}$$

According to definition of maximal function we have

$$M_G f(chx)\le M_{G,1}f(chx) + M_{G,2}f(chx),$$

where

$$M_{G,1}f(chx) = \sup_{0<r<2}\frac{1}{|H(0,r)|_\lambda}\int_0^r |f(cht)|d\mu_\lambda(t),$$
$$M_{G,2}f(chx) = \sup_{2\le r<\infty}\frac{1}{|H(0,r)|_\lambda}\int_0^r |f(cht)|d\mu_\lambda(t).$$

Let $0<r<2$, then taking into account Lemmas 1.1 and 1.2 (a), for (2.13) with $0\le x<r<2$ we get

$$\begin{aligned} M_{G,1}f(ch\ x) &= \sup_{0<r<2}\frac{1}{|H(0,r)|_\lambda}\int_0^r A^\lambda_{ch\ t}|f(ch\ x)|\,d\mu_\lambda(t)\\ &= \sup_{0<r<2}\frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda}\cdot\frac{1}{|H(x,r)|_\lambda}\int_{|x-r|}^{x+r}|f(ch\ t)|\,sh^{2\lambda}\ tdt\\ &\le c_\lambda\sup_{0<r<2}\frac{1}{|H(x,r)|_\lambda}\int_{H(x,r)}|f(ch\ t)|\,d\mu(t) = c_\lambda M_\mu f(ch\ x). \end{aligned}\tag{2.16}$$

For $r<2\le x<\infty$ from Lemmas 1.1, 1.2(a), (2.15) and (2.8) we obtain

$$\begin{aligned} M_{G,1}f(ch\ x) &\le \sup_{0<r<2}\frac{A(x,r)\,|H(x,r)|_\lambda}{|H(0,r)|_\lambda\,|H(x,r)|_\lambda}\int_{x-r}^{x+r}|f(ch\ t)|\,sh^{2\lambda}\ tdt\\ &\le c_\lambda\sup_{0<r<2}\frac{sh\ \frac{r}{2}ch^{2\lambda}\ x\cdot sh^{2\lambda}\ r}{|H(x,r)|_\lambda\left(sh\ \frac{r}{2}\right)^{2\lambda+1}sh^{2\lambda}\ x}\int_{x-r}^{x+r}|f(ch\ t)|\,d\mu(t)\\ &\le c_\lambda\left(\frac{ch\ x}{sh\ x}\right)^{2\lambda}\sup_{0<r<2}ch^{2\lambda}\ \frac{r}{2}\cdot\frac{1}{|H(x,r)|_\lambda}\int_{x-r}^{x+r}|f(ch\ t)|\,d\mu(t)\\ &\le c_\lambda\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^{2\lambda}ch^{2\lambda}\ \frac{1}{2}\sup_{0<r<2}\frac{1}{|H(x,r)|_\lambda}\int_{H(x,r)}|f(ch\ t)|\,d\mu(t)\\ &\le c_\lambda\cdot 4^\lambda\cdot e\cdot M_\mu f(ch\ x), \end{aligned}\tag{2.17}$$

as $\frac{e^{2x}+1}{e^{2x}-1}\le 2\Leftrightarrow e^{2x}+1\le 2e^{2x}-2\Leftrightarrow e^{2x}\ge 3$ at $x\ge 1$.

From (2.16) and (2.17) it follows that

$$M_{G,1}f(ch\ x)\le c_\lambda M_\mu f(ch\ x),\ 0<r<2,\quad 0\le x<\infty. \tag{2.18}$$

Now we consider the case $2\le r<\infty$.

Point that for $ch\ (x-r) \le z \le ch\ (x+r)$ and $x > r$ the function $f(z) = \frac{ch\ r - zch\ x}{\sqrt{z^2-1} sh\ x}$ has maximum equal $-\frac{\sqrt{ch^2\ x - ch^2\ r}}{sh\ x}$.

In fact

$$\begin{aligned} f'(z) &= -\frac{\sqrt{z^2-1} sh\ xch\ x + \frac{z}{\sqrt{z^2-1}} sh\ x(ch\ r - zch\ x)}{(z^2-1) sh^2\ x} \\ &= -\frac{(z^2-1) sh\ xch\ x + zsh\ xch\ r - z^2 sh\ xch\ x}{(z^2-1)^{\frac{3}{2}} sh^2\ x} = \frac{ch\ x - zch\ r}{(z^2-1)^{\frac{3}{2}} sh\ x} = 0 \quad \Leftrightarrow \quad z = \frac{ch\ x}{ch\ r}. \end{aligned}$$

In this point the function $f(z)$ has a maximum:

$$\begin{aligned} f_{\max}(z) &= f\left(\frac{ch\ x}{ch\ r}\right) = \frac{ch^2\ r - ch^2\ x}{\sqrt{ch^2\ x - ch^2\ r} \cdot sh\ x} = -\frac{\sqrt{ch^2\ x - ch^2\ r}}{sh\ x} \\ &= -\frac{ch\ x}{sh\ x}\sqrt{1 - \left(\frac{ch\ r}{ch\ x}\right)^2} \sim -\frac{sh\ x}{ch\ x}, \end{aligned} \tag{2.19}$$

as

$$\lim_{x\to\infty} \frac{sh\ x}{ch\ x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1.$$

From (2.15) and (2.19) we obtain

$$\begin{aligned} A(x,r) &\le \int_{-1}^{\frac{ch\ r - zch\ x}{\sqrt{z^2-1}\ sh\ x}} (1+u)^{\lambda-1} du \le \int_{-1}^{-\frac{\sqrt{ch^2\ x - ch^2 z}}{sh\ x}} (1+u)^{\lambda-1} du \\ &\sim \int_{-1}^{-\frac{sh\ x}{ch\ x}} (1+u)^{\lambda-1} du = \frac{1}{\lambda}\left(1 - \frac{sh\ x}{ch\ x}\right)^{\lambda} \le \frac{1}{\lambda}\left(1 - \frac{sh^2\ x}{ch^2\ x}\right)^{\lambda} = \frac{1}{\lambda}(ch\ x)^{-2\lambda},\ x \to \infty. \end{aligned} \tag{2.20}$$

Now, taking into account Lemmas 1.1 and 1.2(b), also inequalities (2.12) and (2.20), for $2 \le r < \infty$ we get

$$A(x,\ r)\frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \le c_\lambda \begin{cases} \dfrac{ch^{2\lambda}\ r}{ch^{4\lambda}\ \frac{r}{2}}, \\ \dfrac{ch^{2\lambda}\ xch^{2\lambda}\ r}{ch^{2\lambda}\ xch^{4\lambda}\ \frac{r}{2}} \end{cases} \le c_\lambda. \tag{2.21}$$

Applying (2.21) we easily obtain

$$\begin{aligned} M_{G,2} f(ch\ x) &= \sup_{r\ge 2} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^{\lambda}_{ch\ t} |f(ch\ x)|\, d\mu_\lambda(t) \\ &= \sup_{r\ge 2} \frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \cdot \frac{A(x,\ r)}{|H(x,r)|_\lambda} \int_{|x-r|}^{x+r} |f(ch\ t)|\, sh^{2\lambda}\ tdt \\ &\le c_\lambda \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(ch\ t)|\, d\mu_\lambda(t) = c_\lambda M_\mu f(ch\ x). \end{aligned} \tag{2.22}$$

Combining (2.18) and (2.22), we get

$$M_G f(ch\ x) \le c_\lambda M_\mu f(ch\ x).$$

Thus Theorem 2.1 is proved. □

Further we need the following lemma, which is a version of Vitali's covering lemma.

Lemma (*[23], Sawano*)**.** Suppose we have a family of n intervals $\{H(x_j,\ r_j)\}_{j\in\{1,\dots,n\}}$. Then we can take a subfamily $\{H(x_j,\ r_j)\}_{j\in A}$ such that

(1) $\{H(x_j, r_j)\}_{j\in A}$ is disjoint.
(2) $\bigcup_{j\in\{1,\ldots,n\}} H(x_j, r_j) \subset \bigcup_{j\in A} H(x_j, 3r_j)$,

where $A = \{j_1, \ldots j_p\}$ and $j_1, \ldots j_p \in \{1, \ldots n\}$.

The following theorem is valid.

Theorem 2.2. (a) *If $f \in L_{1,\lambda}[0,\infty)$, then for all $\alpha > 0$*

$$|\{x : M_G f(ch\ x) > \alpha\}|_\lambda \le \frac{c_\lambda}{\alpha}\int_0^\infty |f(ch\ t)|\, sh^{2\lambda}\ tdt = \frac{c_\lambda}{\alpha}\|f\|_{L_{1,\lambda}[0,\infty)}$$

holds, where $c_\lambda > 0$ depends only on λ.

(b) *If $f \in L_{p,\lambda}[0,\infty)$, $1 < p \le \infty$, then $M_G f(ch\ x) \in L_{p,\lambda}[0,\infty)$ and $\|M_G f\|_{L_{p,\lambda}[0,\infty)} \le c_{p,\lambda}\|f\|_{L_{p,\lambda}[0,\infty)}$.*

Corollary 2.1. *If $f \in L_{p,\lambda}[0,\infty)$, $1 \le p \le \infty$, then*

$$\lim_{r\to 0}\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} A^\lambda_{ch\ t} f(ch\ x) sh^{2\lambda}\ tdt = f(ch\ x)$$

for a. e. $x \in [0,\infty)$.

Proof of Theorem 2.2. We define $E_\alpha = \{t : M_\mu f(ch\,t) > \alpha\}$. We introduce the function $h(\alpha)$ which is equal to the measure of the set E_α, i.e.

$$h(\alpha) = |E_\alpha|_\lambda = \left|\{t : M_\mu f(ch\,t) > \alpha\}\right|_\lambda.$$

By the definition of function $M_\mu f$ it follows that, for all $x_j \in E_\alpha$ there exists an interval $H(x_j, r_j) \subset E_\alpha$ with centered x_j such that

$$\int_{2H(x_j,r_j)} |f(ch\,t)|\, sh^{2\lambda}\,tdt \ge \int_{\{t\in H(x_j,r_j):|f(ch\,t)|>\alpha\}} |f(ch\,t)|\, sh^{2\lambda} tdt$$
$$> \alpha\int_{\{t\in H(x_j,r_j):|f(ch\,t)|>\alpha\}} sh^{2\lambda} tdt \ge \alpha\left|H(x_j, r_j)\right|_\lambda. \tag{2.23}$$

Further, since

$$H(x, r) = \begin{cases}(0,\ x+r) & \text{if } x < r\\ (x-r,\ x+r) & \text{if } x > r,\end{cases}$$

then for $x_j < 3r_j$ we have

$$\left|H(x_j, 3r_j)\right|_\lambda = \int_0^{x_j+3r_j} sh^{2\lambda}\,tdt > \int_0^{x_j+r_j} sh^{2\lambda}\,tdt = \left|H(x_j, r_j)\right|_\lambda. \tag{2.24}$$

Let $x_j > 3r_j$, then

$$\left|H(x_j, 3r_j)\right|_\lambda = \int_{x_j-3r_j}^{x_j+3r_j} sh^{2\lambda}\,tdt > \int_{x_j-r_j}^{x_j+r_j} sh^{2\lambda}\,tdt. \tag{2.25}$$

From (2.24) and (2.25) we find that for all $r_j > 0$, $j \in \{1,\ldots,n\}$

$$\left|H(x_j, 3r_j)\right|_\lambda \ge \left|H(x_j, r_j)\right|_\lambda. \tag{2.26}$$

By the previous lemma which was proved by Sawano, there exists a set $A \subset \{1, \dots n\}$ such that $\bigcup_{j=1,\dots n} H(x_j, r_j) \subset \bigcup_{l\in A} H(x_l, 3r_l)$ and the intervals $H(x_l, 3r_l)$ are disjoint, moreover by (2.26)

$$\left|\bigcup_{j=1,\dots n} H(x_j, r_j)\right|_\lambda \le \sum_{j=1}^{n} \left|H(x_j, r_j)\right|_\lambda \le \sum_{l\in A} |H(x_l, 3r_l)|_\lambda .$$

From this and (2.23) we have

$$\sum_{j=1}^{n} \left|H(x_j, r_j)\right|_\lambda \le \frac{1}{\alpha}\sum_{l\in A}\int_{H(x_l,3r_l)} |f(ch\, t)|\, sh^{2\lambda}\, t dt. \tag{2.27}$$

We show that $E_\alpha = \{t \in [0, \infty) : M_\mu f(ch\, t) > \alpha\}$ is an open set. For this we need double-sided estimates for the $|H(x, r)|_\lambda$.

At first we consider case $0 < x < r$. Then $H(x, r) = (0, x + r)$.

Let $0 < x + r < 2$, then we have

$$\begin{aligned}|H(x, r)|_\lambda &= \int_0^{x+r} sh^{2\lambda}\, t dt = 2^{2\lambda+1}\int_0^{x+r} \frac{\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}} \\ &\ge \frac{2^{2\lambda+1}}{\left(ch\,\frac{x+r}{2}\right)^{1-2\lambda}}\int_0^{x+r}\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda-1}}{2\lambda+1}\frac{\left(sh\,\frac{x+r}{2}\right)^{2\lambda+1}}{\left(ch\,\frac{x+r}{2}\right)^{1-2\lambda}} \\ &\ge \frac{2^{2\lambda+1}}{(2\lambda+1)\, ch\, 1}\left(sh\,\frac{x+r}{2}\right)^{2\lambda+1} \ge \frac{1}{(2\lambda+1)\, ch\, 1}(x+r)^{2\lambda+1}.\end{aligned} \tag{2.28}$$

On the other hand, since $ch\,\frac{t}{2} \ge 1$ for $t \ge 0$, then

$$|H(x, r)|_\lambda \le 2^{2\lambda+1}\int_0^{x+r}\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda+1}}{2\lambda+1}\left(sh\,\frac{x+r}{2}\right)^{2\lambda+1} \le \frac{e^{2\lambda+1}}{2\lambda+1}(x+r)^{2\lambda+1}. \tag{2.29}$$

At the end we use the inequality (1.3).

