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The boundedness of Hilbert transform in the local Morrey–Lorentz spaces

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ABSTRACT
In this paper, we investigate the boundedness of the Hilbert transform H in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}$, $q/(q+\lambda) \le p \le q/\lambda$, $1 \le q \le \infty$. We prove that the operator H is bounded in $M^{\mathrm{loc}}_{p,q;\lambda}$ under the condition $q/(q+\lambda) < p < q/\lambda$, $1 \le q < \infty$. In the limiting case $p = q/(q+\lambda)$, $1 < q < \infty$, we prove that the operator H is bounded from the space $M^{\mathrm{loc}}_{p,q;\lambda}$ to the weak local Morrey–Lorentz space $WM^{\mathrm{loc}}_{p,q;\lambda}$. Also we show that for the limiting case $p = q/\lambda$, $0 < q \le \infty$, the modified Hilbert transform $\tilde{H}$ is bounded from the space $M^{\mathrm{loc}}_{p,q;\lambda}$ to the bounded mean oscillation space.

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1. Introduction

The local Morrey–Lorentz spaces denoted by $M^{\mathrm{loc}}_{p,q;\lambda}$, $(\mathbb{R}^n)$ are new class of functions which were introduced by Aykol et al.[1] These spaces are a very natural generalization of the Lorentz spaces such that $M^{\mathrm{loc}}_{p,q;0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$.

Recall that the local Morrey-type space $LM_{p\theta,w}$

$$\|f\|_{LM_{p\theta,w}} = \|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)}$$

is introduced by V.S. Guliyev in his doctoral thesis in 1994 (see [2]). In [2] the sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in local Morrey-type space $LM_{p\theta,w}$ are given. In a series of papers by V. Burenkov, H.V. Guliyev, V.S. Guliyev, etc. (see, for example [3]) some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,w}$ are obtained.

Let $0 < p, q \le \infty$ and $0 \le \lambda \le 1$. We define the local Morrey–Lorentz spaces as the spaces of all measurable functions with finite quasinorm

$$\|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}} := \sup_{r>0} r^{-\lambda/q}\|t^{1/p-1/q}f^*(t)\|_{L_q(0,r)}.$$

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Aykol et al. [1] proved that in the case $\lambda < 0$ or $\lambda > 1$ the space $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ is trivial, and in the limiting case $\lambda = 1$ the space $M^{\rm loc}_{p,q;1}(\mathbb{R}^n)$ is the classical Lorentz space $\Lambda_{\infty,t^{1/p-1/q}}(\mathbb{R}^n)$. For $0 < q \le p < \infty$ and $0 < \lambda \le q/p$, the local Morrey–Lorentz spaces $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ are equal to weak Lebesgue spaces $WL_{1/p-\lambda/q}(\mathbb{R}^n)$. In [1] the basic properties of $M^{\rm loc}_{p,q;\lambda}(\mathbb{R}^n)$ were given and the boundedness of the maximal operator was proved.

Let f be a locally integrable function on $\mathbb{R}$. The Hilbert transform Hf of f is defined by the principal-value integral

$$Hf(x) = pv\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,{\rm d}y$$

provided it exists almost everywhere. The modified Hilbert transform $\tilde{H}$ is defined as

$$\tilde{H}f(x) := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\left[\frac{1}{x-y}+\frac{\chi(y)}{y}\right]f(y)\,{\rm d}y,$$

where $\chi(y)$ is the characteristic function of $|y| > 1$. The boundedness of the modified Hilbert transform $\tilde{H}$ from the space L_∞ to the bounded mean oscillation (BMO) space was proved by Muckenhoupt and Wheeden.[4]

The aim of this paper is to prove the boundedness of the Hilbert transform H in the local Morrey–Lorentz spaces $M^{\rm loc}_{p,q;\lambda} = M^{\rm loc}_{p,q;\lambda}(\mathbb{R})$. We prove that the operator H is bounded in $M^{\rm loc}_{p,q;\lambda}$ under the condition $q/(q+\lambda) < p < q/\lambda$, $1 \le q < \infty$. In the limiting case $p = q/(q+\lambda)$, $1 < q < \infty$, we prove that the operator H is bounded from the space $M^{\rm loc}_{p,q;\lambda}$ to the weak local Morrey–Lorentz space $WM^{\rm loc}_{p,q;\lambda}$. Also we show that for the limiting case $p = q/\lambda$, $0 < q \le \infty$, the modified Hilbert transform $\tilde{H}$ is bounded from the space $M^{\rm loc}_{p,q;\lambda}$ to the BMO space. As a result of these we give the boundedness of the maximal Hilbert operator $\mathcal{H}$ in the local Morrey–Lorentz space $M^{\rm loc}_{p,q;\lambda}$.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence. If $p \in [1,\infty]$, the conjugate number p' is defined by $pp' = p + p'$. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities.

