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Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in Lorentz spaces

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ABSTRACT

In this paper, we study the anisotropic maximal commutator M_b^d and the commutator of the anisotropic maximal operator $[b, M^d]$ in the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$. We obtain necessary and sufficient conditions for the boundedness of the operators M_b^d and $[b, M^d]$ on the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

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1. Introduction

Note to CE: Enunciations and equations followed as per section-wise. Please check.

The aim of this paper is to study anisotropic maximal commutator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ in the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$ when b belongs to the anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. We give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ on the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of the anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

Let \mathbb{R}^n be the n -dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$ and S^{n-1} denote the unit sphere on \mathbb{R}^n . For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x, r)$ denote the open ball centered at x of radius r and ${}^c\mathcal{E}(x, r)$ denote the set $\mathbb{R}^n \setminus \mathcal{E}(x, r)$. Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv (t^{d_1} x_1, \dots, t^{d_n} x_n)$. By Refs [1,2], the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x - y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus \mathbb{R}^n , endowed with the metric ρ , defines a homogeneous metric space (see Refs [1–3]). The balls with respect to ρ , centered at x of radius r , are just

the ellipsoids

$$\mathcal{E}_d(x, r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x, r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in \mathbb{R}^n . Let also $\Pi_d(x, r) = \{y \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelepiped, ${}^c\mathcal{E}_d(x, r) = \mathbb{R}^n \setminus \mathcal{E}_d(x, r)$ be the complement of $\mathcal{E}_d(0, r)$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_1(x, r) = B(x, r)$. Note that in the standard parabolic case $d = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behaviour are very informative, in particular, in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, Ref. [4]). For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the anisotropic fractional maximal operator M_α^d is defined by

$$M_\alpha^d f(x) = \sup_{r>0} |\mathcal{E}(x, r)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}(x, r)} |f(y)| dy, \quad 0 \leq \alpha < |d|,$$

where $\mathcal{E}(x, r)$ is the ball of radius r centred at $x \in \mathbb{R}^n$, ${}^c\mathcal{E}(x, r)$ is its complement and $|\mathcal{E}(x, r)|$ denotes the Lebesgue measure of $\mathcal{E}(x, r)$. If $\alpha = 0$, then $M = M_0$ is the well-known Hardy-Littlewood maximal operator.

The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance, Refs [4,5]).

The maximal commutator generated by $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the operator M^d is defined by

$$M_b^d(f)(x) = \sup_{r>0} |\mathcal{E}(x, r)|^{-1} \int_{\mathcal{E}(x, r)} |b(y) - b_{\mathcal{E}(x, r)}| |f(y)| dy.$$

The commutator generated by a suitable function b and the operator M is defined by

$$[b, M^d](f)(x) = b(x)M^d(f)(x) - M^d(bf)(x).$$

In 1978, Janson [6] gave some characterizations of the Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n) \equiv \dot{\Lambda}_{\beta, \mathbf{1}}(\mathbb{R}^n)$ via commutator $[b, T]$ and the author proved that $b \in \dot{\Lambda}_{\beta, d}(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\beta$, $1/p - 1/q = \beta/n$ and T is the classical singular integral operator (see also Ref. [7]). In 2017, Zhang [8] considered some new characterizations of the Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the boundedness of maximal commutator M_b and the (non-linear) commutator $[b, M]$ in Lebesgue spaces and Morrey spaces on Euclidean spaces.

In Refs [9–11] was obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and commutators of maximal operator $[b, M]$ on the Lorentz spaces $L^{p, q}$, see also Ref. [12].

Our main aim is to characterize the commutator functions b in anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$, involved in the boundedness on Lorentz spaces of the anisotropic maximal commutator M_b^d (Theorem 3.1) and the commutator of the maximal operator $[b, M^d]$ (Theorem 4.1).