Now let $2 \le x + r < \infty$. Then

$$\begin{aligned}|H(x, r)|_\lambda &= \int_0^{x+r} sh^{\,2\lambda} t dt \ge \int_{\frac{x+r}{2}}^{x+r}\frac{sh^{2\lambda}\, t d\,(sh\, t)}{ch\, t} \ge \frac{1}{2}\int_{\frac{x+r}{2}}^{x+r}(sh\, t)^{2\lambda-1} d\,(sh\, t) \\ &= \frac{1}{4\lambda}\left(sh^{2\lambda}\,(x+r) - sh^{2\lambda}\,\frac{x+r}{2}\right) \ge \frac{1}{4\lambda}\left(sh^{2\lambda}\,(x+r) - \frac{1}{4^\lambda}sh^{2\lambda}\,(x+r)\right) \\ &= \frac{1}{4\lambda}\left(1-\frac{1}{4\lambda}\right)sh^{2\lambda}\,(x+r) = \frac{4^\lambda-1}{4\lambda\cdot 4^\lambda}\left(2sh\,\frac{x+r}{2}ch\,\frac{x+r}{2}\right)^{2\lambda} \\ &\ge \frac{4^\lambda-1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}.\end{aligned} \tag{2.30}$$

On the other hand

$$\begin{aligned}|H(x, r)|_\lambda &= \int_0^{x+r} sh^{\,2\lambda} t dt = 2^{2\lambda+1}\int_0^{x+r}\frac{\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}} \\ &\le 2^{2\lambda+1}\int_0^{x+r}\left(sh\,\frac{t}{2}\right)^{4\lambda-1} d\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda+1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}.\end{aligned} \tag{2.31}$$

Combining (2.28)–(2.31) we obtain for $0 < x < r$ and $0 < x + r < 2$

$$\frac{1}{(2\lambda+1)\, ch\, 1}(x+r)^{2\lambda+1} \le |H(x, r)|_\lambda \le \frac{e^{2\lambda+1}}{2\lambda+1}(x+r)^{2\lambda+1}, \tag{2.32}$$

and for $0 < x < r$ and $2 \le x + r < \infty$

$$\frac{4^\lambda - 1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda} \le |H\,(x,\ r)|_\lambda \le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.33}$$

Now we consider the case $r \le x < \infty$.

Then $H\,(x,\ r) = (x - r,\ x + r)$.

Let $0 < x + r < 2$. Then we have

$$\begin{aligned} |H\,(x,\ r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda}\,t dt \ge \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda}\,t dt \ge \frac{x+r}{2}\left(sh\,\frac{x+r}{2}\right)^{2\lambda} \\ &\ge \frac{1}{2^{2\lambda+1}}\,(x+r)^{2\lambda+1}. \end{aligned} \tag{2.34}$$

At the end we use the inequality (1.3).

On the other hand according to (2.29) we have

$$|H\,(x,\ r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda}\,t dt \le \int_0^{x+r} sh^{2\lambda}\,t dt \le \frac{1}{4\,(2\lambda+1)}\,(x+r)^{2\lambda+1}.$$

It remains to consider the case $2 \le x + r < \infty$.

For inequality (2.30) we have

$$|H\,(x,\ r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda}\,t dt \ge \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda}\,t dt \ge \frac{4^\lambda - 1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}.$$

On the other hand

$$\begin{aligned} |H\,(x,\ r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda}\,t dt = 2^{2\lambda+1}\int_{x-r}^{x+2} \frac{\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}} \\ &\le 2^{2\lambda+1}\int_{x-r}^{x+r}\left(sh\,\frac{t}{2}\right)^{4\lambda-1}\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda-1}}{\lambda}\left(\left(sh\,\frac{x+r}{2}\right)^{4\lambda} - \left(sh\,\frac{x-r}{r}\right)^{4\lambda}\right) \\ &\le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \end{aligned} \tag{2.35}$$

Combining (2.32)–(2.35) we obtain for $r \le x < \infty$ and $0 < x + r < 2$

$$\frac{1}{(2\lambda+1)ch1}\,(x+r)^{2\lambda+1} \le |H\,(x,\ r)|_\lambda \le \frac{e^{2\lambda+1}}{2\lambda+1}\,(x+r)^{2\lambda+1}, \tag{2.36}$$

and for $2 \le x + r < \infty$

$$\frac{4^\lambda - 1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda} \le |H\,(x,\ r)|_\lambda \le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.37}$$

Now from (2.32) and (2.36) for $0 < x + r < 2$ we have

$$\frac{1}{(2\lambda+1)\,ch\,1}\,(x+r)^{2\lambda+1} \le |H\,(x,\ r)|_\lambda \le \frac{e^{2\lambda+1}}{2\lambda+1}\,(x+r)^{2\lambda+1}. \tag{2.38}$$

But from (2.33) and (2.35) for $2 \le x + r < \infty$

$$\frac{4^\lambda - 1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda} \le |H\,(x,\ r)|_\lambda \le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.39}$$

Now we will prove that the set E_α is open. By the definition of the maximal operator there exists $r > 0$ such that for some $u > \alpha$

$$\int_{H(x,r)} |f(ch\, t)| sh^{2\lambda} t dt = u |H(x, r)|_\lambda .$$

We consider the case $0 < x + r < 2$. There exists $\delta_1 > 0$ such that

$$\frac{u}{\alpha} > e^{2\lambda+1} (ch\, 1) \left(\frac{r+\delta_1}{r}\right)^{2\lambda+1} . \tag{2.40}$$

Let $|x - y| < \delta_1$, then $H(x, r) \subset H(y, r + \delta_1)$. If $z \in H(x, r)$, then $|z - y| \le |z - x| + |x - y| \le r + \delta_1$, from this it follows that $z \in H(y, r + \delta_1)$.

Then

$$\int_{H(y,r+\delta_1)} |f(ch\, t)| sh^{2\lambda} t dt \ge \int_{H(y,r)} |f(ch\, t)| sh^{2\lambda} t dt = u|H(y, r)|_\lambda . \tag{2.41}$$

Now by (2.36), we have

$$\begin{aligned} |H(y, r + \delta_1)|_\lambda &\le \frac{e^{2\lambda+1}}{2\lambda+1} (y + r + \delta_1)^{2\lambda+1} \\ &\le \frac{e^{2\lambda+1}}{2\lambda+1} (y + r)^{2\lambda+1} \left(\frac{r+\delta_1}{r}\right)^{2\lambda+1} \le e^{2\lambda+1} (ch\, 1) \left(\frac{r+\delta_1}{r}\right)^{2\lambda+1} |H(y, r)|_\lambda . \end{aligned}$$

From this it follows that

$$|H(y, r)|_\lambda \ge \left(\frac{r+\delta_1}{r}\right)^{-(2\lambda+1)} \left(e^{2\lambda+1} ch\, 1\right)^{-1} |H(y, r + \delta_1)|_\lambda . \tag{2.42}$$

From (2.41) and (2.42) it follows that

$$\frac{1}{|H(y, r + \delta_1)|_\lambda} \int_{H(y,r+\delta_1)} \left|f\left(e^{2\lambda+1} ch\, t\right)\right| sh^{2\lambda} t dt \ge (ch\, 1)^{-1} \left(\frac{r+\delta_1}{r}\right)^{-(2\lambda+1)} u > \alpha,$$

if

$$\frac{u}{\alpha} > e^{2\lambda+1} (ch\, 1) \left(\frac{r+\delta_1}{r}\right)^{2\lambda+1} .$$

In the case $0 < x + r < 2$ we obtain that $\exists\delta_1 > 0$ by condition (2.40) such that for $\forall t \in H(y, \delta_1)$ the inequality $M_\mu f(ch\, t) > \alpha$ holds, from this it follows that $H(y, \delta_1) \subset E_\alpha$, that is the set E_α is open.

It remains consider the case $2 \le x + r < \infty$. There exists $\delta_2 > 0$ such that

$$\frac{u}{\alpha} \ge \frac{2^{2\lambda+1} \cdot 3^{4\lambda}}{4^\lambda - 1} \left(ch\, (r + 1) \frac{r+\delta_2}{r}\right)^{4\lambda} > \frac{2^{2\lambda+1} \cdot 3^{4\lambda}}{4^\lambda - 1} \left(\frac{r+\delta_2}{r}\right)^{4\lambda} . \tag{2.43}$$

From (2.39) we have

$$\begin{aligned} |H(y, r + \delta_2)|_\lambda &\le \frac{2^{2\lambda-1}}{\lambda} \left(sh\, \frac{y + r + \delta_2}{2}\right)^{4\lambda} \le \frac{2^{2\lambda-1}}{\lambda} \left[sh \left(\frac{y + r}{2} + \frac{r+\delta_2}{2}\right)\right]^{4\lambda} \\ &\le \frac{2^{2\lambda-1}}{\lambda} \left[sh \left(\frac{y + r}{2} + (r + 1) \frac{r+\delta_2}{2}\right)\right]^{4\lambda} \\ &= \frac{2^{2\lambda-1}}{\lambda} \left(sh\, \frac{y + r}{2} ch\, (r + 1) \frac{r+\delta_2}{2} + ch\, \frac{y + r}{2} sh\, (r + 1) \frac{r+\delta_2}{2}\right)^{4\lambda} \\ &\le \frac{3^{4\lambda} \cdot 2^{2\lambda-1}}{\lambda} \left(sh\, \frac{y + r}{2}\right)^{4\lambda} \left(ch\, (r + 1) \frac{r+\delta_2}{2}\right)^{4\lambda} . \end{aligned} \tag{2.44}$$

At the end we use the inequality $ch\,t \le 2sh\,t$ at $t \ge 1$. From (2.44) and (2.39) we have

$$|H(y,\ r+\delta_2)|_\lambda \le \frac{2^{2\lambda+1}\cdot 3^{4\lambda}}{4^\lambda - 1}\left(ch\ (r+1)\,\frac{r+\delta_2}{2}\right)^{4\lambda}|H(y,\ r)|_\lambda.$$

From this it follows that

$$|H(y,\ r)|_\lambda \ge \frac{4^\lambda - 1}{2^{2\lambda+1}\cdot 3^{4\lambda}}\left(ch\ (r+1)\,\frac{r+\delta_2}{2}\right)^{-4\lambda}|H(y,\ r+\delta_2)|_\lambda.$$

Now from (2.41) we have

$$\frac{1}{|H(y,\ r+\delta_2)|_\lambda}\int_{H(y,r+\delta)}|f(ch\,t)|sh^{2\lambda}\,tdt \ge \frac{4^\lambda - 1}{2^{2\lambda+1}\cdot 3^{4\lambda}}\left(ch\ (r+1)\,\frac{r+\delta_2}{r}\right)^{-4\lambda}u > \alpha,$$

if (2.43) is true.

In the case $2 \le x + r < \infty$ we prove that $\exists\delta_2 > 0$ by condition (2.43) such that for $\forall t \in H(y,\ \delta_2)$ the inequality $M_\mu f\,(ch\,t) > \alpha$ holds, from this it follows that $H(y,\ \delta_2) \subset E_\alpha$, that is the set E_α is open.

As above it follows that $\exists\delta > 0$, where $\delta = \min\{\delta_1,\ \delta_2\}$ such that for $\forall t \in H(y,\ \delta)$ the inequality $M_\mu f\,(ch\,t) > \alpha$ holds, from this it follows that $H(y,\ \delta) \subset E_\alpha$, that is the set E_α is open.

Since $[0,\ \infty)$ is separable, so with the help of the Lindelöf (see [24]) covering theorem $E_\alpha \subset \bigcup_{j\in N} H(x_j, r_j)$. Then, letting n tends to infinity in (2.27), we obtain

$$|E_\alpha|_\lambda \le \frac{1}{\alpha}\int_0^\infty |f(ch\,t)|\,sh^{2\lambda}\,tdt,$$

and this is the assertion (a) of theorem.

Further, since by Theorem 2.1 $M_G f(ch\,x) \le c_\lambda M_\mu f(ch\,x)$, then

$$F_\alpha = \{x : M_G f(ch\,x) > \alpha\} \subset E_\alpha = \left\{x : M_\mu f(ch\,x) > \frac{\alpha}{c_\lambda}\right\};$$

consequently

$$|F_\alpha|_\lambda \le \left|\left\{x : M_\mu f(ch\,x) > \frac{\alpha}{c_\lambda}\right\}\right|_\lambda \le \frac{c_\lambda}{\alpha}\|f\|_{p,\lambda},$$

and this is the assertion (a) of theorem.