2. Preliminaries

We shall use the following notation. For a finite interval $E \subset \mathbb{R}$ and $0 < p \le \infty$, $L_p(E)$ is the standard Lebesgue space of all functions f Lebesgue measurable on E for which

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p\,{\rm d}y\right)^{1/p} < \infty,$$

if $0 < p < \infty$ and

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \ge \alpha\}| > 0\},$$

if $p = \infty$. Also, for a finite interval $E \subset \mathbb{R}$, $L_p^{\rm loc}(E)$ is the set of all functions f such that $f \in L_p(K)$ for any compact $K \subset E$. If $E = \mathbb{R}$, then, for brevity, we write L_p for $L_p(\mathbb{R})$ and

$L_p^{\rm loc}$ for $L_p^{\rm loc}(\mathbb{R})$. The same convention refers to the case of weak Lebesgue spaces $WL_p(E)$, the space of all functions f Lebesgue measurable on E for which

$$\|f\|_{WL_p(E)} := \sup_{0<t\le|E|} t^{1/p} f^*(t), \quad 1 \le p < \infty.$$

Here $|E|$ is the Lebesgue measure of E, and f^* denotes the right continuous non-increasing rearrangement of f

$$f^*(t) := \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \quad \forall t \in (0,\infty),$$

where

$$\mu_f(\lambda) := |\{y \in \mathbb{R} : |f(y)| > \lambda\}|$$

is the distribution function of the function f.

Now we recall definitions of Morrey spaces, Lorentz spaces and local Morrey–Lorentz spaces.

Morrey spaces were introduced by Morrey [5] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier–Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory. Morrey spaces were widely studied during last decades, including the study of classical operators of harmonic analysis maximal, singular and potential operators (see [3, 6–8]).

Definition 2.1 ([5]): *We denote by $L_{p,\lambda} \equiv L_{p,\lambda}(\mathbb{R})$ Morrey space for $0 \le \lambda \le 1$, $0 \le p < \infty$, $f \in L_{p,\lambda}$ if $f \in L_p^{\rm loc}$ and*

$$\|f\|_{L_{p,\lambda}} := \sup_{x\in\mathbb{R}, r>0} r^{-\lambda/p} \|f\|_{L_{p(I(x,r))}} < \infty,$$

where $I = I(x,r) = \{y : x - r < y < x + r\}$.

If $\lambda = 0$, then $L_{p,0} = L_p$, if $\lambda = 1$, then $L_{p,1} = L_\infty$, if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}$.

Also by $WL_{p,\lambda} \equiv WL_{p,\lambda}(\mathbb{R})$ we denote the weak Morrey space of all functions $f \in WL_p^{\rm loc}$ for which

$$\|f\|_{WL_{p,\lambda}} := \sup_{x\in\mathbb{R}, r>0} r^{-\lambda/p} \|f\|_{WL_{p(I(x,r))}} < \infty,$$

where $WL_p(I(x,r))$ denotes the weak L_p space of measurable functions f for which

$$\|f\|_{WL_p(I(x,r))} \equiv \|f\chi_{(I(x,r))}\|_{WL_p(\mathbb{R})} = \sup_{\tau>0} \tau |\{y \in I(x,r) : |f(y)| > \tau\}|^{1/p}.$$

Lorentz spaces are introduced by Lorentz in 1950. Lorentz spaces, which are Banach spaces and generalizations of the more familiar L_p spaces, appear to be useful in the general interpolation theory.

Definition 2.2: *The Lorentz space* $L_{p,q} \equiv L_{p,q}(\mathbb{R})$, $0 < p, q \le \infty$ *is the collection of all measurable functions f on* $\mathbb{R}$ *such that the quantity*

$$\|f\|_{L_{p,q}} := \|t^{1/p-1/q} f^*(t)\|_{L_q(0,\infty)} \tag{2.1}$$

is finite. The functional $\|\cdot\|_{L_{p,q}}$ *is a norm if and only if either* $1 \le q \le p$ *or* $p = q = \infty$.

More information about Lorentz spaces can be found in [9].

