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V. S. Guliyev

To cite this article: V. S. Guliyev (22 Jan 2026): Local Morrey–Lorentz spaces and commutators of fractional maximal operator with Lipschitz functions on them, Integral Transforms and Special Functions, DOI: [10.1080/10652469.2026.2617863](https://doi.org/10.1080/10652469.2026.2617863)

To link to this article: <https://doi.org/10.1080/10652469.2026.2617863>



Published online: 22 Jan 2026.



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Research Article



# Local Morrey–Lorentz spaces and commutators of fractional maximal operator with Lipschitz functions on them

V. S. Guliyev <sup>a,b,c,d</sup>

<sup>a</sup>Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan; <sup>b</sup>Department of Mathematics, Kirsehir Ahi Evran University, Kirsehir, Turkey; <sup>c</sup>Peoples Friendship University of Russia (RUDN University), Moscow, Russian Federation; <sup>d</sup>Institute of Mathematics, Ministry of Science and Education of Azerbaijan, Baku, Azerbaijan

## ABSTRACT

In this paper, we give necessary and sufficient conditions for the boundedness of the fractional maximal commutator operator  $M_{b,\alpha}$  and the commutators of the fractional maximal operator  $[b, M_\alpha]$  in the local Morrey–Lorentz spaces  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  when  $b$  belongs to Lipschitz spaces  $\dot{\Lambda}_\beta(\mathbb{R}^n)$ , whereby some new characterizations for certain subclasses of  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  spaces are obtained.

## ARTICLE HISTORY

Received 13 September 2025  
Accepted 12 January 2026

## KEYWORDS

Fractional maximal operator; commutator; local Morrey–Lorentz space; Lipschitz space

## 2020 MATHEMATICS SUBJECT CLASSIFICATIONS


42B20; 42B25; 42B35

## 1. Introduction

Let  $0 < p, q \leq \infty$  and let  $0 \leq \lambda \leq 1$ . We define the local Morrey–Lorentz spaces as the spaces of all measurable functions with finite quasinorm

$$\|f\|_{M_{p,r,\lambda}^{\text{loc}}} := \sup_{t>0} t^{-\frac{\lambda}{r}} \|(\cdot)^{\frac{1}{p}-\frac{1}{r}} f^*(\cdot)\|_{L_r(0,t)}.$$

The purpose of this paper is to give necessary and sufficient conditions for the boundedness of the fractional maximal commutators  $M_{b,\alpha}$  and the commutators of the fractional maximal operator  $[b, M_\alpha]$  on the local Morrey–Lorentz spaces  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  when  $b$  belongs to Lipschitz spaces  $\dot{\Lambda}_\beta(\mathbb{R}^n)$ . We obtain some new characterizations for certain subclasses of  $\dot{\Lambda}_\beta(\mathbb{R}^n)$ . Local Morrey–Lorentz spaces  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$ , which are natural generalizations of the Lorentz spaces  $L^{p,q}(\mathbb{R}^n) \equiv M_{p,q,0}^{\text{loc}}(\mathbb{R}^n)$  and the classical Lorentz spaces  $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n) \equiv M_{p,q,1}^{\text{loc}}(\mathbb{R}^n)$ , were introduced and their main properties were obtained in [1], see also [2–4]. For  $0 < q \leq p < \infty$  and  $0 < \lambda \leq \frac{q}{p}$ , the local Morrey–Lorentz spaces  $M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)$  are equal to weak Lebesgue spaces  $WL_{\frac{1}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$ . In [1] the basic properties of  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  were given

**CONTACT** V. S. Guliyev  vagif@guliyev.com

and the boundedness of the maximal operator was proved. Generally speaking, local Morrey spaces were also introduced separately by Guliyev [5] and Garcia-Cuerva and Herrero [6] (see also [7]).

For  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  and  $0 \leq \alpha < n$ , the fractional maximal operator  $M_\alpha$  is defined by

$$M_\alpha f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls  $B \subset \mathbb{R}^n$  containing  $x$ , and  $|B|$  is the Lebesgue measure of the ball  $B$ .

The fractional maximal commutator generated by the operator  $M_\alpha$  and  $b \in L^1_{\text{loc}}(\mathbb{R}^n)$  is defined by

$$M_{b,\alpha} f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x) - b(y)| |f(y)| \, dy.$$

The commutators generated by the operator  $M_\alpha$  and a suitable function  $b$  is defined by

$$[b, M_\alpha] f(x) = b(x) M_\alpha f(x) - M_\alpha (bf)(x).$$

Obviously, the operators  $M_{b,\alpha}$  and  $[b, M_\alpha]$  essentially differ from each other since  $M_{b,\alpha}$  is positive and sublinear and  $[b, M_\alpha]$  is neither positive nor sublinear.