The structure of the paper is as follows. In Section 2 we give some definitions and auxiliary results. In Section 3, we find necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M_b^d on $L^{p,r}(\mathbb{R}^n)$ spaces when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on $L^{p,r}(\mathbb{R}^n)$ spaces when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and some basic properties

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L^p spaces, also they appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on \mathbb{R}^n , then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\},$$

where

$$d_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad \forall s > 0.$$

Definition 2.1 ([13]): The Lorentz space $L^{p,r} \equiv L^{p,r}(\mathbb{R}^n)$, $0 < p, r \leq \infty$ is the collection of all measurable functions f on \mathbb{R}^n such the quantity

$$\|f\|_{L^{p,r}} := \|t^{\frac{1}{p} - \frac{1}{r}} f^*(t)\|_{L^r(0,\infty)} \quad (2.1)$$

is finite. Clearly $L^{p,p} \equiv L^p$ and $L^{p,\infty} \equiv WL^p$. The functional $\|\cdot\|_{L^{p,r}}$ is a norm if and only if either $1 \leq r \leq p$ or $p = r = \infty$.

Lemma 2.1 ([13]): Let $1 < p, p', r, r' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Suppose that $f \in L^{p,r}(\mathbb{R}^n)$ and $g \in L^{p',r'}(\mathbb{R}^n)$. Then

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq 2\|f\|_{L^{p,r}(\mathbb{R}^n)}\|g\|_{L^{p',r'}(\mathbb{R}^n)}.$$

The following result completely characterizes the boundedness of M^d on Lorentz spaces.

Lemma 2.2 ([13]): Let $1 \leq p, r \leq \infty$.

- (i) If $1 < p \leq \infty$, then the operator M^d is bounded on the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$.
- (ii) If $p = 1$, $1 \leq r \leq \infty$, then M^d is bounded on from $L^{1,r}(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$.

The following result completely characterizes the boundedness of M_α^d on Lorentz spaces.

Lemma 2.3 ([14]): Let $0 \leq \alpha < n$, $1 \leq p < \frac{|d|}{\alpha}$ and $p \leq q < \infty$.

- (i) If $1 < p < \frac{|d|}{\alpha}$, $1 \leq r \leq s \leq \infty$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$ is necessary and sufficient for the boundedness of the operator M_α^d from the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$.
- (ii) If $p = 1$, $1 \leq r \leq \infty$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{|d|}$ is necessary and sufficient for the boundedness of the operator M_α^d from the Lorentz spaces $L^{1,r}(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n) \equiv L^{q,\infty}(\mathbb{R}^n)$.

3. Anisotropic maximal commutator operator in Lorentz spaces

In this section, as an application of the theorems of the previous section, we consider the boundedness of an anisotropic maximal commutator M_b^d on the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$ when b belongs to an anisotropic Lipschitz space. We give some new characterizations of the anisotropic Lipschitz spaces are given. Such a characterization was given in Refs [8,15] for the boundedness of M_b^d on Lebesgue and Morrey spaces.

Definition 3.1: (1) Let $0 < \beta < 1$, we say a function b belongs to the anisotropic Lipschitz space $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C\rho(x - y)^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)}$.

- (2) Let $0 < \beta < 1$ and $1 \leq p < \infty$. A locally integrable function b is said to belong to the space $\text{Lip}_{\beta,p}(\mathbb{R}^n)$ if there exists a constant C such that

$$\sup_{\mathcal{E} \ni x} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}|^p dy \right)^{\frac{1}{p}} \leq C,$$

where the supremum is taken over every anisotropic ball $\mathcal{E} \subset \mathbb{R}^n$ containing x and $b_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} b(y) dy$. The least constant C satisfying the conditions above shall be denoted by $\|b\|_{\text{Lip}_{\beta,p}(\mathbb{R}^n)}$.

- (3) In (2), when $p = 1$, we have

$$\|b\|_{\text{Lip}_{\beta,1}(\mathbb{R}^n)} = \sup_{\mathcal{E} \ni x} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}| dy \right) := \|b\|_{\text{Lip}_{\beta,d}(\mathbb{R}^n)}.$$

To prove the theorems, we need auxiliary results. It is known that the Lipschitz space $\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ coincides with some Morrey-Campanato space (see Ref. [16] for example) and can be characterized by mean oscillation as follows, which is due to DeVore and Sharpley [17] and Janson et al. [16] (see also Ref. Paluszyński [18]).