We will prove the approval (b). Suppose

$$f_1(ch\,x) = \begin{cases} f(ch\,x), & \text{if } |f(ch\,x)| \ge \dfrac{\alpha}{2},\\ 0, & \text{if } 0 \le |f(ch\,x)| < \dfrac{\alpha}{2}. \end{cases} \tag{2.45}$$

Then we have

$$|f(ch\,x)| \le |f_1(ch\,x)| + \frac{\alpha}{2} \quad \text{and} \quad M_\mu f(ch\,x) \le M_\mu f_1(ch\,x) + \frac{\alpha}{2},$$

so $\{x : M_\mu f(ch\,x) > \alpha\} \subset \{x : M_\mu f_1(ch\,x) > \frac{\alpha}{2}\}$ and at last

$$|E_\alpha|_\lambda = \big|\{x : M_\mu f(ch\,x) > \alpha\}\big|_\lambda \le \frac{2}{\alpha}\|f_1\|_{1,\lambda},$$

from here and (2.45) it follows that

$$|E_\alpha|_\lambda = \big|\{x : M_\mu f(ch\,x) > \alpha\}\big|_\lambda \le \frac{2}{\alpha}\int_{\{x:|f(ch\,x)|>\frac{\alpha}{2}\}}|f(ch\,x)|\,sh^{2\lambda}\,xdx. \tag{2.46}$$

Suppose that the function $f\,(ch\,x)$ is defined on $[0,\ \infty)$. We consider for each $\alpha > 0$ the set E_α such that $|f| > \alpha$; $E_\alpha = \{x : |f(ch\,x)| > \alpha\}$.

Let $h(\alpha)$ be the measure of the set E_α, i.e.

$$h(\alpha) = |E_\alpha|_\lambda = |\{x : |f\,(ch\,x)| > \alpha\}|_\lambda\,.$$

The function $h(\alpha)$ is called the distribution of the function $|f\,(ch\,x)|$. Every quantity, depended only on ëxtent" f, can be expressed over of distribution of function $h(\alpha)$ (see [25], p. 15). For example if $f \in L_{p,\lambda}$, then by Fubini's theorem we obtain

$$\begin{aligned}\int_0^\infty |f\,(ch\,t)|^p\, sh^{2\lambda}\, tdt &= p\int_0^\infty \left(\int_0^{|f(ch\,t)|} \alpha^{p-1} d\alpha\right) sh^{2\lambda}\, tdt \\ &= p\int_0^\infty \alpha^{p-1}\left(\int_{\{t\in[0,\,\infty):|f(ch\,t)|>\alpha\}} sh^{2\lambda}\, tdt\right) \\ &= p\int_0^\infty \alpha^{p-1}|\{t\in[0,\,\infty) : |f(ch\,t)|>\alpha\}|_\lambda d\alpha = p\int_0^\infty \alpha^{p-1}h(\alpha)d\alpha. \end{aligned} \tag{2.47}$$

Now, if $M_\mu f \in L_{1,\lambda}$, then by (2.46) and (2.47) applying Fubini's theorem, we will have

$$\begin{aligned}\|M_\mu f\|_{p,\lambda}^p &= p\int_0^\infty \alpha^{p-1}\Big|\{x : M_\mu f(ch\,x)>\alpha\}\Big|_\lambda d\alpha \\ &\le p\int_0^\infty \alpha^{p-1}\left(\frac{2}{\alpha}\int_{\{x:|f(ch\,x)|>\frac{\alpha}{2}\}} |f(ch\,x)|\, sh^{2\lambda}\, xdx\right) d\alpha \\ &= 2p\int_0^\infty |f(ch\,x)|\left(\int_0^{2|f(ch\,x)|}\alpha^{p-2}d\alpha\right) sh^{2\lambda}\, xdx \\ &= \frac{2p}{p-1}\int_0^\infty |f(ch\,x)|\left(\alpha^{p-1}\Big|_0^{2|f(ch\,x)|}\right) sh^{2\lambda}\, xdx \\ &= \frac{p\cdot 2^p}{p-1}\int_0^\infty |f(ch\,x)|^p\, sh^{2\lambda}\, xdx = c_p\|f\|_{p,\lambda}^p,\end{aligned}$$

from this it follows that

$$\|M_\mu f\|_{p,\lambda} \le c_p\|f\|_{p,\lambda}, \quad 1<p<\infty. \tag{2.48}$$

The assertion (b) follows from Theorem 2.1 and inequality (2.48):

$$\|M_G f\|_{p,\lambda} \le c_\lambda \|M_\mu f\|_{p,\lambda} \le c_{p,\lambda}\|f\|_{p,\lambda}.$$

In the case $p=\infty$ last inequality is obtained evidently.

Thus Theorem 2.2 is proved. □

Proof of Corollary 2.1. At first let us show that for any function $f \in L_{p,\lambda}[0,\infty)$, $1\le p\le\infty$,, representation $ch\ t \mapsto A^\lambda_{ch\ t} f$ from $\mathbb{R}$ into $L_{p,\lambda}$ continuous, that is

$$\|A^\lambda_{ch\ t}f - f\|_{L_{p,\lambda}} \to 0 \quad \text{at } t\to 0. \tag{2.49}$$

Let $f(x)$ be a continuous function defined for $[a,b]\subset[0,\infty)$. Consider the function

$$y(t,x,\varphi) = ch\ tch\ x - sh\ tsh\ x\cos\ \varphi.$$

Hence we have

$$\begin{aligned}|y(t,x,\varphi)-y(0,x,\varphi)| &= |ch\ tch\ x - sh\ tsh\ x\cos\ \varphi - ch\ x| \\ &= |(ch\ t-1)ch\ x - sh\ tsh\ x\cos\ \varphi - ch\ x| \le 2sh^2\frac{t}{2}ch\ x + 2sh\ \frac{t}{2}ch\ \frac{t}{2}sh\ x \\ &\le 2sh\ \frac{t}{2}\left(sh\ \frac{t}{2}ch\ x + ch\ \frac{t}{2}sh\ x\right) \\ &= 2sh\ \frac{t}{2}sh\left(\frac{t}{2}+x\right) \le 2sh\ \frac{t}{2}sh\left(\frac{t}{2}+b\right) \to 0 \quad t\to 0.\end{aligned} \tag{2.50}$$

On the strength of uniform continuity of the function $f(x)$ on segment $[a, b]$ for any $\varepsilon > 0$ one may choose the number $\delta > 0$, such that

$$|f[y(t,x,\varphi)] - f[y(0,x,\varphi)]| < \varepsilon, \text{ if } |y(t,x,\varphi) - y(0,x,\varphi)| < \delta, \text{ (that follows from (2.50))}.$$

Then we have

$$\left|A^{\lambda}_{ch\ t} f(ch\ x) - f(ch\ x)\right| \le \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)} \int_0^{\pi} |f[y(t,x,\varphi)] - f[y(0,x,\varphi)]|(\sin\ \varphi)^{2\lambda-1} d\varphi < \varepsilon.$$

It follows, that

$$\|A^{\lambda}_{ch\ t} f - f\|_{\infty,\lambda} = \sup_{x\in[a,b]} |A^{\lambda}_{ch\ t} f(ch\ x) - f(ch\ x)| < \varepsilon.$$

And for $1 \le p < \infty$

$$\|A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[a,b]} = \left(\int_a^b |A^{\lambda}_{ch\ t} f(ch\ x) - f(ch\ x)|^p sh^{2\lambda} x dx\right)^{\frac{1}{p}} < \varepsilon \left(\int_a^b sh^{2\lambda}\ x dx\right)^{\frac{1}{p}} < c_{p,\lambda}\varepsilon.$$

Thus for any continuous function defined on the segment $[a, b] \subset [0, \infty)$ and for any number $\varepsilon > 0$ the following inequality is valid:

$$\|A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[a,b]} < \varepsilon \quad 1 \le p \le \infty. \tag{2.51}$$

It is known the set of all continuous functions with compact support in $[0, \infty)$ is dense in $L_{p,\lambda}[0, \infty)$ (see [26], Theorem 4.2). Therefore for any number $\varepsilon > 0$ there exists a continuous function with compact support in $[0, \infty)$, such that

$$\|f - f_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} < \varepsilon. \tag{2.52}$$

We denote $g_{\varepsilon} = f - f_{\varepsilon}$. Then $g_{\varepsilon} \in L_{p,\lambda}[0, \infty)$ and

$$\|g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} < \varepsilon. \tag{2.53}$$

Thus, if $f \in L_{p,\lambda}[0, \infty)$, then for any number $\varepsilon > 0$ there exists a continuous function f_{ε} with the compact support and function $g_{\varepsilon} \in L_{p,\lambda}[0, \infty)$ with condition $\|g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} < \varepsilon$, such that $f = f_{\varepsilon} + g_{\varepsilon}$.

Hence we have $A^{\lambda}_{ch\ t} f(ch\ x) = A^{\lambda}_{ch\ t} f_{\varepsilon}(ch\ x) + A^{\lambda}_{ch\ t} g_{\varepsilon}(ch\ x) - f(ch\ x) + f_{\varepsilon}(ch\ x) - f_{\varepsilon}(ch\ x)$, from which it follows that

$$A^{\lambda}_{ch\ t} f - f\ \|_{L_{p,\lambda}[0,\infty)} \le \|A^{\lambda}_{ch\ t} f_{\varepsilon} - f_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} + \|f - f_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} + \|A^{\lambda}_{ch\ t} g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)}.$$

Now, taking into account that (see [22], Lemma 2)

$$\|A^{\lambda}_{ch\ t} g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} \le \|g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)}, \qquad t \in [0, \infty), 1 \le p \le \infty$$

and also the inequalities (2.51)–(2.53), we get

$$\|A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[0,\infty)} \le 3\varepsilon,$$

from which (2.49) follows.

By the locality of the problem, one can account that $f \in L_{1,\lambda}[0, \infty)$. In general case one can multiply f by characteristic function of interval $H(0, r) = [0, r)$ and obtain required convergence almost everywhere interior to this interval and by tending r to infinity one could obtain it on the whole interval $[0, \infty)$.

Suppose for any $r > 0$ and for any $x \in [0, \infty)$

$$f_r(ch\ x) = \frac{1}{|H(0,r)|_{\lambda}} \int_{H(0,r)} A^{\lambda}_{ch\ t} f(ch\ x) sh^{2\lambda}\ t dt.$$

Let $r_0 > 0$, $H = H(0, r_0)$. According to the generalized Minkowski generalized inequality and discount (2.49), we obtain

$$\begin{aligned} f_r - f\,\|_{L_{1,\lambda}(H)} &= \left|\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)}\left(A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\right) sh^{2\lambda}\ t dt\right|_{L_{1,\lambda}(H)} \\ &\le \frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} \|A^\lambda_{ch\ t} f - f\|_{L_{1,\lambda}(H)} sh^{2\lambda}\ t dt \\ &\le \sup_{|t|\le r_0} \|A^\lambda_{ch\ t} f - f\|_{L_{1,\lambda}(H)} \to 0, \quad \text{at } r_o \to +0. \end{aligned}$$

It means that there is a sequence r_k such that $r_k \to +0$, $(k \to \infty)$ and

$$\lim_{k\to\infty} f_{r_k}(ch\ x) = f(ch\ x)$$

almost everywhere at $x \in [0, \infty)$.

Now, let us prove that $\lim_{r\to+0} f_r(ch\ x)$ exists almost everywhere. For this purpose for any $x \in [0, \infty)$ we consider

$$\Omega_f(ch\ x) = \left|\overline{\lim_{r\to+0}} f_r(ch\ x) - \underline{\lim_{r\to+0}}\ f_r(ch\ x)\right|$$

the oscillation of f_r at the point x as $r \to +0$.

If g is a continuous function with compact support on $[0, \infty)$, then g_r is convergent to g and consequently $\Omega_g \equiv 0$. Further, if $g \in L_{1,\lambda}[0, \infty)$, then according to the statement of Theorem 2.2 we get

$$|\{x \in [0, \infty) : M_G g(ch\ x) > \varepsilon\}|_\lambda \le \frac{c}{\varepsilon}\|g\|_{L_{1,\lambda}[0,\infty)}, \quad g \in L_{1,\lambda}[0, \infty).$$

On the other hand it is obvious that $\Omega g(ch\ x) \le 2M_G g(ch\ x)$. Thus

$$|\{x \in [0, \infty) : \Omega_g(ch\ x) > \varepsilon\}|_\lambda \le \frac{2c}{\varepsilon}\|g\|_{L_{1,\lambda}[0,\infty)},\ \ g \in L_{1,\lambda}[0, \infty).$$

By the same way as it was proved above, any function $f \in L_{p,\lambda}[0, \infty)$ can be written in form $f = h + g$, where h is continuous function and has a compact support on $[0, \infty)$, and $g \in L_{p,\lambda}[0, \infty)$, moreover $\|g\|_{L_{p,\lambda}[0,\infty)} < \varepsilon$, for any $\varepsilon > 0$. But $\Omega \le \Omega_h + \Omega_g$ $\Omega_h \equiv 0$, however is continuous by h. Therefore it follows that

$$|\{x \in [0, \infty) : \Omega_g(ch\ x) > \varepsilon\}|_\lambda \le \frac{c}{\varepsilon}\|g\|_{L_{1,\lambda}[0,\infty)}.$$

Taking in inequality $\|g\|_{L_{1,\lambda}[0,\infty)} < \varepsilon$ the number ε arbitrary small, we get $\Omega f = 0$ almost everywhere on $[0, \infty)$. Consequently, $\lim_{r\to0} f_r(ch\ x)$ exists almost everywhere on $[0, \infty)$, which was required to prove. □

Remark 2.1. Theorem 2.2 was proved earlier by W.C. Connett and A.L. Schwartz [8] for the Jacobi-type hypergroups.