The commutator estimates have many important applications, for example, in studying the regularity and boundedness of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and in characterizing certain function spaces (see, for instance [8,9]). The nonlinear commutator of maximal function  $[b, M]$  can be used in studying the product of a function in  $H^1$  and a function in BMO (see [10] for instance). Note that, the boundedness of the operator  $M_b$  on  $L^p(\mathbb{R}^n)$  spaces was proved by Garcia-Cuerva et al. [11]. In [12] by Bastero et al. studied the necessary and sufficient condition for the boundedness of  $[b, M]$  on  $L^p(\mathbb{R}^n)$  spaces.

The mapping properties of  $M_{b,\alpha}$  and  $[b, M_\alpha]$  have been studied extensively by many authors (see, for instance, [11–19]). The operators  $M_\alpha$ ,  $[b, M_\alpha]$  and  $M_{b,\alpha}$  play an important role in real and harmonic analysis and applications (see, for instance, [19–25]). In [18] Zhang and Wu studied the necessary and sufficient condition for the boundedness of  $[b, M_\alpha]$  on  $L^p(\mathbb{R}^n)$  spaces, see also [19]. In [26] Janson gave some characterizations of the Lipschitz space  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  via commutator  $[b, T]$  and the author proved that  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  if and only if  $[b, T]$  is bounded from  $L_p(\mathbb{R}^n)$  to  $L_q(\mathbb{R}^n)$ , where  $1 < p < n/\beta$ ,  $1/p - 1/q = \beta/n$  and  $T$  is the classical singular integral operator. In [15] the author recently gave necessary and sufficient conditions for the boundedness of the commutator of a fractional maximal operator in Orlicz spaces on any stratified Lie group when the commutator function belongs to Lipschitz spaces, and obtained some new characteristics for some subclasses of Lipschitz spaces (see also [27]).

In the works [21,22], necessary and sufficient conditions for the boundedness of the maximal commutator operator  $M_b$  and the commutators of the maximal operator  $[b, M]$  on the Lorentz spaces  $L_{p,q}$  were obtained, see also [23,28].

The structure of the paper is as follows. In Sec. 2 we give some definitions and auxiliary results. In Sec. 3 we obtain necessary and sufficient conditions for the boundedness of the fractional maximal commutator  $M_{b,\alpha}$  on  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  spaces. In Sec. 4 we give necessary

and sufficient conditions for the boundedness of the commutator of fractional maximal operator  $[b, M]$  on  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  spaces.

By  $A \lesssim B$  we mean that  $A \leq CB$  with some positive constant  $C$  independent of appropriate quantities. If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2. Definition and some basic properties

Let us start with the definition of Lorentz spaces, see, for example, [29]. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar  $L_p$  spaces, also they are appeared to be useful in the general interpolation theory.

Suppose that  $f$  is a measurable function on  $\mathbb{R}^n$ , then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\},$$

where

$$d_f(s) := |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad s > 0.$$

The Lorentz space  $L_{p,q} \equiv L_{p,q}(\mathbb{R}^n)$ ,  $0 < p, q \leq \infty$  is the collection of all measurable functions  $f$  on  $\mathbb{R}^n$  such that the quantity

$$\|f\|_{L_{p,q}} := \|(\cdot)^{\frac{1}{p}-\frac{1}{q}} f^*(\cdot)\|_{L_q(0,\infty)} \quad (2.1)$$

is finite. Clearly  $L_{p,p} \equiv L_p$  and  $L_{p,\infty} \equiv WL_p$ . The functional  $\|\cdot\|_{L_{p,q}}$  is a norm if and only if either  $1 \leq q \leq p$  or  $p = q = \infty$ .

**Definition 2.1 ([1]):** Let  $0 < p, r \leq \infty$  and  $0 \leq \lambda \leq 1$ . We denote by  $M_{p,r,\lambda}^{\text{loc}} \equiv M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  the local Morrey–Lorentz space, the space of all measurable functions with finite quasinorm

$$\|f\|_{M_{p,r,\lambda}^{\text{loc}}} := \sup_{t>0} t^{-\frac{\lambda}{r}} \|(\cdot)^{\frac{1}{p}-\frac{1}{r}} f^*(\cdot)\|_{L_r(0,r)}.$$