Definition 2.3 ([1]): *Let* $0 < p, q \le \infty$ *and* $0 \le \lambda \le 1$. *We denote by* $M^{\text{loc}}_{p,q;\lambda} \equiv M^{\text{loc}}_{p,q;\lambda}(\mathbb{R})$ *the local Morrey–Lorentz spaces, the spaces of all measurable functions with finite quasinorm*

$$\|f\|_{M^{\text{loc}}_{p,q;\lambda}} := \sup_{r>0} r^{-\lambda/q} \|s^{1/p-1/q} f^*(s)\|_{L_q(0,r)}.$$

If $\lambda < 0$ *or* $\lambda > 1$, *then* $M^{\text{loc}}_{p,q;\lambda} = \Theta$, *where* Θ *is the set of all functions equivalent to 0 on* $\mathbb{R}$. *Also* $M^{\text{loc}}_{p,q;0} = L_{p,q}$ *and* $M^{\text{loc}}_{p,p;\lambda} \equiv M^{\text{loc}}_{p;\lambda}$. *In the limiting case* $\lambda = 1$ *the space* $M^{\text{loc}}_{p,q;1}$ *is the classical Lorentz space* $\Lambda_{\infty, t^{1/p-1/q}}$.

We denote by $WM^{\text{loc}}_{p,q;\lambda} \equiv WM^{\text{loc}}_{p,q;\lambda}(\mathbb{R})$ *the weak local Morrey–Lorentz spaces of all measurable functions with finite quasinorm*

$$\|f\|_{WM^{\text{loc}}_{p,q;\lambda}} := \sup_{r>0} r^{-\lambda/q} \|s^{1/p-1/q} f^*(s)\|_{WL_q(0,r)}.$$

Lemma 2.4 ([1]): *Let* $0 < q \le p < \infty$, $1/s = 1/p - \lambda/q$ *and* $0 < \lambda \le q/p$. *Then*

$$\left(\frac{q}{p}\right)^{-1/q} \|f\|_{WL_s} \le \|f\|_{M^{\text{loc}}_{p,q;\lambda}} \le \lambda^{-1/q} \|f\|_{WL_s}.$$

In particular, $\|f\|_{WL_\infty} = \|f\|_{M^{\text{loc}}_{q/\lambda,q;\lambda}}$.

Definition 2.5 ([10]): *Let* $f \in L^{\text{loc}}_1(\mathbb{R})$. *The space of functions with BMO,* $\text{BMO} \equiv \text{BMO}(\mathbb{R})$, *consists of those functions* f *for which*

$$\|f\|_{\text{BMO}} := \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f - f_I| \, dx$$

is finite, where the supremum is taken over open intervals $I \subset \mathbb{R}$ *and*

$$f_I = \frac{1}{|I|} \int_I f \, dx.$$

We need the following two definitions about Hardy operators which are used in the proof of Theorem 3.1. These operators are very important in analysis and have been widely studied.

Definition 2.6 ([11]): *Let* f *be a measurable function on* $(0, \infty)$ *and* β *be a real number. The weighted Hardy operators* A_β *and* $\mathcal{A}_\beta$ *with power weights acting on* f *are defined by*

$$A_\beta f(t) = t^{\beta-1} \int_0^t \frac{f(s)}{s^\beta} \, ds, \quad \mathcal{A}_\beta f(t) = t^\beta \int_t^\infty \frac{f(s)}{s^{\beta+1}} \, ds. \tag{2.2}$$

Definition 2.7 ([12]): *Let f be a measurable function on* $(0,\infty)$ *and* η *be a real number. The Hardy operators* P_η *and* $\mathcal{P}_\eta$ *are defined by*

$$P_\eta f(t) = t^{-\eta}\int_0^t f(s)\,\mathrm{d}s, \quad \mathcal{P}_\eta f(t) = t^{-\eta}\int_t^\infty f(s)\,\mathrm{d}s. \tag{2.3}$$

The following theorem was proved in [12] by Andersen and Muckenhoupt.

Theorem A ([12]): *Suppose* $1 \le p \le q < \infty$, u *and* v *are nonnegative weight functions. Then the following* (p,q) *weak type inequalities are valid.*

(i) *For* $\eta > 0$ *if*

$$B(\eta,a) = \sup_{\xi>0}\left(\int_\xi^\infty (\xi/x)^a (u(x)/x^{\eta q})\,\mathrm{d}x\right)^{1/q}\left(\int_0^\xi v(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{1/p'} \tag{2.4}$$

is finite for some $a > 0$, *then* (u,v) *is a* (p,q) *weak type weight pair for* P_η

$$\left(\int_{\{t\in(0,\infty):|P_\eta f(t)|>\tau\}} u(t)\,\mathrm{d}t\right)^{1/q} \le C\tau^{-1}\left(\int_0^\infty |f(t)|^p v(t)\,\mathrm{d}t\right)^{1/p}. \tag{2.5}$$