Remark 2.2. If $f \in L_{1,\lambda}[0, \infty)$, then (see [27], Theorem 2.1)

$$\lim_{r\to0}\frac{1}{\left(sh\ \frac{r}{2}\right)^{2\lambda+1}}\int_0^r \left|A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\right| sh^{2\lambda}\ t dt = 0,$$

almost everywhere for $x \in [0, \infty)$.

This implies that for any $\varepsilon > 0$ one can find $\delta > 0$, such that for all $r < \delta$ the following inequality is just:

$$\frac{1}{\left(sh\ \frac{r}{2}\right)^{2\lambda+1}}\int_0^r \left|A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\right| sh^{2\lambda}\ t dt < \varepsilon.$$

Then from Lemma 1.1, we obtain

$$\begin{aligned} &\left|\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)}\left[A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\right] sh^{2\lambda}\ t dt\right| \\ &\quad \le \frac{1}{\left(sh\ \frac{r}{2}\right)^{2\lambda+1}}\int_0^r \left|A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\right| sh^{2\lambda}\ t dt < \varepsilon, \end{aligned}$$

for all $r < \delta$, which means that Corollary 2.1 is valid under assumption $f \in L_{1,\,\lambda}\,[0,\ \infty)$.

3. *G*-Riesz potential

In this section the concept of Riesz–Gegenbauer potential associated with the Gegenbauer differential operator G is introduced and its integral representation is found. For the functions $f, g \in L_{1,\lambda}[1,\infty)$ in [9], the Gegenbauer transformation is defined as follows:

$$F_P : f(t) \mapsto \hat{f}_P(\gamma) = \int_1^{\infty} f(t)\, P_\gamma^\lambda(t) \left(t^2-1\right)^{\lambda-\frac{1}{2}} dt, \tag{3.1}$$

$$F_Q : f(t) \mapsto \hat{f}_Q(\gamma) = \int_1^{\infty} f(t)\, Q_\gamma^\lambda(t) \left(t^2-1\right)^{\lambda-\frac{1}{2}} dt,$$

where the functions $P_\gamma^\lambda(x)$ and $Q_\gamma^\lambda(x)$ are eigenfunctions of operator G.

The inverse of the Gegenbauer transformations is defined by the formulas

$$F_P^{-1} : \hat{f}_P(\alpha) \mapsto f(x) = c_\lambda^* \int_1^{\infty} \hat{f}_P(\gamma)\, Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma, \tag{3.2}$$

$$F_Q^{-1} : \hat{f}_Q(\alpha) \mapsto f(x) = c_\lambda^* \int_1^{\infty} \hat{f}_Q(\gamma)\, P_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma, \tag{3.3}$$

where $c_\lambda^* = \frac{2^{\frac{3}{2}-\lambda}\sqrt{\pi}\,\Gamma(\lambda+1)\Gamma\left(\frac{1}{2}-\gamma\right)\Gamma\left(\frac{3+2\lambda}{4}\right)\left(\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{5-2\lambda}{4}\right)\cos\pi\lambda\right)^{-1}}{{}_2F_1\left(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1}{2}\right)-{}_2F_1\left(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1-2\lambda}{2}\right)}$, and ${}_2F_1(\alpha;\ \beta;\ \gamma;\ x)$ is Gauss function.

For $f \in D(\mathbb{R}_+)$ the transformations (3.1)–(3.3) are defined, where $D(\mathbb{R}_+)$ is the set of infinitely differentiable even functions on $\mathbb{R}_+ = [0, \infty)$ with compact supports.

Preliminary we prove the following lemma.

Lemma 3.1. *Let $f, g \in L_{1,\lambda}[1,\infty) \cap L_{2,\lambda}[1,\infty)$. Then the following equality is true:*

$$\int_1^{\infty} f(x)\, A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx = c_\lambda^* \int_1^{\infty} \hat{f}_P(\gamma) \left(\widehat{A_t^\lambda g}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma. \tag{3.4}$$

Proof. From (3.4) we have

$$\int_1^{\infty} f(x)\, A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx$$
$$= c_\lambda^* \int_1^{\infty} A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^{\infty} \hat{f}_P(\gamma)\, Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma. \tag{3.5}$$

Since (see the proof of Lemma 8 in [22])

$$\int_1^{\infty} \hat{f}_P(\gamma)\, Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma \lesssim \|f\|_{L_{2,\lambda}},$$

then taking into account the inequality (see [22], Lemma 1.2)

$$\left\|A_t^\lambda g\right\|_{L_{1,\lambda}} \le \|g\|_{L_{1,\lambda}},$$

we obtain

$$\left|\int_1^{\infty} A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^{\infty} \hat{f}_P(\gamma)\, Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma\right|$$
$$\lesssim \|f\|_{L_{2,\lambda}} \int_1^{\infty} \left|A_t^\lambda g(x)\right| \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx = \|f\|_{L_{2,\lambda}} \left\|A_t^\lambda g\right\|_{L_{1,\lambda}} \le \|f\|_{L_{2,\lambda}} \|g\|_{L_{1,\lambda}}.$$

By the Fubini theorem we have

$$c_\lambda^* \int_1^{\infty} A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^{\infty} \hat{f}_P(\gamma)\, Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$= c_\lambda^* \int_1^\infty A_t^\lambda g(x)\, Q_\gamma^\lambda(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$= c_\lambda^* \int_1^\infty \left(\widehat{A_t^\lambda g}\right)_Q (\gamma)\, \hat{f}_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma. \tag{3.6}$$

Taking into account (3.5) in (3.6), we obtain (3.4).

Thus Lemma 3.1 is proved. □

Definition 3.1. For $0 < \alpha < 2\lambda + 1$ Riesz–Gegenbauer potential (G-Riesz potential) $I_G^\alpha f(ch\, x)$ is defined by the equality

$$I_G^\alpha f(ch\ x) = G_\lambda^{-\frac{\alpha}{2}} f(ch\ x). \tag{3.7}$$

Such (see [28], p. 1933)

$$G_\lambda P_\gamma^\lambda(ch\ x) = \gamma(\gamma + 2\lambda)\, P_\gamma^\lambda(ch\ x),$$

then taking into account selfadjoint of operator G (see [21], Lemma 4), we obtain for (3.5)

$$\left(\widehat{G_\lambda f}\right)_P(\gamma) = \int_1^\infty P_\gamma^\lambda(ch\ x)\, G_\lambda f(ch\ x)\, sh^{2\lambda}\ x dx$$

$$= \int_0^\infty f(ch\ x) \left(G_\lambda P_\gamma^\lambda(ch\ x)\right) sh^{2\lambda}\ x dx$$

$$= \gamma(\gamma+2\lambda) \int_\lambda^\infty f(ch\ x)\, P_\gamma^\lambda(ch\ x)\, sh^{2\lambda} x dx = \gamma(\gamma+2\lambda)\, \hat{f}_P(\gamma).$$

Obviously, by induction we have

$$\left(\widehat{G_\lambda^k f}\right)_P(\gamma) = (\gamma(\gamma+2\lambda))^k\, \hat{f}_P(\lambda), \quad k = 1, 2, \ldots.$$

This formula is naturally spread for the fractional indexes in the following form:

$$\left(\widehat{G_\lambda^{-\frac{\alpha}{2}} f}\right)_P(\gamma) := (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}}\, \hat{f}_P(\lambda). \tag{3.8}$$

But then for (3.7) and (3.8) we have

$$\left(\widehat{I_G^\alpha f}\right)_p(\gamma) = (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}}\, \hat{f}_p(\lambda). \tag{3.9}$$

Lemma 3.2. *Let $h_r(ch\ x)$ be the kernel associated with G_λ and $0 < \alpha < 2\lambda + 1$. Then*

$$I_G^\alpha f(ch\ t) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x)\, dr\right) A_{ch\ t}^\lambda f(ch\ x)\, sh^{2\lambda} x dx. \tag{3.10}$$

Proof. Let

$$\left(\hat{h}_r\right)_Q(\gamma) = e^{-\gamma(\gamma+2\lambda)r},$$

then from (3.3) it follows, that

$$h_r(ch\ x) = \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} P_\gamma^\lambda(ch\ x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma.$$

By Lemma 3.1

$$\int_0^\infty h_r(ch\ x)\, A_{ch\ x}^\lambda f(ch\ x)\, sh^{2\lambda}\ x dx = c_\lambda^* \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} \left(\widehat{A_{ch\ t}^\lambda f}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma.$$

Thus we have

$$\int_0^\infty \int_0^\infty r^{\frac{\alpha}{2}-1} h_r\,(ch\ x)\, A_{ch\ t}^{\lambda} f\,(ch\ x)\, sh^{2\lambda}\ xdxdr$$

$$= c_\lambda^* \int_1^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} e^{-\gamma(\gamma+2\lambda)r} dr \right) \left(\widehat{A_{ch\ t}^{\lambda} f} \right)_P (\gamma) \left(\gamma^2 - 1 \right)^{\lambda-\frac{1}{2}} d\gamma$$

$$\left[\gamma\,(\gamma + 2\lambda)\, r = t, dr = \frac{dt}{\gamma\,(\gamma+2\lambda)} \right]$$

$$= c_\lambda^* \int_1^\infty \left(\int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt \right) (\gamma\,(\gamma+2\lambda))^{-\frac{\alpha}{2}} \left(\widehat{A_{ch\ t}^{\lambda} f} \right)_P (\gamma) \left(\gamma^2 - 1 \right)^{\lambda-\frac{1}{2}} d\gamma$$

$$= c_\lambda^* \Gamma \left(\frac{\alpha}{2} \right) \int_1^\infty (\gamma\,(\gamma+2\lambda))^{-\frac{\alpha}{2}} \left(\widehat{A_{ch\ t}^{\lambda} f} \right)_P (\gamma) \left(\gamma^2 - 1 \right)^{\lambda-\frac{1}{2}} d\gamma.$$

Taking into account that (see [22], Lemma 1.2)

$$\left(\widehat{A_{ch\ t}^{\lambda} f} \right)_P (\gamma) = \hat{f}_P\,(\gamma)\, Q_\gamma^\lambda\,(ch\, t)$$

for (3.9) and (3.2) we obtain

$$\int_0^\infty \int_0^\infty r^{\frac{\alpha}{2}-1} h_r\,(ch\ x)\, A_{ch\ t}^{\lambda} f\,(ch\ x)\, sh^{2\lambda}\ xdxdr$$

$$= c_\lambda^* \Gamma \left(\frac{\alpha}{2} \right) \int_1^\infty (\gamma\,(\gamma+2\lambda))^{-\frac{\alpha}{2}}\, \hat{f}_P\,(\gamma)\, Q_\gamma^\lambda\,(ch\ t) \left(\gamma^2 - 1 \right)^{\lambda-\frac{\alpha}{2}} d\gamma$$

$$= \Gamma \left(\frac{\alpha}{2} \right) \int_0^\infty \left(\widehat{I_G^\alpha f} \right)_P (\gamma)\, Q_\gamma^\lambda\,(ch\ t) \left(\gamma^2 - 1 \right)^{\lambda-\frac{1}{2}} d\lambda = \Gamma \left(\frac{\alpha}{2} \right) I_G^\alpha f\,(ch\ t)\,,$$

from this and for (3.2) it follows, that

$$I_G^\alpha f\,(ch\ t) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r\,(ch\ x)\, dr \right) A_{ch\ t}^{\lambda} f\,(ch\ x)\, sh^{2\lambda}\ xdx.$$

Thus Lemma 3.2 is proved. □

Corollary 3.1. *The following equality is true*

$$\left| I_G^\alpha f\,(ch\ t) \right| \lesssim \int_0^\infty \left| A_{ch\ t}^{\lambda} f\,(ch\ x) \right| (sh\ x)^{\alpha-2\lambda-1}\, sh^{2\lambda}\ xdx. \tag{3.11}$$

In fact from formula (see [28], p. 1933)

$$P_\gamma^\lambda\,(ch\ x) = \frac{\Gamma\,(\gamma+2\lambda)\cos \pi\lambda}{\Gamma\,(\gamma)\,\Gamma\,(\gamma+\lambda+1)} (2ch\ x)^{-\gamma-2\lambda} \times {}_2F_1 \left(\frac{\gamma}{2} + \lambda, \frac{\gamma}{2} + \lambda + \frac{1}{2}; \gamma + \lambda + 1; \frac{1}{ch^2\ x} \right)$$

we have

$$\left| P_\gamma^\alpha\,(ch\ x) \right| \lesssim (ch\ x)^{-\gamma-2\lambda}.$$

The function of Gauss ${}_2F_1\,(\alpha, \beta; \gamma; x)$ is convergent by appointed importance of parameters on the interval $[0, \infty)$ (see [29], p. 1054).