Lemma 3.1 (see [16–18]): Let $0 < \beta < 1$, $1 \leq p < \infty$ and b the locally integrable function on \mathbb{R}^n , then

$$\|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} = \|b\|_{\text{Lip}_{\beta,d}(\mathbb{R}^n)} \approx \|b\|_{\text{Lip}_{\beta,p}(\mathbb{R}^n)}.$$

Lemma 3.2 ([15]): Let $0 < \beta < 1$ and $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$, then the following pointwise estimate holds

$$M_b^d f(x) \lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} M_{\beta}^d f(x).$$

Theorem 3.1: Let $0 < \beta < 1$, $b \in L_{loc}^1(\mathbb{R}^n)$, $1 < p < \frac{|d|}{\beta}$, $1 \leq r \leq s \leq \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|}$. The following assertions are equivalent

- (i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.
- (ii) The operator M_b^d is bounded from $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$.
- (iii) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|(b(\cdot) - b_{\mathcal{E}}) \chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}} \leq C. \quad (3.1)$$

- (iv) There exist a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|(b(\cdot) - b_{\mathcal{E}}) \chi_{\mathcal{E}}\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \leq C. \quad (3.2)$$

Proof: (i) \Rightarrow (ii). Suppose that $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. Combining Lemmas 2.2 and 3.2, we get

$$\begin{aligned} \|M_b^d f\|_{L^{q,s}(\mathbb{R}^n)} &\lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|M_{\beta}^d f\|_{L^{q,s}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L^{p,r}(\mathbb{R}^n)}, \end{aligned}$$

which implies that the operator M_b^d is bounded from $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$.

(ii) \Rightarrow (iii). Assume that the operator M_b^d is bounded from $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed ball. We consider $f = \chi_{\mathcal{E}}$. It is easy to compute that

$$\|\chi_{\mathcal{E}}\|_{L^{p,r}(\mathbb{R}^n)} \approx r^{\frac{|d|}{p}}. \quad (3.3)$$

On the other hand, for all $x \in \mathcal{E}$, we have

$$\begin{aligned} |b(x) - b_{\mathcal{E}}| &\leq \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(x) - b(y)| dy \\ &\leq M_b^d(\chi_{\mathcal{E}})(x) \lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} M_{\beta}^d(\chi_{\mathcal{E}})(x). \end{aligned} \quad (3.4)$$

Since $M_{b,\beta}$ is bounded from $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$, then by (3.3) and (3.4) we obtain

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|(b - b_{\mathcal{E}}) \chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}} &\leq \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|M_{\beta}^d(\chi_{\mathcal{E}})\|_{L^{q,s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}} \\ &\lesssim \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|\chi_{\mathcal{E}}\|_{L^{p,r}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}} = \frac{\|\chi_{\mathcal{E}}\|_{L^{p,r}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{p,r}(\mathbb{R}^n)}} = 1, \end{aligned} \quad (3.5)$$

which implies that (3.1) holds since the ball $\mathcal{E} \subset \mathbb{R}^n$ is arbitrary.

(iii) \Rightarrow (iv). Assume that (3.1) holds, we will prove (3.2). For any fixed anisotropic ball \mathcal{E} , by Lemma 2.1, (3.1) and (3.3), it is easy to see that

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{1+\frac{\beta}{|d|}}} \int_{\mathcal{E}} |\mathcal{E}(y) - b_B| dy &\lesssim \frac{1}{|\mathcal{E}|^{1+\frac{\beta}{|d|}}} \| (b - b_{\mathcal{E}}) \chi_{\mathcal{E}} \|_{L^{q,s}(\mathbb{R}^n)} \| \chi_{\mathcal{E}} \|_{L^{q',s'}} \\ &\lesssim \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\| (b - b_{\mathcal{E}}) \chi_{\mathcal{E}} \|_{L^{q,s}(\mathbb{R}^n)}}{\| \chi_{\mathcal{E}} \|_{L^{q,s}(\mathbb{R}^n)}} \\ &\lesssim 1. \end{aligned}$$

(iv) \Rightarrow (i). For any fixed anisotropic ball \mathcal{E} , we have

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{1+\frac{\beta}{|d|}}} \int_{\mathcal{E}} |\mathcal{E}(y) - b_{\mathcal{E}}| dy &= \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\| (b(\cdot) - b_{\mathcal{E}}) \chi_{\mathcal{E}} \|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \\ &\leq \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\| (b(\cdot) - b_{\mathcal{E}}) \chi_{\mathcal{E}} \|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \\ &\lesssim 1, \end{aligned}$$

which implies that $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

Thus the proof of Theorem 3.1 is finished. ■

4. Commutator of anisotropic maximal operator in Lorentz spaces

In this section, we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ from $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ spaces.