(ii) *For* $\eta \ge 0$ *if*

$$B(\eta) = \sup_{\xi>0}\xi^{-\eta}\left(\int_0^\xi u(x)\,\mathrm{d}x\right)^{1/q}\left(\int_\xi^\infty v(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{1/p'} \tag{2.6}$$

is finite, then (u,v) *is a* (p,q) *weak type weight pair for* $\mathcal{P}_\eta$

$$\left(\int_{\{t\in(0,\infty):|\mathcal{P}_\eta f(t)|>\tau\}} u(t)\,\mathrm{d}t\right)^{1/q} \le C\tau^{-1}\left(\int_0^\infty |f(t)|^p v(t)\,\mathrm{d}t\right)^{1/p}. \tag{2.7}$$

Note that, taking $u(t) = v(t) = \chi_{(0,r)}(t)$ in the inequalities (2.5) and (2.7) we get the following inequalities:

$$\left(\int_{\{t\in(0,r):|P_\eta f(t)|>\tau\}} \mathrm{d}t\right)^{1/q} \le C\tau^{-1}\left(\int_0^r |f(t)|^p\,\mathrm{d}t\right)^{1/p}, \tag{2.8}$$

$$\left(\int_{\{t\in(0,r):|\mathcal{P}_\eta f(t)|>\tau\}} \mathrm{d}t\right)^{1/q} \le C\tau^{-1}\left(\int_0^r |f(t)|^p\,\mathrm{d}t\right)^{1/p}. \tag{2.9}$$

We will use the following two lemmas to obtain the boundedness of the Hilbert transform H and maximal Hilbert transform $\mathcal{H}$ in the local Morrey–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R})$. In this lemma the boundedness of the operators A_β and $\mathcal{A}_\beta$ in Morrey spaces are given.

Lemma 2.8: (i) *Let* $1 \le q < \infty$, $0 \le \lambda < 1$ *and* $\beta < \lambda/q + 1/q'$. *Then the operator* A_β *is bounded in the Morrey space* $L_{q,\lambda}(0,\infty)$.

(ii) *Let* $1 < q < \infty, 0 \le \lambda < 1$ *and* $\beta = \lambda/q + 1/q'$. *Then the operator* A_β *is bounded from the Morrey space* $L_{q,\lambda}(0,\infty)$ *to the weak Morrey space* $WL_{q,\lambda}(0,\infty)$.

Proof: (i) The proof of the first part of the lemma was given by Samko.[11]

(ii) Let $\beta = \lambda/q + 1/q'$

$$\begin{aligned}\|A_\beta f\|_{WL_{q,\lambda}(0,\infty)} &= \sup_{r>0} r^{-\lambda/q} \|\chi_{(0,r)} A_\beta f(t)\|_{WL_q(0,\infty)} \\ &= \sup_{r>0} r^{-\lambda/q} \sup_{\tau>0} \tau \left(\int_{\{t\in(0,r):|A_\beta f(t)|>\tau\}} \mathrm{d}t \right)^{1/q} \\ &= \sup_{r>0} r^{-\lambda/q} \sup_{\tau>0} \tau \left(\int_{\left\{t\in(0,r):\left|t^{\beta-1}\int_0^t \frac{f(s)}{s^\beta}\,\mathrm{d}s\right|>\tau\right\}} \mathrm{d}t \right)^{1/q}.\end{aligned}$$

In (2.4), if we take $p = q, \eta = 1-\beta > 0, u(t) = \chi_{(0,r)}(t), v(t) = \chi_{(0,r)}(t)t^{\beta q}$, then we get the constant $B(\eta, a)$ as follows:

$$\begin{aligned}B(\eta,a) &= \sup_{\xi>0} \xi^{a/q} \left(\int_\xi^\infty \chi_{(0,r)}(s) s^{-a} s^{-q(1-\beta)}\,\mathrm{d}s \right)^{1/q} \\ &\quad \left(\int_0^\xi (\chi_{(0,r)}(s) s^{\beta q})^{-1/(q-1)}\,\mathrm{d}s \right)^{1/q'} \\ &= \sup_{0<\xi<r} \xi^{a/q} \left(\int_\xi^r s^{-a}\, s^{-q(1-\beta)}\,\mathrm{d}s \right)^{1/q} \left(\int_0^\xi \chi_{(0,r)}(s) s^{-\beta q/(q-1)}\,\mathrm{d}s \right)^{1/q'} \\ &\le C \sup_{\xi>0} \xi^{a/q} \xi^{-a/q+\beta-1+1/q-\beta+1-1/q} < \infty\end{aligned}$$