Taking into account the last inequality, we estimate from above $h_r\,(ch\ x)$

$$|h_r\,(ch\ x)| \lesssim \int_1^\infty e^{-\gamma(\gamma+2\lambda)r}\,(ch\ x)^{-\gamma-2\lambda} \left(\gamma^2 - 1 \right)^{\lambda-\frac{1}{2}} d\gamma$$

$$\lesssim \int_0^\infty e^{-(\gamma+1)(\gamma+1+2\lambda)r} \gamma^{\lambda-\frac{1}{2}} (ch\ x)^{-\gamma-2\lambda-1} d\gamma$$

$$\lesssim e^{-r}\,(ch\ x)^{-2\lambda-1}\int_0^\infty \gamma^{\lambda-\frac{1}{2}}\,(ch\ x)^{-\gamma}\,d\gamma\left[\frac{1}{ch\ x}\le\frac{e}{e^{x+1}}\right]$$
$$\lesssim e^{-r}\,(ch\ x)^{-2\lambda-1}\int_0^\infty e^{-(x+1)\gamma}\gamma^{\lambda-\frac{1}{2}}d\gamma\,[(x+1)\,\gamma=u]$$
$$\lesssim e^{-r}\,(ch\ x)^{-2\lambda-1}\int_0^\infty e^{-u}u^{\lambda-\frac{1}{2}}du=\Gamma\left(\lambda+\frac{1}{2}\right)e^{-r}\,(ch\ x)^{-2\lambda-1}.$$

Hence we have

$$\int_0^\infty r^{\frac{\alpha}{2}-1}h_r\,(ch\ x)\,dr\lesssim(ch\ x)^{-2\lambda-1}\int_0^\infty r^{\frac{\alpha}{2}-1}e^{-r}dr$$
$$=\Gamma\left(\frac{\alpha}{2}\right)(ch\ x)^{-2\lambda-1}\le\Gamma\left(\frac{\alpha}{2}\right)(ch\ x)^{\alpha-2\lambda-1}\le\Gamma\left(\frac{\alpha}{2}\right)(sh\ x)^{\alpha-2\lambda-1}.$$

Taking into account this inequality on (3.10), we obtain our approval.

4. The Hardy–Littlewood–Sobolev theorem for G-Riesz potential

We consider the G-fractional integral

$$\Im_G^\alpha f(ch\ x)=\int_0^\infty A_{ch\ t}^\lambda(sh\ x)^{\alpha-2\lambda-1}f(ch\ t)sh^{2\lambda}\ tdt,\quad 0<\alpha<2\lambda+1.$$

We denote by $WL_{p,\lambda}\,[0,\ \infty)$ the weak $L_{p,\lambda}$ space of measurable functions f for which

$$\|f\|_{WL_{p,\lambda}[0,\infty)}=\sup_{t>0}t\ |\{x\in[0,\ \infty):|f\,(ch\,x)|>t\}|^{\frac{1}{p}}.$$

The next examples show that for $p\ge\frac{2\lambda+1}{\alpha}$ the integral $\Im_G^\alpha$ does not exist for $f\in L_{p,\lambda}\,[0,\ \infty)$.

Example 1. Let $x\in[0,\ \infty)$, $0<\alpha<2\lambda+1$,
$f\,(x)=\frac{1}{sh^\alpha\,x ln^2(sh\,x)}\chi_{\left(0,\,\frac{1}{2}\right)}\,(x)$. For $p=\frac{2\lambda+1}{\alpha}$ $f\in L_{p,\lambda}\,[0,\ \infty]$ and $\Im_G^\alpha f\,(x)=+\infty$.

In fact

$$\|f\|_{L_{p,\lambda}}=\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,xdx}{(sh\,x)^{\alpha p}\left(ln^2sh\,x\right)^p}\le\int_0^{\frac{1}{2}}\frac{ch\,xdx}{sh\,x\left(ln^2sh\,x\right)^p}$$
$$=-\int_0^{\frac{1}{2}}(-lnsh\,x)^{-2p}\,d\,(-lnsh\,x)=-\frac{1}{1-2p}\,(-lnsh\,x)^{1-2p}\Big|_0^{\frac{1}{2}}$$
$$=\frac{1}{2p-1}\left|lnsh\,\frac{1}{2}\right|^{1-2p}.$$

On the other hand

$$\Im_G^\alpha f\,(x)=\int_0^\infty(sh\,t)^{\alpha-2\lambda-1}\,A_{ch\,t}f\,(x)\,sh^{2\lambda}\,tdt$$
$$=\int_0^\infty A_{ch\,t}\,(sh\,x)^{\alpha-2\lambda-1}\,f\,(t)\,sh^{2\lambda}\,tdt$$
$$=\int_0^\infty\frac{A_{ch\,t}\,(sh\,x)^{\alpha-2\lambda-1}\,sh^{2\lambda}\,tdt}{sh^\alpha\,tln^2\,(sh\,t)}.$$

Since

$$A_{ch\,t}\,(sh\,x)^{\alpha-2\lambda-1} = A_{ch\,t}\left(ch^2\,x-1\right)^{\frac{\alpha-2\lambda-1}{2}}$$
$$= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}\left((ch\,xch\,t - sh\,xsh\,t\cos\varphi)^2-1\right)^{\frac{\alpha-2\lambda-1}{2}}(\sin\varphi)^{2\lambda-1}\,d\varphi$$

(as $ch\;(x-t)\le ch\,xch\,t - sh\,xsh\,t\cos\varphi \le ch\;(x+t)$)

$$\ge \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\frac{1}{2}}\int_0^{\pi}\left(ch^2\,(x+t)-1\right)^{\frac{\alpha-2\lambda-1}{2}}(\sin\varphi)^{2\lambda-1}\,d\varphi \ge (sh\;(x+t))^{\alpha-2\lambda-1}$$
$$= (sh\,xch\,t + ch\,xsh\,t)^{\alpha-2\lambda-1} \ge (2ch\,xch\,t)^{\alpha-2\lambda-1},$$

then for any fixed $x\in[0,\infty)$

$$\Im_G^{\alpha} f\,(x) \ge (2ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,tdt}{(ch\,t)^{2\lambda+1-\alpha}\,sh^{\alpha}\,tln^2\,(sh\,t)}$$
$$\ge \frac{4^{\alpha-2\lambda-1}}{\left(ch\,\frac{1}{2}\right)^{2\lambda+1}}(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{ch^{\alpha}\,tsh^{2\lambda}\,tdt}{sh^{\alpha}\,tln^2\,(sh\,t)}$$
$$\ge c_{\alpha,\lambda}\,(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,tdt}{ln^2\,(sh\,t)}$$
$$\ge c_{\alpha,\,\lambda}\,(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,tdt}{sh^2\,t} \ge c_{\alpha,\,\lambda}\,(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}t^{2\lambda-2}dt = +\infty.$$

At the end we use the inequality (1.3).

Example 2. Let $x\in[0,\;\infty)$, $0<\alpha<2\lambda+1$, $f\,(x)=\frac{1}{sh^{\alpha}\,x}\chi_{(2,\,\infty)}\,(x)$. For $p>\frac{2\lambda+1}{\alpha}$ $f\in L_{p,\,\lambda}\,[0,\;\infty)$ and $\Im_G^{\alpha} f\,(x)=+\infty$.

In fact

$$\|f\|_{L_{p,\,\lambda}} = \int_2^{\infty}\frac{sh^{2\lambda}\,xdx}{(sh\,x)^{\alpha p}} \le \int_2^{\infty}\frac{ch\,xsh^{2\lambda}\,xdx}{(sh\,x)^{\alpha p}}$$
$$= \int_2^{\infty}\frac{sh^{2\lambda}\,xd\,(sh\,x)}{(sh\,x)^{\alpha p}} = \left.\frac{(sh\,x)^{2\lambda+1-\alpha p}}{2\lambda+1-\alpha p}\right|_2^{\infty} = \frac{(sh\,2)^{2\lambda+1-\alpha p}}{\alpha p-2\lambda-1}.$$

On other hand for any fixed $x\in[0,\;\infty)$

$$\Im_G^{\alpha} f\,(x) \ge (2ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{sh^{2\lambda}\,tdt}{(ch\,t)^{2\lambda+1-\alpha}\,sh^{\alpha}\,t}$$

(as, $\frac{1}{2}ch\,t<sh\,t<ch\,t,\;\;t\ge 2$)

$$\ge 2^{2\alpha-4\lambda-2}\,(ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{dt}{sh\,t} \ge 4^{\alpha-2\lambda-2}\,(ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{dt}{ch\,t}$$
$$= 2^{2\alpha-4\lambda-2}\,(ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{e^t\,dt}{e^{2t}+1} = 2^{2\alpha-4\lambda-1}\,(ch\,x)^{\alpha-2\lambda-1}\left.\mathrm{arct}ge^t\right|_2^{\infty} = +\infty.$$

For the G-Riesz potential the following Hardy–Littlewood–Sobolev theorem is valid.

Theorem 4.1. *Let* $1-2\lambda<\alpha<1+2\lambda$, $1\le p<\frac{2\lambda+1}{\alpha}$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$.

(a) *If* $f\in L_{p,\lambda}[0,\infty)$, *then the integral* $\Im_G^\alpha f$ *is convergent absolutely for almost every* $x\in[0,\infty)$.

(b) *If* $1<p<\frac{2\lambda+1}{\alpha}$, $f\in L_{p,\lambda}[0,\infty)$, *then* $\Im_G^\alpha f\in L_{q,\lambda}[0,\infty)$ *and*

$$\|I_G^\alpha f\|_{L_{q,\lambda}[0,\infty)}\lesssim\left\|\Im_G^\alpha f\right\|_{L_{q,\lambda}[0,\infty)}\lesssim\|f\|_{L_{p,\lambda}[0,\infty)},$$

(c) *If* $f\in L_{1,\lambda}[0,\infty)$, $\frac{1}{q}=1-\frac{\alpha}{2\lambda+1}$, *then*

$$\|I_G^\alpha f\|_{WL_{q,\lambda}[0,\infty)}\lesssim\|\Im_G^\alpha f\|_{WL_{q,\lambda}[0,\infty)}\lesssim\|f\|_{L_{1,\lambda}}[0,\infty). \tag{4.1}$$

Proof. Let $f\in L_{p,\lambda}[0,\infty)$, $1\le p<\frac{2\lambda+1}{\alpha}$, $f_1(ch\ x)=f(ch\ x)\chi_{(0,1)}(x)$,

$$f_2(ch\ x)=f(ch\ x)-f_1(ch\ x),\qquad \chi_{(0,1)}(x)=\begin{cases}1, & x\in(0,1),\\ 0, & x\in(1,\infty).\end{cases}$$

Then

$$\Im_G^\alpha f(ch\ x)=\Im_G^\alpha f_1(ch\ x)+\Im_G^\alpha f_2(ch\ x)=\Im_1(ch\ x)+\Im_2(ch\ x).$$

We estimate above the $\Im_1(ch\ x)$.

$$\begin{aligned}|\Im_1(ch\ x)|&\le\int_0^1(sh\ x)^{\alpha-2\lambda-1}A_{ch\ t}^\lambda|f(ch\ x)|\,sh^{2\lambda}\ tdt\\&=\int_0^\infty(sh\ x)^{\alpha-2\lambda-1}\chi_{(0,1)}(t)A_{ch\ t}^\lambda|f(ch\ x)|\,sh^{2\lambda}\ tdt.\end{aligned}$$

By Young inequality (see [22], Lemma 4) we have

$$\|\Im_1(ch)(\cdot)\|_{L_{p,\lambda}[0,\infty)}\le\|f(ch)(\cdot)\|_{L_{p,\lambda}[0,\infty)}\cdot\left\||\cdot|^{\alpha-2\lambda-1}\chi_{(0,1)}\right\|_{L_{1,\lambda}[0,\infty)}. \tag{4.2}$$

Here

$$\begin{aligned}\left\||\cdot|^{\alpha-2\lambda-1}\chi_{(0,1)}\right\|_{L_{1,\lambda}}&=\int_0^1(sh\ t)^{\alpha-2\lambda-1}sh^{2\lambda}\ tdt\\&\le\int_0^1(sh\ t)^{\alpha-1}ch\ tdt=\int_0^1(sh\ t)^{\alpha-1}d(sh\ t)=\frac{1}{\alpha}sh^\alpha 1.\end{aligned}$$

From (4.1) and (4.2) it follows, that $\Im_1(ch\ x)$ for almost every $x\in[0,\infty)$ is convergent absolutely.

By using the Hölder inequality

$$\begin{aligned}|\Im_2(ch\ x)|&\le\int_1^\infty(sh\ t)^{\alpha-2\lambda-1}A_{ch\ t}^\lambda|f(ch\ x)|\,sh^{2\lambda}\ tdt\\&\le\left\|A_{ch\ t}^\lambda f\right\|_{L_{p,\lambda}}\cdot\left(\int_1^\infty(sh\ t)^{(\alpha-2\lambda-1)q}\,sh^{2\lambda}\ tdt\right)^{\frac{1}{q}}\\&\le\|f\|_{L_{p,\lambda}}\left(\int_1^\infty(sh\ t)^{(\alpha-2\lambda-1)q+2\lambda}\,ch\ tdt\right)^{\frac{1}{q}}\\&=\|f\|_{L_{p,\lambda}}\left(\int_1^\infty(sh\ t)^{(\alpha-2\lambda-1)q+2\lambda}\,d(sh\ t)\right)^{\frac{1}{q}}\\&=\left(\frac{(sh1)^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)\,q-2\lambda-1}\right)^{\frac{1}{q}}\cdot\|f\|_{L_{p,\lambda}}=c_{\alpha,\lambda,p}\,\|f\|_{L_{p,\lambda}},\end{aligned}$$

from this it follows the absolutely convergence of $\Im_2(ch\ x)$ for almost every $x\in[0,\infty)$.

Thus, for all $f\in L_{p,\lambda}[0,\infty)$, $1\le p<\frac{2\lambda+1}{\alpha}$, G-Riesz potential $\Im_G^\alpha f(ch\ x)$ is convergent absolutely for almost every $x\in[0,\infty)$.