For a function b defined on \mathbb{R}^n , we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between M_b^d and $[b, M^d]$ are valid.

Let b be any non-negative locally integrable function. Then for all $f \in L_{loc}^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the following inequality is valid

$$\begin{aligned} |[b, M^d]f(x)| &= |b(x)M^d f(x) - M^d(bf)(x)| \\ &= |M^d(b(x)f)(x) - M^d(bf)(x)| \\ &\leq M^d(|b(x) - b|f)(x) = M_b^d f(x). \end{aligned} \tag{4.1}$$

If b is any locally integrable function on \mathbb{R}^n , then

$$|[b, M^d]f(x)| \leq M_b^d f(x) + 2b^-(x) M^d f(x), \quad x \in \mathbb{R}^n \tag{4.2}$$

holds for all $f \in L_{loc}^1(\mathbb{R}^n)$ (see, for example, Refs [19,20]).

Obviously, the operators M_b^d and $[b, M^d]$ are essentially different from each other because M_b^d is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

Let $\mathcal{E} = \mathcal{E}(x, r)$ be a fixed anisotropic ball. Denote by $M_b^d f$ the local fractional anisotropic maximal function of f

$$M_b^d f(x) := \sup_{\mathcal{E}' \ni x: \mathcal{E}' \subset \mathcal{E}} \frac{1}{|\mathcal{E}'|} \int_{\mathcal{E}'} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

Applying Theorem 3.1, we obtain the following result.

Theorem 4.1: *Let $0 < \beta < 1$, $b \in L_{\text{loc}}^1(\mathbb{R}^n)$, $1 < p < \frac{|d|}{\beta}$, $1 \leq r \leq s \leq \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|}$. The following assertions are equivalent.*

- (i) $b \in \dot{\Lambda}_{\beta, d}(\mathbb{R}^n)$ and $b \geq 0$.
- (ii) The operator $[b, M^d]$ is bounded from $L^{p, r}(\mathbb{R}^n)$ to $L^{q, s}(\mathbb{R}^n)$.
- (iii) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|(b(\cdot) - M_b^d(b)(\cdot)) \chi_{\mathcal{E}}\|_{L^{q, s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q, s}(\mathbb{R}^n)}} \leq C. \quad (4.3)$$

- (iv) There exists a constant $C > 0$ such that

$$\sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|(b(\cdot) - M_b^d(b)(\cdot)) \chi_{\mathcal{E}}\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \leq C. \quad (4.4)$$

Proof: (i) \Rightarrow (ii). Suppose that $b \in \dot{\Lambda}_{\beta, d}(\mathbb{R}^n)$ and $b \geq 0$. Combining Lemma 2.2 and Theorem 3.1, and inequality (4.1), we get

$$\begin{aligned} \|[b, M^d]f\|_{L^{q, s}(\mathbb{R}^n)} &\leq \|M_b^d f\|_{L^{q, s}(\mathbb{R}^n)} \lesssim \|b\|_{\dot{\Lambda}_{\beta, d}(\mathbb{R}^n)} \|M_b^d f\|_{L^{q, s}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\dot{\Lambda}_{\beta, d}(\mathbb{R}^n)} \|f\|_{L^{p, r}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we obtain that $[b, M^d]$ is bounded from $L^{p, r}(\mathbb{R}^n)$ to $L^{q, s}(\mathbb{R}^n)$.

- (ii) \Rightarrow (iii). Assume that $[b, M^d]$ is bounded from $L^{p, r}(\mathbb{R}^n)$ to $L^{q, s}(\mathbb{R}^n)$.