for all $a > 0$. Therefore we get

$$\begin{aligned}&\sup_{r>0} r^{-\lambda/q} \sup_{\tau>0} \tau \left(\int_{\left\{t\in(0,r):\left|t^{\beta-1}\int_0^t \frac{f(s)}{s^\beta}\,\mathrm{d}s\right|>\tau\right\}} \mathrm{d}t \right)^{1/q} \\ &\le C \sup_{r>0} r^{-\lambda/q} \left(\int_0^\infty \chi_{(0,r)}(t) \left(\frac{f(t)}{t^\beta} \right)^q t^{\beta q}\,\mathrm{d}t \right)^{1/q} \\ &= C \sup_{r>0} r^{-\lambda/q} \left(\int_0^r f(t)^q\,\mathrm{d}t \right)^{1/q} \\ &= C\|f\|_{L_{q,\lambda}(0,\infty)},\end{aligned}$$

which completes the proof. ■

Lemma 2.9: (i) *Let* $1 \le q < \infty, 0 \le \lambda < 1$ *and* $\beta > \lambda/q - 1/q$. *Then the operator* $\mathcal{A}_\beta$ *is bounded in the Morrey space* $L_{q,\lambda}(0,\infty)$.

(ii) *Let* $1 < q < \infty, 0 \le \lambda < 1$ *and* $\beta = \lambda/q - 1/q$. *Then the operator* $\mathcal{A}_\beta$ *is bounded from the Morrey space* $L_{q,\lambda}(0,\infty)$ *to the weak Morrey space* $WL_{q,\lambda}(0,\infty)$.

Proof: (i) The proof of the first part of the lemma was given by Samko.[11]
(ii) Let $\beta = \lambda/q - 1/q$. We will make the proof by using the similar methods as in the proof of the boundedness of A_β

$$\|\mathcal{A}_\beta f\|_{WL_{q,\lambda}(0,\infty)} = \sup_{r>0} r^{-\lambda/q} \left\|\chi_{(0,r)}\mathcal{A}_\beta f(t)\right\|_{WL_q(0,\infty)}$$
$$= \sup_{r>0} r^{-\lambda/q} \sup_{\tau>0} \tau \left(\int_{\{t\in(0,r):|\mathcal{A}_\beta f(t)|>\tau\}} \mathrm{d}t\right)^{1/q}$$
$$= \sup_{r>0} r^{-\lambda/q} \sup_{\tau>0} \tau \left(\int_{\left\{t\in(0,r):\left|t^\beta \int_t^\infty \frac{f(s)}{s^{\beta+1}}\,\mathrm{d}s\right|>\tau\right\}} \mathrm{d}t\right)^{1/q}.$$

In (2.6), if we take $p = q$, $\eta = -\beta > 0$, $u(t) = \chi_{(0,r)}(t)$ and $v(t) = \chi_{(0,r)}(t)t^{q(\beta+1)}$, then we get the constant $B(\eta)$ as the following:

$$B(\eta) = \sup_{\xi>0} \xi^\beta \left(\int_0^\xi \chi_{(0,r)}(s)\,\mathrm{d}s\right)^{1/q} \left(\int_\xi^\infty (\chi_{(0,r)}(s)s^{q(\beta+1)})^{-1/(q-1)}\,\mathrm{d}s\right)^{1/q'}$$
$$\leq C \sup_{0<\xi<r} \xi^{\beta+\frac{1}{q}-\beta-1+1-\frac{1}{q}} < \infty.$$

Therefore we get

$$\sup_{r>0} r^{-\lambda/q} \sup_{\tau>0} \tau \left(\int_{\{t\in(0,r):|t^\beta \int_t^\infty f(s)/s^{\beta+1}\,\mathrm{d}s|>\tau\}} \mathrm{d}t\right)^{1/q}$$
$$\leq C \sup_{r>0} r^{-\lambda/q} \left(\int_0^\infty \chi_{(0,r)}(t) \left(\frac{f(t)}{t^{\beta+1}}\right)^q t^{q(\beta+1)}\,\mathrm{d}t\right)^{1/q}$$
$$= C \sup_{r>0} r^{-\lambda/q} \left(\int_0^r f(t)^q\,\mathrm{d}t\right)^{1/q}$$
$$= C\|f\|_{L_{q,\lambda}(0,\infty)},$$

which completes the proof.

■

3. Boundedness of the Hilbert transform on the spaces $M^{\mathrm{loc}}_{p,q;\lambda}$

In this section, we prove the boundedness of the Hilbert transform H and maximal Hilbert transform $\mathcal{H}$ in the local Morre–Lorentz spaces $M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R})$.