(b) We have

$$\Im_G^\alpha f(ch\ x) = \left(\int_0^r + \int_r^\infty\right) A_{ch\ t}^\lambda f(ch\ x)(sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda}\ tdt = A_1(x,r) + A_2(x,r). \tag{4.3}$$

We consider $A_1(x,r)$. Let $0 < r < 2$.

$$\begin{aligned}
|A_1(x,r)| &\le \int_0^r \left|A_{ch\ t}^\lambda f(ch\ x)\right| (sh\ t)^{2\lambda} (sh\ t)^{\alpha-2\lambda-1} dt \\
&\le \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\ t}^\lambda |f(ch\ x)|\, sh^{2\lambda}\ tdt}{(sh\ t)^{2\lambda+1-\alpha}} \\
&\le \sum_{k=0}^\infty \left(sh \frac{r}{2^{k+1}}\right)^\alpha \left(sh \frac{r}{2^{k+1}}\right)^{-2\lambda-1} \int_0^{\frac{r}{2^k}} A_{ch\ t}^\lambda |f(ch\ x)|\, sh^{2\lambda}\ tdt \\
&\lesssim M_{G,1} f\,(ch\,x) \sum_{k=1}^\infty \left(\frac{1}{2^k} sh\ \frac{r}{2}\right)^\alpha \lesssim \left(sh\ \frac{r}{2}\right)^\alpha M_{G,1} f\,(ch\,x) \sum_{k=1}^\infty \frac{1}{2^{k\alpha}} \\
&\lesssim \left(sh\ \frac{r}{2}\right)^\alpha M_{G,1} f(ch\ x),
\end{aligned} \tag{4.4}$$

as $sh\ \frac{t}{a} \le \frac{1}{a} sh\,t$ for $a \ge 1$.

We consider $A_2(x,r)$. By Hölder inequality

$$\begin{aligned}
|A_2(x,r)| &\le \left(\int_r^\infty \left|A_{ch\ t}^\lambda f(ch\ x)\right|^p sh^{2\lambda}\ tdt\right)^{\frac{1}{p}} \left(\int_r^\infty (sh\ t)^{(\alpha-2\lambda-1)q} sh^{2\lambda}\ tdt\right)^{\frac{1}{q}} \\
&\le \left\|A_{ch\ t}^\lambda f\right\|_{L_{p,\lambda}} \left(\int_{r/2}^\infty (sh\ t)^{(\alpha-2\lambda-1)q+2\lambda} ch\ tdt\right)^{\frac{1}{q}} \\
&\le \|f\|_{L_{p,\lambda}} \left(\frac{\left(sh\ \frac{r}{2}\right)^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)\,q - 2\lambda - 1}\right)^{\frac{1}{q}} < \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{2\lambda+1}{q}} \\
&= \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{\alpha-2\lambda-1+(2\lambda+1)\left(\frac{1}{p}-\frac{\alpha}{2\lambda+1}\right)} \\
&= \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{(2\lambda+1)\left(\frac{1}{p}-1\right)} = \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}}.
\end{aligned} \tag{4.5}$$

Taking into account (4.4) and (4.5) in (4.3), we obtain

$$\left|\Im_G^\alpha f(ch\ x)\right| \le \left(\left(sh\ \frac{r}{2}\right)^\alpha M_{G,1} f(ch\ x) + \left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{p,\lambda}}\right). \tag{4.6}$$

Minimum of the right-hand side of the inequality (4.6) reaches to

$$sh\ \frac{r}{2} = \left(\frac{2\lambda+1}{\alpha q} \cdot \frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)}\right)^{\frac{p}{2\lambda+1}}.$$

Then from (4.6) we have

$$\begin{aligned}
\left|\Im_G^\alpha f(ch\ x)\right| &\le \left\{\left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)}\right)^{\frac{\alpha p}{2\lambda+1}} M_{G,1} f(ch\ x) + \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)}\right)^{-\frac{p}{q}} \|f\|_{L_{p,\lambda}}\right\} \\
&= \left(M_{G,1} f(ch\ x)\right)^{\frac{p}{q}} \|f\|_{L_{p,\lambda}}^{1-\frac{p}{q}},
\end{aligned}$$

for the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1} \Rightarrow 1 - \frac{p}{q} = \frac{\alpha p}{2\lambda+1}$.

From this we have

$$\int_0^\infty \left|\Im_G^\alpha f(ch\ t)\right|^q sh^{2\lambda}\ tdt \le \left\|M_{G,1}f(ch(\cdot))\right\|_{L_{p,\lambda}}^p \cdot \|f\|_{L_{p,\lambda}}^{q-p} \le \|f\|_{L_{p,\lambda}}^{q-p} \cdot \|f\|_{L_{p,\lambda}}^p = \|f\|_{L_{p,\lambda}}^q,$$

from this it follows that for $0 < r < 2$

$$\|I_G^\alpha\|_{L_{q,\lambda}} \lesssim \left\|\Im_G^\alpha f\right\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.7}$$

Now let $2 \le r < \infty$. Then from (4.3) and by Lemma 1.1 we have

$$\begin{aligned}
|A_1(x, r)| &\le \int_0^r \frac{A_{ch\,t}^\lambda |f(ch\,x)|sh^{2\lambda}\,tdt}{(sh\,t)^{2\lambda+1-\alpha}} = \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\,t}^\lambda |f(ch\,t)|\,sh^{2\lambda}\,tdt}{(sh\,t)^{2\lambda+1-\alpha}} \\
&\le \sum_{k=0}^\infty \left(sh\,\frac{r}{2^{k+1}}\right)^{\alpha+2\lambda-1}\left(sh\,\frac{r}{2^{k+1}}\right)^{-4\lambda}\int_0^{\frac{r}{2^k}} A_{ch\,t}^\lambda |f(ch\,x)|sh^{2\lambda}\,tdt \\
&\lesssim \sum_{k=0}^\infty \left(sh\,\frac{r}{2^{k+1}}\right)^{\alpha+2\lambda-1}\left(ch\,\frac{r}{2^{k+1}}\right)^{-4\lambda}\int_0^{\frac{r}{2^k}} A_{ch\,t}^\lambda |f(ch\,x)|\,sh^{2\lambda}\,tdt \\
&\lesssim M_{G,2}f(ch\,x)\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1}\sum_{k=1}^\infty \frac{1}{2^{(\alpha+2\lambda-1)k}} \lesssim \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} M_{G,2}f(ch\,x).
\end{aligned} \tag{4.8}$$

Taking into account Hölder inequality from (4.3) we obtain

$$\begin{aligned}
|A_1(x, r)| &\le \|f\|_{L_{p,\lambda}}\left(\int_r^\infty (sh\,t)^{(\alpha-2\lambda-1)q}\,sh^{2\lambda}\,tdt\right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{L_{p,\lambda}}\left(\int_r^\infty \frac{\left(sh\,\frac{t}{2}\right)^{(\alpha-2\lambda-1)q}}{\left(ch\,\frac{t}{2}\right)^{(2\lambda+1-\alpha)q}}\,\frac{sh^{2\lambda}\,\frac{t}{2}d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}}\right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{L_{p,\lambda}}\left(\int_r^\infty \left(sh\,\frac{t}{2}\right)^{(\alpha-2\lambda-1)q}\left(sh\,\frac{t}{2}\right)^{4\lambda-1}d\left(sh\,\frac{t}{2}\right)\right)^{\frac{1}{r}} \\
&\lesssim \|f\|_{L_{p,\lambda}}\left(sh\,\frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}}.
\end{aligned} \tag{4.9}$$

Now from (4.3), (4.8) and (4.9) we have

$$|\Im_G^\alpha f(ch\,x)| \lesssim \left(\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} M_{G,2}f(ch\,x) + \left(sh\,\frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}}\|f\|_{L_{p,\lambda}}\right). \tag{4.10}$$

Minimum of the right-hand side of the inequality (4.10) reaches to

$$sh\,\frac{r}{2} = \left(\frac{(2\lambda+1-\alpha)\,q-4\lambda}{(\alpha+2\lambda-1)\,q}\,\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2}f(ch\,x)}\right)^{\frac{p}{4\lambda}}.$$

Then from (4.10) we have

$$\begin{aligned}
|\Im_G^\alpha f(ch\,x)| &\lesssim \left\{\left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2}f(ch\,x)}\right)^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} M_{G,2}f(ch\,x) + \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2}f(ch\,x)}\right)^{\frac{(\alpha+2\lambda-1)p-4\lambda}{4\lambda}}\|f\|_{L_{p,\lambda}}\right\} \\
&= \left(M_{G,2}f(ch\,x)\right)^{\frac{(1+2\lambda-\alpha)p+4\lambda}{4\lambda}}\|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}}.
\end{aligned}$$

From this we obtain

$$\int_0^\infty |\Im_G^\alpha f(ch\,x)|^q sh^{2\lambda}\,xdx \le \|M_G f\|^{\frac{(1-2\lambda-\alpha)q}{4\lambda}+q}\|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} \lesssim \|f\|_{L_{p,\lambda}}^{q+\frac{(1-2\lambda-\alpha)q}{4\lambda}}\|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)q}{4\lambda}} = \|f\|_{L_{p,\lambda}}^q.$$

From this it follows that for $2 \le r < \infty$

$$\|I_G^\alpha f\|_{L_{q,\lambda}} \lesssim \|\mathfrak{I}_G^\alpha f\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}.$$

Combining last inequality and (4.7) we obtain the approval (b).

c Let $f \in L_{1,\lambda}[0,\infty)$ and $0 < r < 2$. By (4.3) we get

$$\left|\left\{x : |\mathfrak{I}_G^\alpha f(ch\ x)| > 2\beta\right\}\right|_\lambda \le |\{x : |A_1(x,r)| > \beta\}|_\lambda + |\{x : |A_2(x,r)| > \beta\}|_\lambda .$$

From inequality (4.4) and Theorem 2.2 we have

$$\begin{aligned}\beta\,|\{x \in [0,\infty) : |A_1(x,r)| > \beta\}|_\lambda &= \beta \int_{\{x\in[0,\infty):|A_1(x,r)|>\beta\}} sh^{2\lambda}\ x dx \\ &\le \beta \int_{\{x\in[0,\infty):c_{\alpha,\lambda}(sh^\alpha\ \frac{r}{2})M_G f(ch\ x)>\beta\}} sh^{2\lambda}\ x dx \\ &= \beta \left|\left\{x \in [0,\infty) : M_G f(ch\ x) > \frac{\beta}{sh^\alpha\ \frac{r}{2}}\right\}\right|_\lambda \\ &\le \beta \cdot \frac{c_\lambda}{\beta} sh^\alpha\ \frac{r}{2} \int_0^\infty |f(ch\ x)|\, sh^{2\lambda}\ x dx = c_\lambda sh^\alpha\ \frac{r}{2}\, \|f\|_{L_{1,\lambda}},\end{aligned}$$

and also

$$\begin{aligned}|A_2(x,r)| &\le \int_r^\infty \left|A_{ch\ t}^\lambda f(ch\ x)\right| (sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda}\ t dt \\ &\le \int_r^\infty \frac{\left|A_{ch\ t}^\lambda f(ch\ x)\right| sh^{2\lambda}\ t dt}{(sh\ t)^{2\lambda+1-\alpha}} \le \int_r^\infty \frac{\left|A_{ch\ t}^\lambda f(ch\ x)\right| sh^{2\lambda}\ t dt}{\left(sh\ \frac{t}{2}\right)^{2\lambda+1-\alpha}} \\ &\le \left(sh\ \frac{r}{2}\right)^{\alpha-2\lambda-1} \int_r^\infty \left|A_{ch\ t}^\lambda f(ch\ x)\right| sh^{2\lambda}\ t dt \le \left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{1,\lambda}}.\end{aligned}$$

Suppose $\left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{1,\lambda}} = \beta$, we obtain $|A_2(x,r)| \le \beta$ and consequently $|\{x \in [0,\infty) : |A_2(x,r)| > \beta\}|_\lambda = 0$.