For any fixed anisotropic ball \mathcal{E} and $x \in \mathcal{E}$, we have

$$b(x) - M_b^d(b)(x) = b(x)M(\chi_{\mathcal{E}})(x) - M(b\chi_{\mathcal{E}})(x) = [b, M^d](\chi_{\mathcal{E}})(x).$$

Assume that $[b, M^d]$ is bounded from $L^{p, r}(\mathbb{R}^n)$ to $L^{p, s}(\mathbb{R}^n)$, then we get

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|(b - M_b^d(b)) \chi_{\mathcal{E}}\|_{L^{q, s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q, s}(\mathbb{R}^n)}} &= \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|[b, M^d](\chi_{\mathcal{E}})\|_{L^{q, s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q, s}(\mathbb{R}^n)}} \lesssim \frac{1}{|\mathcal{E}|^{\frac{\beta}{|d|}}} \frac{\|\chi_{\mathcal{E}}\|_{L^{p, r}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q, s}(\mathbb{R}^n)}} \\ &\lesssim \frac{\|\chi_{\mathcal{E}}\|_{L^{p, r}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{p, r}(\mathbb{R}^n)}} = 1, \end{aligned}$$

which implies that (4.3) holds since the anisotropic ball $\mathcal{E} \subset \mathbb{R}^n$ is arbitrary.

(iii) \Rightarrow (iv). Assume that (4.3) holds, then for any fixed anisotropic ball \mathcal{E} , by Lemma 2.1, we conclude that

$$\begin{aligned} & \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|\mathbf{d}|}}} \frac{\|(b(\cdot) - M_b^d(b)(\cdot)) \chi_{\mathcal{E}}\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \\ & \lesssim \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{\frac{\beta}{|\mathbf{d}|}}} \frac{\|(b(\cdot) - M_b^d(b)(\cdot)) \chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)} \|\chi_{\mathcal{E}}\|_{L^{q',s'}}}{|\mathcal{E}|} \\ & \lesssim \frac{1}{|\mathcal{E}|^{\frac{\beta}{|\mathbf{d}|}}} \frac{\|(b - M_b^d(b)) \chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^{q,s}(\mathbb{R}^n)}} \\ & \lesssim 1. \end{aligned}$$

(iv) \Rightarrow (i). Assume that (4.4) holds, we will prove $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ and $b \geq 0$. Denote by

$$E := \{x \in \mathcal{E} : b(x) \leq b_{\mathcal{E}}\}, \quad F := \{x \in \mathcal{E} : b(x) > b_{\mathcal{E}}\}.$$

Since

$$\int_E |b(t) - b_{\mathcal{E}}| dt = \int_F |\mathcal{E}(t) - b_{\mathcal{E}}| dt,$$

in view of the inequality $b(x) \leq b_{\mathcal{E}} \leq M_b^d(b)(x)$, $x \in \mathcal{E}$, we get

$$\begin{aligned} \frac{1}{|\mathcal{E}|^{1+\frac{\beta}{|\mathbf{d}|}}} \int_{\mathcal{E}} |b(t) - b_{\mathcal{E}}| dt &= \frac{2}{|\mathcal{E}|^{1+\frac{\beta}{|\mathbf{d}|}}} \int_E |b(t) - b_{\mathcal{E}}| dt \\ &\leq \frac{2}{|\mathcal{E}|^{1+\frac{\beta}{|\mathbf{d}|}}} \int_E |b(t) - M_b^d(b)(t)| dt \\ &\leq \frac{2}{|\mathcal{E}|^{1+\frac{\beta}{|\mathbf{d}|}}} \int_{\mathcal{E}} |b(t) - M_b^d(b)(t)| dt \lesssim 1. \end{aligned}$$

Consequently, $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

In order to show that $b \geq 0$, note that $M_b^d(b) \geq |\mathcal{E}|$. Hence

$$0 \leq b^- = |\mathcal{E}| - b^+ \leq M_b^d(b) - b^+ + b^- = M_b^d(b) - b.$$

Thus

$$\begin{aligned} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} b^-(x) dx &\leq \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |M_b^d(b)(x) - b(x)| dx \\ &= \frac{\|(b(\cdot) - M_b^d(b)(\cdot)) \chi_{\mathcal{E}}\|_{L^1(\mathbb{R}^n)}}{|\mathcal{E}|} \\ &\lesssim |\mathcal{E}|^{\frac{\beta}{|\mathbf{d}|}}, \end{aligned}$$

and by the Lebesgue Differentiation theorem, we get that

$$0 \leq b^-(x) = \lim_{|\mathcal{E}| \rightarrow 0} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} b^-(y) dy = 0 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus the proof of Theorem 4.1 is completed. ■

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