It is well known that Hf exist almost everywhere whenever f is a step function. The almost everywhere existence of the limit (of certain integral averages) was known for dense subset of L_1 and the result was extended to all of L_1 by establishing control over the corresponding maximal operator. For the Hilbert transform, the dense subset of L_1 consists of the step functions, and in order to extend to all of L_1 the almost everywhere existence of the limit of

$$H_\varepsilon f(x) = \frac{1}{\pi}\int_{\{y\in\mathbb{R}:|x-y|>\varepsilon\}} \frac{f(y)}{x-y}\,\mathrm{d}y, \quad x \in \mathbb{R}$$

as $\varepsilon \to 0$, we need to consider the maximal Hilbert transform $\mathcal{H}f$ of f

$$\mathcal{H}f(x) = \sup_{\varepsilon>0} |H_\varepsilon f(x)|, \quad x \in \mathbb{R}.$$

For each measurable function f on $(0,\infty)$ and each $t > 0$, let

$$\begin{aligned}(Sf)(t) &= \int_0^\infty \min(1, \frac{s}{t}) f(s)\frac{\mathrm{d}s}{s} \\ &= \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s + \int_t^\infty f(s)\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.1}$$

It is clear that S is linear. For the aim, its importance based on the fact that it dominates the maximal Hilbert transform.

Theorem B ([9]): *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R})$ and suppose*

$$(Sf^*)(1) = \int_0^1 f^*(s)\,\mathrm{d}s + \int_1^\infty f^*(s)\frac{\mathrm{d}s}{s} < \infty. \tag{3.2}$$

Then

$$(\mathcal{H}f)^*(t) \le CS(f^*)(t), \quad 0 < t < \infty, \tag{3.3}$$

where C *is a constant independent of* f *and* t.

Theorem C ([9]): *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R})$ and* f *satisfies* (3.2). *Then the Hilbert transform $Hf(x)$, $x \in \mathbb{R}$ exists almost everywhere. Furthermore*

$$(Hf)^*(t) \le CS(f^*)(t), \quad 0 < t < \infty, \tag{3.4}$$

where C *is a constant independent of f and t.*

Remark 1: Note that, the weak-type inequality (3.3) is due to Bennett and Rudnick,[13] the integrated form (3.4) was known previously to O'Neil and Weiss [14] and Calderon.[15]

The following is a well-known theorem about Hilbert transform.

Theorem D ([4,16,17]): *Let $0 < q \le \infty$ and $1 \le p \le \infty$.*

(i) *If $1 < p < \infty$ $1 \le q \le \infty$, then the operator* H *is bounded in the Lorentz space $L_{p,q}$.*
(ii) *If $p = 1$ and $0 < q \le 1$, then the operator* H *is bounded from $L_{1,q}$ to the space WL_1.*
(iii) *If $p = \infty$, then the modified Hilbert operator $\tilde{H}$ is bounded from L_∞ to BMO.*

The following theorem is the main result of our paper, in which we get the analogue of Theorem D for the boundedness of the Hilbert transform in the local Morrey–Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}$.

Theorem 3.1: *Suppose that $f \in M^{\text{loc}}_{p,q;\lambda}(\mathbb{R})$ satisfies Equation* (3.2), $0 < q \le \infty$, $0 \le \lambda < 1$ *and* $q/(q+\lambda) \le p \le q/\lambda$. *Then the Hilbert transform* Hf *exists almost everywhere. Furthermore,*

(i) *If* $q/(q+\lambda) < p < q/\lambda$, $1 \le q < \infty$, *then the operator* H *is bounded in the local Morrey–Lorentz space* $M^{\text{loc}}_{p,q;\lambda}$.
(ii) *If* $p = q/(q+\lambda)$, $1 < q < \infty$, *then the operator* H *is bounded from* $M^{\text{loc}}_{p,q;\lambda}$ *to the weak space* $WM^{\text{loc}}_{p,q;\lambda}$.
(iii) *If* $p = q/\lambda$, $0 < q \le \infty$, *then the modified Hilbert operator* $\tilde{\text{H}}$ *is bounded from* $M^{\text{loc}}_{p,q;\lambda}$ *to BMO.*

Proof: Since f satisfies (3.2), from Theorem C the Hilbert transform $Hf(x)$, $x \in \mathbb{R}$ exists almost everywhere.

(i) Suppose that $1 \le q < \infty$, $0 \le \lambda < 1$, $q/(q+\lambda) < p < q/\lambda$ and $f \in M^{\text{loc}}_{p,q;\lambda}$.