Finally we get

$$\begin{aligned}&\left|\left\{x \in [0,\infty) : \left|\mathfrak{I}_G^\alpha f(ch\ x)\right| > \beta\right\}\right|_\lambda \le c_{\alpha,\lambda} \cdot \frac{1}{\beta} sh^\alpha\ \frac{r}{2}\, \|f\|_{L_{1,\lambda}} \\ &\le c_\lambda \left(sh\ \frac{r}{2}\right)^{\alpha+\frac{2\lambda+1}{q}} = c_\lambda \left(sh\ \frac{r}{2}\right)^{2\lambda+1} = c_\lambda \left(\frac{1}{\beta}\|f\|_{L_{1,\lambda}}\right)^q.\end{aligned}$$

From this and (3.11) for $0 < r < 2$ it follows (4.1). Now we consider the case $2 \le r < \infty$. From the inequality (4.8) and Theorem 2.2 we have

$$\begin{aligned}|\{x \in [0,\ \infty) : |A_1(x,\ r) > \beta|\}|_\lambda &= \beta \int_{\{x\in[0,\ \infty):A_1(x,r)>\beta\}} sh^{2\lambda}\, x dx \\ &\le \beta \int_{\{x\in[0,\ \infty):(sh\ \frac{r}{2})^{\alpha+2\lambda-1} M_G f(ch\, x)>\beta\}} \\ &= \beta \left|\left\{x \in [0,\ \infty) : M_G f\,(ch\, x) > \frac{\beta}{\left(sh\ \frac{r}{2}\right)^{\alpha+2\lambda-1}}\right\}\right|_\lambda \\ &\le c_\lambda \cdot \beta \cdot \frac{1}{\beta}\left(sh\ \frac{r}{2}\right)^{\alpha+2\lambda-1} \int_0^\infty |f\,(ch\, x)\,|sh^{2\lambda}\, x dx \\ &= c_\lambda \left(sh\ \frac{r}{2}\right)^{\alpha+2\lambda-1} \|f\|_{L_{1,\lambda}}.\end{aligned}$$

On the other hand

$$|A_2(x,r)| \le \int_r^\infty \frac{|A_{ch\,t} f(ch\,x)\, sh^{2\lambda} t dt|}{(sh\,t)^{2\lambda+1-\alpha}} \le \int_r^\infty \frac{A_{ch\,t}|f(ch\,x)|sh^{2\lambda} t dt}{(sh\,t)^{2\lambda+1-\alpha}}$$
$$\le \int_r^\infty \frac{A_{ch\,t}|f(ch\,x)|sh^{2\lambda} t dt}{\left(sh\,\frac{t}{2}\right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}}} \le \left(sh\,\frac{r}{2}\right)^{-\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}},$$

since

$$2\lambda+1-\alpha > \frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q} = \frac{(\alpha+2\lambda-1)(2\lambda+1)(2\lambda+1-\alpha)}{\alpha(2\lambda+1)}$$
$$= \frac{(\alpha+2\lambda-1)(2\lambda+1-\alpha)}{\alpha} \Leftrightarrow \alpha > \alpha+2\lambda-1 \Leftrightarrow \lambda < \frac{1}{2}.$$

Putting $\left(sh\,\frac{r}{2}\right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}} = \beta$, we obtain $|A_2(x,r)| \le \beta$ and consequently

$$|\{x \in [0,\infty) : |A_2(x,r)| > \beta\}|_\lambda = 0.$$

Now by Theorem 2.2 we have

$$|\{x \in [0,\infty) : |\Im_G^\alpha f(ch\,x)| > 2\beta\}|_\lambda \le c_\lambda \frac{1}{\beta}\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} \|f\|_{L_{1,\lambda}}$$
$$\le c_\lambda \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1+\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}}$$
$$= c_\lambda \left(sh\,\frac{r}{2}\right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha}} \|f\|_{L_{1,\lambda}} \le c_\lambda \left(\frac{1}{\beta}\|f\|_{L_{1,\lambda}}\right)^q.$$

From this and (3.11) for $2 \le r < \infty$ it follows (4.10).
Thus, $f \mapsto \Im_G^\alpha f$ is weak type $(1,q)$ and Theorem 4.1 is proved. □

Definition 4.1 (*[14]*). We denote by $BMO_G[0,\infty)$ the BMO-Gegenbauer space (G-BMO space) as the set of functions locally integrable on $[0,\infty)$, with finite norm

$$\|f\|_{BMO_G} = \sup_{x,r\in(0,\infty)} \frac{1}{|H(0,r)|_\lambda} \int_{H(0,r)} \left|A_{ch\,t}^\lambda f(ch\,x) - f_{H(0,r)}(ch\,x)\right| sh^{2\lambda} t dt,$$

where

$$f_{H(0,r)}(ch\,x) = \frac{1}{|H(0,r)|_\lambda} \int_{H(0,r)} A_{ch\,t}^\lambda |f(ch\,x)|\, sh^{2\lambda} t dt.$$

As an analogue of [14] we introduce modified fractional Riesz–Gegenbauer integral (G-Riesz integral) by

$$\widetilde{\Im}_G^\alpha f(ch\,x) = \int_0^\infty (A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1}\chi_{(\frac{1}{4},\infty)}(ch\,t))\, f(ch\,t) sh^{2\lambda} t dt$$

and modified Riesz–Gegenbauer potential (G-Riesz potential) by

$$\widetilde{I}_G^\alpha f(ch\,x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\,t)\, dr\right)$$
$$\times (A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1}\chi_{(\frac{1}{4},\infty)}(ch\,t))\, f(ch\,t) sh^{2\lambda} t dt,$$

where $\chi_{\left(\frac{1}{4},\infty\right)}(ch\,t)$ is a characteristic of function of the interval $\left(\frac{1}{4},\infty\right)$. Also in the proof of inequality (3.11) we obtain that

$$|\widetilde{I}_G^\alpha f(ch\,x)| \lesssim \int_0^\infty \left|A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1}\chi_{\left(\frac{1}{4},\infty\right)}(ch\,t)\right| |f(ch\,t)|\, sh^{2\lambda} t dt,$$

from this we have

$$\|\widetilde{I}_G^{\alpha} f\|_{BMO_G} \lesssim \|\widetilde{\Im}_G^{\alpha} f\|_{BMO_G}. \tag{4.11}$$

The following theorem is valid.

Theorem 4.2. *Let* $1-2\lambda<\alpha<2\lambda+1$, $p\alpha=2\lambda+1$ *and* $f\in L_{p,\lambda}[0,\infty)$. *Then* $\widetilde{\Im}_G^{\alpha} f\in BMO_G[0,\infty)$ *and the inequality*

$$\left\|\widetilde{\Im}_G^{\alpha} f\right\|_{BMO_G} \lesssim \|f\|_{L_{p,\lambda}}$$

is valid.

Proof. Suppose that

$$f_1(ch\ x)=f(ch\ x)\,\chi_{(0,r/4)}(ch\ x), \qquad f_2(ch\ x)=f(ch\ x)-f_1(ch\ x),$$

where $\chi_{(0,r/4)}(ch\ x)$ is the characteristic function of the interval $\left(0,\ \frac{r}{4}\right)$, that is,

$$\chi_{(0,r/4)}(ch\ x)=\begin{cases}1, & 0<x<\dfrac{r}{4}\\ 0, & x>\dfrac{r}{4}.\end{cases}$$

Then

$$\widetilde{\Im}_G^{\alpha} f(ch\ x)=\widetilde{\Im}_G^{\alpha} f_1(ch\ x)+\widetilde{\Im}_G^{\alpha} f_2(ch\ x)=F_1(ch\ x)+F_2(ch\ x),$$

where

$$F_1(ch\ x)=\int_0^{r/4}\left(A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1}-(sh\ t)^{\alpha-2\lambda-1}\chi_{\left(\frac{1}{4},\infty\right)}(ch\ t)\right)f(ch\ t)\,sh^{2\lambda}\ tdt$$

and

$$F_2(ch\ x)=\int_{r/4}^{\infty}\left(A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1}-(sh\ t)^{\alpha-2\lambda-1}\chi_{\left(\frac{1}{4},\infty\right)}(ch\ t)\right)f(ch\ t)\,sh^{2\lambda}\ tdt.$$

Since the function $f_1(ch\ x)$ has compact support, then the number

$$a_1=-\int_{(0,r/4)/\left(0,\min\left\{\frac{1}{4},\frac{r}{4}\right\}\right)}(sh\ t)^{\alpha-2\lambda-1}f(ch\ t)\,sh^{2\lambda}\ tdt$$

is finite. We can write

$$\begin{aligned}F_1(ch\ x)-a_1&=\int_0^{r/4}A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1}f(ch\ x)sh^{2\lambda}\ tdt\\&\quad-\int_{(0,r/4)/\left(0,\min\left\{\frac{1}{4},\frac{r}{4}\right\}\right)}(sh\ t)^{\alpha-2\lambda-1}f(ch\ t)\,sh^{2\lambda}\ tdt\\&\quad+\int_{(0,r/4)\left(0,\min\left\{\frac{1}{4},\frac{r}{4}\right\}\right)}(sh\ t)^{\alpha-2\lambda-1}f(ch\ t)\,sh^{2\lambda}\ tdt\\&=\int_0^{r/4}A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1}f(ch\ t)\,sh^{2\lambda}\ tdt\\&=\int_0^{\infty}A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1}f_1(ch\ t)\,sh^{2\lambda}\ tdt.\end{aligned} \tag{4.12}$$

Consider the integral

$$\begin{aligned}A_{ch\ t}^{\lambda}f_1(ch\ x)&=c_{\lambda}\int_0^{\pi}f(ch\ xch\ t-sh\ xsh\ t\cos\ \varphi)\\&\quad\times\chi_{(0,r/4)}(ch\ xch\ t-sh\ xsh\ t\cos\ \varphi)(\sin\ \varphi)^{2\lambda-1}d\varphi.\end{aligned}$$

So far as, $ch\ (x-t) \le ch\ xch\ t - sh\ xsh\ t \cos\ \varphi \le ch\ (x+t)$, then for $|x-t| > \frac{r}{4}$

$$\chi_{(0,r/4)}\left(ch\ xch\ t - sh\ xsh\ t \cos\ \varphi\right) = 0,$$

and then

$$\begin{aligned} A^{\lambda}_{ch\ t} f_1 (ch\ x) &= c_\lambda \int_{\{\varphi\in[0,\pi],|x-t|\le r/4\}} f\left(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi\right)(\sin\ \varphi)^{2\lambda-1} d\varphi \\ &= A^{\lambda}_{ch\ t} f(ch\ x). \end{aligned}$$

Then for (4.12) we have

$$|F_1(ch\ x) - a_1| \le \int_{\{t\in[0,\infty);|x-t|\le r/4\}} (sh\ t)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ tdt. \tag{4.13}$$

We consider the estimation (4.13).

Let $\left(x-\frac{r}{4}, x+\frac{r}{4}\right)\cap[0,\infty) = \left(0, x+\frac{r}{4}\right)$, then $0 \le x \le r/4$ and we have from (4.13) and (4.14)

$$\begin{aligned} |F_1(ch\ x) - a_1| &\le \int_0^{x+r/4} (sh\ t)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ tdt \\ &\le \int_0^{r} (sh\ t)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ tdt \le \left(sh\ \frac{r}{2}\right)^{\alpha} M_{G,1} f(ch\ x). \end{aligned} \tag{4.14}$$

Let now $\left(x-\frac{r}{4}, x+\frac{r}{4}\right)\cap[0,\infty) = \left(x-\frac{r}{4}, x+\frac{r}{4}\right)$, then $x > \frac{r}{4}$. Consider the case $\frac{r}{4} \le x \le \frac{3r}{4}$. Then

$$\begin{aligned} |F_1(ch\ x) - a_1| &\le \int_{x-r/4}^{x+r/4} (sh\ t)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ tdt \\ &\le \int_0^{r} (sh\ t)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ tdt \le \left(sh\ \frac{r}{2}\right)^{\alpha} M_{G,1} f(ch\ x). \end{aligned} \tag{4.15}$$

Finally, let $\frac{3r}{4} \le x < \infty$, then by Hölder inequality, we have

$$\begin{aligned} |F_1(ch\ x) - a_1| &= \int_{x-\frac{r}{4}}^{x+\frac{r}{4}} (sh\ t)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t} |f(ch\ x)|\, sh^{2\lambda}\ tdt \\ &\le \left\| A^{\lambda}_{ch\ t} f\right\|_{L_{p,\lambda}} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} (sh\ t)^{(\alpha-2\lambda-1)q}\, sh^{2\lambda}\ tdt\right)^{\frac{1}{q}} \\ &\le \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} \left(2sh\ \frac{t}{2} ch\ \frac{t}{2}\right)^{2\lambda} dt\right)^{\frac{1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} \left(sh\ \frac{t}{2}\right)^{2\lambda} \left(ch\ \frac{t}{2}\right)^{2\lambda-1} d\left(sh\ \frac{t}{2}\right)\right)^{\frac{1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} sh^{2\lambda}\ \frac{t}{2} d\left(sh\ \frac{t}{2}\right)\right)^{\frac{1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(sh\left(\frac{x}{2}+\frac{r}{8}\right)\right)^{\frac{2\lambda+1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(\frac{x}{2}+\frac{r}{8}\right)\right)^{\alpha-2\lambda-1+(2\lambda+1)\left(1-\frac{\alpha}{2\lambda+1}\right)} = c_{\lambda,p} \|f\|_{L_{p,\lambda}}. \end{aligned} \tag{4.16}$$

Combining (4.14)–(4.16), we obtain that for $0 < r < 2$

$$|F_1(ch\ x) - a_1| \le c_{\alpha,\lambda} \left(sh\ \frac{r}{2}\right)^{\alpha} M_{G,1} f(ch\ x) + c_{\lambda,p} \|f\|_{L_{p,\lambda}}.$$

From this it follows that

$$\sup_{0<r<2} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A^\lambda_{ch\ t} (F_1(ch\ x) - a_1) \right| sh^{2\lambda}\ tdt$$
$$\leq \sup_{0<r<2} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\ t} |F_1(ch\ x) - a_1| sh^{2\lambda}\ tdt$$
$$\leq \sup_{0<r<2} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \int_0^r A^\lambda_{ch\ t} \left| M_{G,1} f(ch\ x) \right| sh^{2\lambda}\ tdt + c_{\lambda,p} \frac{\|f\|_{L_{p,\lambda}}}{|H(0,r)|_\lambda} \int_0^r sh^{2\lambda}\ tdt$$
$$\leq \sup_{0<r<2} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(\int_0^r sh^{2\lambda}\ tdt \right)^{\frac{1}{q}} \left\| A^\lambda_{ch\ t} \left| M_{G,1} f(\cdot) \right| \right\|_{L_{p,\lambda}} + c_{\lambda,p} \|f\|_{L_{p,\lambda}}$$
$$\leq \sup_{0<r<2} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(sh\ \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \|M_{G,1} f\|_{L_{p,\lambda}} + c_{\lambda,p} \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.17}$$

We consider the case $2 \leq r < \infty$. Let $0 \leq x \leq \frac{3r}{4}$.
Then

$$|F_1(ch\,x) - a_1| \leq \int_0^r (sh\,t)^{\alpha-2\lambda-1} A_{ch\,t} |f(ch\,x)| sh^{2\lambda}\,tdt \leq \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\,t} |f(ch\,x)\, sh^{2\lambda} tdt|}{(sh\,t)^{2\lambda+1-\alpha}}$$
$$\leq \sum_{k=0}^{\infty} \left(sh\,\frac{r}{2^{k+1}} \right)^{\alpha+2\lambda-1} \left(sh\,\frac{r}{2^{k+1}} \right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{ch\,t} |f(ch\,x)| sh^{2\lambda} tdt$$
$$\lesssim \left(sh\,\frac{r}{2} \right)^{\alpha+2\lambda-1} M_{G,2} f(ch\,x). \tag{4.18}$$

Using (4.18), and by Lemma 1.1 we obtain

$$\sup_{r\geq 2} \frac{1}{|H(0,\ r)|_\lambda} \int_0^r |A_{ch\,t} f(ch\,x)| sh^{2\lambda}\,tdt$$
$$\leq \|M_{G,2} f\|_{L_{p,\lambda}} \sup_{r\geq 2} \left(sh\,\frac{r}{2} \right)^{\alpha-2\lambda-1} \left(\int_0^r sh^{2\lambda}\,tdt \right)^{\frac{1}{q}} + \|f\|_{L_{p,\lambda}}$$
$$\lesssim \|f\|_{L_{p,\lambda}} \sup_{r\geq 2} \left(sh\,\frac{r}{2} \right)^{\alpha-2\lambda-1} \left(ch\,\frac{r}{2} \right)^{\frac{4\lambda}{q}} + \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}}$$
$$\lesssim \|f\|_{L_{p,\lambda}} \sup_{r\geq 2} \left(ch\,\frac{r}{2} \right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}} \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}, \tag{4.19}$$

since $\alpha - 2\lambda - 1 + \frac{4\lambda}{q} = \alpha - 2\lambda - 1 + 4\lambda\left(1 - \frac{1}{2\lambda+1}\right) = \alpha + 2\lambda - 1 - \frac{4\lambda\alpha}{2\lambda+1} = \frac{\alpha-2\lambda}{1+2\lambda}\alpha + 2\lambda - 1 < 0 \Leftrightarrow \frac{1-2\lambda}{1+2\lambda}\alpha < 1 - 2\lambda \Leftrightarrow 0 < \alpha < 2\lambda + 1$.

Combining (4.17) and (4.19) we obtain that

$$\sup_{r>0} \frac{1}{|H(0,\ r)|_\lambda} \int_0^r |A_{ch\,t}(F_1(ch\,x) - a_1)| sh^{2\lambda}\,tdt \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.20}$$

Suppose $a_2 = \int_{\left(0,\max\left\{\frac{1}{4},\frac{r}{4}\right\}\right)/(0,\frac{r}{4})} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ tdt$.

We estimate above the difference

$$|F_2(ch\ x) - a_2| = \left| \int_{\frac{r}{4}}^{\infty} \left(A^\lambda_{ch\ t} (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \chi_{\left(\frac{1}{4},1\right)}(ch\ t) \right) f(ch\ t)\, sh^{2\lambda}\ tdt \right.$$
$$\left. - \int_{(0,\max\left\{\frac{1}{4},\frac{r}{4}\right\})/(0,\frac{r}{4})} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ td\ t \right|$$

$$= \left| \int_{\frac{r}{4}}^{\infty} \left(A_{ch\ t}^{\lambda} (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right) f\,(ch\ t)\, sh^{2\lambda}\ tdt \right|$$
$$\le \int_{\frac{r}{4}}^{\infty} |f\,(ch\ t)|\, B\,(x,t)\, sh^{2\lambda}\ tdt = \tau\,(x,r)\,. \tag{4.21}$$

We consider expansion

$$B\,(x,t) = \left| A_{ch\ t}^{\lambda} (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right|$$
$$= c_\lambda \left| \int_0^{\pi} \left((ch\ xch\ t - sh\ xsh\ t\cos\ \varphi)^2 - 1 \right)^{\frac{\alpha-2\lambda-1}{2}} - ((sh\ t)^{\alpha-2\lambda-1})\,(\sin\ \varphi)^{2\lambda-1}\, d\varphi \right|$$
$$\le c_\lambda \int_0^{\pi} \left| (\max\,(sh\,(x+t)\,,\,|sh\,(x-t)|))^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right| (\sin\ \varphi)^{2\lambda-1}\, d\varphi.$$

We estimate above the expression $B(x,t)$. It is easy to notice that

$$B(x,t) \lesssim \left| \max\,(\{sh\ (x+t), |sh\ (x-t)|\})^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right| \equiv V(x,t).$$

I. If $0 < t < x - t < \infty$, then $0 < t < \frac{x}{2} < x + t$.
From this it follows, that

$$(sh\ t)^{\alpha-2\lambda-1} > (sh\,(x+t))^{\alpha-2\lambda-1}\,. \tag{4.22}$$

II. If $0 < x - t < t < \infty$, then $\frac{x}{2} < t < x < x + t$, and in this case the inequality (4.22) is just.
III. If $0 < t - x < \infty$, then $x < t < x + t < \infty$.
Again the inequality (4.22) takes place.
IV. If $0 < x + t < \infty$, since $t < x + t$, then (4.22) is valid.
Combining all these cases, we obtain that

$$V\,(x,t) = (sh\ t)^{\alpha-2\lambda-1} - (sh\,(x+t))^{\alpha-2\lambda-1}\,.$$

Applying the Lagrange formula to segment $[t, x+t]$, we obtain

$$V\,(x,t) \equiv V_\xi\,(x,t) = \frac{(2\lambda+1-\alpha)\, xch\ \xi}{(sh\ \xi)^{2\lambda+2-\alpha}}, \quad t < \xi < t + x.$$

From this we have

$$\begin{aligned} v_\xi\,(x,t) &\lesssim x\,(sh\ t)^{\alpha-2\lambda-2} \quad \xi < 1, \\ v_\xi\,(x,t) &\lesssim x\,(sh\ t)^{\alpha-2\lambda-1} \quad \xi \ge 1, \end{aligned} \tag{4.23}$$

At first we consider the case $\xi < 1$.
Applying the Hölder inequality and also (4.22) and (4.23), from (4.21) for $x \le r$ we obtain

$$\tau\,(x,r) = \int_{r/4}^{\infty} |f\,(ch\ t)|\, B\,(x,t)\, sh^{2\lambda}\ tdt \lesssim \|f\|_{L_{p,\lambda}}\, x \left(\int_{r/4}^{\infty} \frac{sh^{2\lambda}\ \ tdt}{(sh\ t)^{(2\lambda+2-\alpha)q}} \right)^{\frac{1}{q}}$$
$$\lesssim \|f\|_{L_{p,\lambda}}\, r \left(\int_{r/4}^{\infty} (sh\ t)^{(\alpha-2\lambda-2)q+2\lambda}\, dsh\ t \right)^{\frac{1}{q}} = \|f\|_{L_{p.\lambda}} \frac{r}{sh\ \frac{r}{4}} \lesssim \|f\|_{L_{p,\lambda}}\,. \tag{4.24}$$

By the hypothesis of theorem $\alpha - 2\lambda - 2 + (2\lambda+1)\,/q = \alpha - 2\lambda - 2 + (2\lambda+1) \left(1 - \frac{\alpha}{2\lambda+1}\right) = \alpha - 2\lambda - 2 + 2\lambda + 1 - \alpha = -1$.
Now we consider the case $\xi \ge 1$.
Let $0 < x \le 8$. Taking into account (4.22) and acting as above for $x \le r$ we obtain

$$\tau\,(x,r) = \int_{r/4}^{\infty} |f\,(ch\ t)|\, B\,(x,t)\, sh^{2\lambda}\ tdt \lesssim \|f\|_{L_{p,\lambda}}\, x \left(\int_{r/4}^{\infty} \frac{sh^{2\lambda}\ tdt}{(sh\ t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{\left(sh^{2\lambda}\ \frac{t}{2}\right)\left(ch\ \frac{t}{2}\right)^{2\lambda-1} d(sh\ \frac{t}{2})}{\left(2sh\ \frac{t}{2} ch\ \frac{t}{2}\right)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{\left(sh\ \frac{t}{2}\right)^{4\lambda-1} d(sh\ \frac{t}{2})}{\left(sh\ \frac{t}{2}\right)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}}$$

$$= \|f\| x \left(\int_{x/4}^{\infty} \left(sh\ \frac{t}{2}\right)^{(\alpha-2\lambda-1)q+4\lambda-1} d\left(sh\ \frac{t}{2}\right) \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{x}{8}\right)^{\alpha-2\lambda-1+4\lambda/q} = \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{x}{8}\right)^{\alpha-2\lambda-1+4\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)}$$

$$= \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{t}{8}\right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda-1} \lesssim \|f\|_{L_{p,\lambda}} \left(sh\ \frac{x}{8}\right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.25}$$

Let now $8 < x < \infty$. Then as above, we obtain

$$\tau(x,r) \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{sh^{2\lambda}\ t dt}{(sh\ t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{x}{8}\right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda-1}$$

$$= \|f\|_{L_{p,\lambda}} \frac{x}{\left(sh\ \frac{x}{8}\right)^{1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha}} = \|f\|_{L_{p,\lambda}} \frac{x}{\left(2sh\ \frac{x}{16} ch\ \frac{x}{16}\right)^{1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha}}$$

$$= \|f\|_{L_{p,\lambda}} \frac{x}{\left(2sh\ \frac{x}{2^n}\right)^{2\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}x\right)}} \lesssim \cdots \lesssim \|f\|_{L_{p,\lambda}} \frac{x}{\left(2^n sh\ \frac{x}{2^{n+3}}\right)^{2^n\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}x\right)}}$$

$$\lesssim \|f\|_{L_{p,\lambda}} \frac{x}{\left(\frac{x}{8}\right)^{2^n\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha\right)}} \lesssim \|f\|_{L_{p,\lambda}} \frac{8}{\left(\frac{x}{8}\right)^{2^n\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha\right)-1}} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.26}$$

For the sufficiently large $n = n_0$

$$2^{n_0}\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha\right)-1 \geq 0 \Leftrightarrow \frac{1-2\lambda}{1+2\lambda}\alpha \leq 1-2\lambda-\frac{1}{2^{n_0}}$$
$$\frac{1-2\lambda}{1+2\lambda}\alpha < 1-2\lambda \Leftrightarrow \alpha < 2\lambda+1.$$

Combining the estimates (4.24)–(4.26), on (4.21) for $0 < x \leq r$ we obtain

$$|F_2(ch\ x) - a_2| \lesssim \|f\|_{L_{p,\lambda}}.$$

Hence we have

$$\left|A_{ch\ t}^{\lambda} F_2(ch\ x) - a_2\right| \leq A_{ch\ t}^{\lambda} |F_2(ch\ x) - a_2| \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.27}$$

From (4.27) it follows, that

$$\sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left|A_{ch\ t}^{\lambda} F_2(ch\ x - a_2)\right| sh^{2\lambda}\ t dt$$
$$\leq \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r A_{ch\ t}^{\lambda} |F_2(ch\ x) - a_2| sh^{2\lambda}\ t dt$$
$$\lesssim \|f\|_{L_{p,\lambda}} \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r sh^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.28}$$

Denote $a_f = a_1 + a_2 = \int_{\left(0,\max\left\{\frac{1}{4},\frac{r}{4}\right\}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ t dt$.

Finally from (4.20) and (4.28) we obtain

$$\sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A^\lambda_{ch\,t} \tilde{\Im}^\alpha_G f(ch\ x) - a_f \right| sh^{2\lambda}\ tdt$$
$$= \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A^\lambda_{ch\,t} F_1(ch\ x) - a_1 + A^\lambda_{ch\,t} F_2(ch\ x) - a_2 \right| sh^{2\lambda}\ tdt$$
$$\le \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A^\lambda_{ch\,t} F_1(ch\ x) - a_1 \right| sh^{2\lambda}\ tdt$$
$$+ \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A^\lambda_{ch\,t} F_2(ch\ x) - a_2 \right| sh^{2\lambda}\ tdt \lesssim \|f\|_{L_{p,\lambda}},$$

from this it follows that

$$\left\| \tilde{\Im}^\alpha_G f \right\|_{BMO_G[0,\infty)} \le 2 \sup_{x,r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A^\lambda_{ch\,t} \tilde{\Im}^\alpha_G f(ch\ x) - a_f \right| sh^{2\lambda}\ tdt \lesssim \|f\|_{L_{p,\lambda}}.$$

Theorem 4.2 is proved. □

Corollary 4.1. *Let $\alpha p = 2\lambda+1$, $1-2\lambda < \alpha < 2\lambda+1$, $f \in L_{p,\lambda}[0,\infty)$. If the integral $\Im^\alpha_G f$ is absolutely convergent, then $\Im^\alpha_G f \in BMO_G[0,\infty)$ and the inequality*

$$\left\| \Im^\alpha_G f \right\|_{BMO_G[0,\infty)} \lesssim \|f\|_{L_{p,\lambda}[0,\infty)}$$

is valid.

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