From the definition of norm in local Morrey–Lorentz spaces and by using the inequality (3.4) we get

$$\begin{aligned}
\|Hf\|_{M^{\text{loc}}_{p,q,\lambda}} &= \sup_{r>0} r^{-\lambda/q} \|t^{1/p-1/q}(Hf)^*(t)\|_{L_q(0,r)} \\
&\le C \sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q}\left(\frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty \frac{f^*(s)}{s}\,\mathrm{d}s\right)\right\|_{L_q(0,r)} \\
&\le C \left(\sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q-1}\int_0^t f^*(s)\,\mathrm{d}s \right\|_{L_q(0,r)} \right. \\
&\quad \left. + \sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q}\int_t^\infty \frac{f^*(s)}{s}\,\mathrm{d}s \right\|_{L_q(0,r)} \right) \\
&= I_1 + I_2.
\end{aligned}$$

Let us estimate I_1

$$\begin{aligned}
I_1 &= C \sup_{r>0} r^{-\lambda/q} \|t^{1/p-1/q-1}\int_0^t f^*(s)\,\mathrm{d}s\|_{L_q(0,r)} \\
&= C\|A_{(1/p-1/q)}g\|_{L_{q,\lambda}(0,\infty)},
\end{aligned}$$

where $g(t) = t^{1/p-1/q}f^*(t)$. Since $1/p - \lambda/q < 1$, for $\beta = 1/p - 1/q$ the inequality $\beta < \lambda/q + 1/q'$ holds. From Lemma 2.8 (i) the operator A_β is bounded in the Morrey spaces $L_{q,\lambda}(0,\infty)$. Then,

$$\begin{aligned}
I_1 &\lesssim \|A_{(1/p-1/q)}g\|_{L_{q,\lambda}(0,\infty)} \lesssim \|g\|_{L_{q,\lambda}(0,\infty)} \\
&= \sup_{r>0} r^{-\lambda/q} \|t^{1/p-1/q}f^*(t)\|_{L_q(0,r)} = \|f\|_{M^{\text{loc}}_{p,q;\lambda}}.
\end{aligned} \tag{3.5}$$

Now we consider I_2

$$I_2 = C \sup_{r>0} r^{-\lambda/q} \left\| t^{1/p-1/q} \int_t^\infty \frac{f^*(s)}{s}\, ds \right\|_{L_q(0,r)}$$
$$= C \|\mathcal{A}_{(1/p-1/q)} g\|_{L_{q,\lambda}(0,\infty)},$$

where, again, $g(t) = t^{1/p-1/q} f^*(t)$. Since $1/p - \lambda/q > 0$, for $\beta = 1/p - 1/q$ the inequality $\beta > \lambda/q - 1/q$ holds. From Lemma 2.9 (i) the operator $\mathcal{A}_\beta$ is bounded in the Morrey spaces $L_{q,\lambda}(0,\infty)$. Then,

$$I_2 \lesssim \|\mathcal{A}_{(1/p-1/q)} g\|_{L_{q,\lambda}(0,\infty)} \lesssim \|g\|_{L_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\lambda/q} \|t^{1/p-1/q} f^*(t)\|_{L_q(0,r)} = \|f\|_{M^{\rm loc}_{p,q;\lambda}}. \tag{3.6}$$

From the inequalities (3.5) and (3.6) we obtain the boundedness of the operator H in $M^{\rm loc}_{p,q;\lambda}$.

(ii) For the limiting case $p = q/(q+\lambda)$, $1 < q < \infty$, suppose $f \in M^{\rm loc}_{p,q;\lambda}$. From the definition of norm in weak local Morrey–Lorentz spaces and by using the inequality (3.4) and Minkowski's inequality we get

$$\|Hf\|_{WM^{\rm loc}_{(q/q+\lambda),q;\lambda}} = \sup_{r>0} r^{-\lambda/q} \|t^{(q+\lambda)/q-1/q} (Hf)^*(t)\|_{WL_q(0,r)}$$
$$= \sup_{r>0} r^{-\lambda/q} \|t^{1+\lambda-1/q} (Hf)^*(t)\|_{WL_q(0,r)}$$
$$\le C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\lambda-1/q} \left(\frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \frac{f^*(s)}{s}\, ds \right) \right\|_{WL_q(0,r)}$$
$$\le C \sup_{r>0} r^{-\lambda/q} \|t^{\lambda-1/q} \int_0^t f^*(s)\, ds\|_{WL_q(0,r)}$$
$$+ C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\lambda-1/q} \int_t^\infty \frac{f^*(s)}{s}\, ds \right\|_{WL_q(0,r)} = N_1 + N_2.$$

Let us estimate N_1

$$N_1 = C \sup_{r>0} r^{-\lambda/q} \|t^{(\lambda-1)/q} \int_0^t f^*(s)\, ds\|_{WL_q(0,r)}$$
$$= C \|A_\beta h\|_{WL_{q,\lambda}(0,\infty)},$$

where $\beta = 1 + (\lambda - 1)/q$ and $h(t) = t^{1+(\lambda-1)/q} f^*(t)$. Therefore we get from Lemma 2.8 (ii)

$$N_1 \lesssim \|A_\beta h\|_{WL_{q,\lambda}(0,\infty)} \lesssim \|h\|_{L_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\lambda/q} \|t^{1+((\lambda-1)/q)} f^*(t)\|_{L_q(0,r)}$$
$$= \|f\|_{M^{\rm loc}_{q/(q+\lambda),q;\lambda}}. \tag{3.7}$$

Now we consider N_2

$$N_2 = C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+((\lambda-1)/q)} \int_t^\infty \frac{f^*(s)}{s} \, \mathrm{d}s \right\|_{WL_q(0,r)}$$
$$= C \|\mathcal{A}_\beta h\|_{WL_{q,\lambda}(0,\infty)},$$

where, again, $\beta = 1 + \lambda - 1/q$ and $h(t) = t^{1+\lambda-1/q} f^*(t)$. From Lemma 2.9 (ii) the operator $\mathcal{A}_\beta$ is bounded from the Morrey spaces $L_{q,\lambda}(0,\infty)$ to $WL_{q,\lambda}(0,\infty)$. Then,

$$N_2 \lesssim \|\mathcal{A}_\beta h\|_{WL_{q,\lambda}(0,\infty)} \lesssim \|h\|_{L_{q,\lambda}(0,\infty)}$$
$$= \sup_{r>0} r^{-\lambda/q} \|t^{1+(\lambda-1)/q} f^*(t)\|_{L_q(0,r)} = \|f\|_{M^{\mathrm{loc}}_{q/(q+\lambda),q;\lambda}}. \tag{3.8}$$

From the inequalities (3.7) and (3.8) we obtain the boundedness of the operator H from $M^{\mathrm{loc}}_{q/(q+\lambda),q;\lambda}$ to $WM^{\mathrm{loc}}_{q/(q+\lambda),q;\lambda}$.

(iii) For the limiting case $p = q/\lambda$, $0 < q \le \infty$, it will be convenient to use the modified Hilbert transform $\tilde{H}$ instead of H. The reason for using $\tilde{H}f$ is that it may exist while Hf may not exist (see [18], p. 210).

Since $\tilde{H}$ is bounded from L_∞ to BMO, the inequality

$$\|\tilde{H}f\|_{\mathrm{BMO}} \le C\|f\|_{L_\infty} \equiv \|f\|_{WL_\infty}$$

holds (see [4,18]).

From Lemma 2.4 we get

$$\|\tilde{H}f\|_{\mathrm{BMO}} \le C\|f\|_{M^{\mathrm{loc}}_{q/\lambda,q;\lambda}},$$

which proves that the modified Hilbert transform $\tilde{H}$ is bounded from $M^{\mathrm{loc}}_{q/\lambda,q;\lambda}$ to BMO. ■

In the following theorem we give the boundedness of the maximal Hilbert operator $\mathcal{H}$ in the local Morrey–Lorentz space $M^{\mathrm{loc}}_{p,q;\lambda}$.

Theorem 3.2: *Let $f \in M^{\mathrm{loc}}_{p,q;\lambda}$ satisfies* (3.2), $1 \le q \le \infty$, $0 \le \lambda < 1$ *and* $q/(q+\lambda) \le p \le q/\lambda$.

(i) *If $q/(q+\lambda) < p < q/\lambda$, $1 \le q < \infty$, then the operator $\mathcal{H}$ is bounded in the local Morrey–Lorentz space $M^{\mathrm{loc}}_{p,q;\lambda}$.*
(ii) *If $p = q/(q+\lambda)$, $1 < q < \infty$, then the operator $\mathcal{H}$ is bounded from $M^{\mathrm{loc}}_{p,q;\lambda}$ to the weak local Morrey–Lorentz space $WM^{\mathrm{loc}}_{p,q;\lambda}$.*

Proof: By using the same method of Theorem 3.1, the proof of the statements (i) and (ii) of this theorem can be easily obtained from the inequality (3.3). ■

In the case $\lambda = 0$ from Theorem 3.1 we get Theorem D and in the case $q \le p$ from Lemma 2.4 proved in [1] we get the following corollary.

Corollary 3.3: *Let $0 < q \le p < \infty$, $1/r = 1/p - \lambda/q$ and $0 < \lambda \le q/p$. Let also $f \in M^{\rm loc}_{p,q;\lambda}$ satisfies* (3.2). *Then the Hilbert transform $Hf(x)$, $x \in \mathbb{R}$ exists almost everywhere. Furthermore, the operator* H *is bounded in WL_r.*

Remark 2: Note that for the limiting case $\lambda = 1$, the space $M^{\rm loc}_{p,q;1}$ is the classical Lorentz space $\Lambda_{\infty,t^{1/p-1/q}}$ (see [1]). The proof of the boundedness of the Hilbert transform H in $\Lambda_{\infty,t^{1/p-1/q}}$ is given in [9].

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