In the cases  $\lambda < 0$  or  $\lambda > 1$ , we have  $M_{p,r,\lambda}^{\text{loc}} = \Theta$ , where  $\Theta$  is the set of all functions equivalent to 0 on  $\mathbb{R}^n$ . Also  $M_{p,r;0}^{\text{loc}} = L_{p,r}$ . In the limiting case  $\lambda = 1$  the space  $M_{p,r;1}^{\text{loc}}$  is the classical Lorentz space  $\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{r}}}$ . For  $0 < r \leq p < \infty$  and  $0 < \lambda \leq \frac{r}{p}$ , the local Morrey–Lorentz spaces  $M_{p,r,\lambda}^{\text{loc}}$  are equal to weak Lebesgue spaces  $WL_{\frac{1}{p}-\frac{\lambda}{r}}$ . Note that, in the case  $r = \infty$  we have  $M_{p,\infty,\lambda}^{\text{loc}} = \Lambda_{\infty, t^{\frac{1}{p}}}$  and in the case  $p = r$  we have  $M_{p,p,\lambda}^{\text{loc}} \equiv WL_{\frac{1-\lambda}{p}}$ .

We denote by  $WM_{p,r,\lambda}^{\text{loc}} \equiv WM_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  the weak local Morrey–Lorentz space of all measurable functions with finite quasinorm

$$\|f\|_{WM_{p,r,\lambda}^{\text{loc}}} := \sup_{t>0} t^{-\frac{\lambda}{r}} \|(\cdot)^{\frac{1}{p}-\frac{1}{r}} f^*(\cdot)\|_{WL_r(0,t)}.$$

**Lemma 2.1 ([1]):** Let  $0 < r \leq p < \infty$ ,  $\frac{1}{s} = \frac{1}{p} - \frac{\lambda}{r}$  and  $0 < \lambda \leq \frac{r}{p}$ . Then

$$\left(\frac{r}{p}\right)^{-\frac{1}{r}} \|f\|_{WL_s} \leq \|f\|_{M_{p,r;\lambda}^{\text{loc}}} \leq \lambda^{-\frac{1}{r}} \|f\|_{WL_s}.$$

In particular,  $\|f\|_{WL_\infty} = \|f\|_{M_{\frac{r}{\lambda},r;\lambda}^{\text{loc}}}$  and  $\|f\|_{WL_{\frac{1-\lambda}{p}}} = \|f\|_{M_{p,p;\lambda}^{\text{loc}}}$ .

**Lemma 2.2 ([24, Lemma 2.4]):** Let  $0 < p, p_1, p_2, r, r_1, r_2 < \infty$ ,  $0 \leq \lambda \leq 1$ ,  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$  and  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ . Suppose that  $f \in M_{p_1,r_1;\lambda}^{\text{loc}}(\mathbb{R}^n)$  and  $g \in M_{p_2,r_2;\lambda}^{\text{loc}}(\mathbb{R}^n)$ . Then

$$\|fg\|_{M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)} \leq 2^{p-\frac{1}{r}} \|f\|_{M_{p_1,r_1;\lambda}^{\text{loc}}(\mathbb{R}^n)} \|g\|_{M_{p_2,r_2;\lambda}^{\text{loc}}(\mathbb{R}^n)}.$$

**Corollary 2.1 ([24, Corollary 2.2]):** Let  $0 \leq \lambda < 1$ ,  $1 < p, p', r, r' < \infty$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{r} + \frac{1}{r'} = 1$ . Suppose that  $f \in M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)$ . Then

$$\|f\|_{L_1(B)} \leq \|f\|_{M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)} |B|^{\frac{1}{p'} + \frac{\lambda}{r}}.$$

The following theorem is the boundedness of the maximal operator in local Morrey–Lorentz spaces  $M_{p,r;\lambda}^{\text{loc}}$ .

**Theorem 2.1 ([3, Theorem 1.1]):** Let  $1 \leq r \leq \infty$ ,  $0 \leq \lambda < 1$  and  $\frac{r}{r+\lambda} \leq p < \infty$ .

- (i) If  $\frac{r}{r+\lambda} < p < \frac{r}{\lambda}$ , then the operator  $M$  is bounded in the local Morrey–Lorentz space  $M_{p,r;\lambda}^{\text{loc}}$ .
- (ii) If  $p = \frac{r}{r+\lambda}$ , then the operator  $M$  is bounded from  $M_{p,r;\lambda}^{\text{loc}}$  to the weak space  $WM_{p,r;\lambda}^{\text{loc}}$ .

The following theorem is the boundedness of the fractional maximal operator in local Morrey–Lorentz spaces  $M_{p,r;\lambda}^{\text{loc}}$ .

**Theorem 2.2 ([4, Theorem 3.1]):** Let  $0 \leq \lambda < 1$ ,  $0 \leq \alpha < n$ ,  $1 \leq r \leq s \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $\frac{r}{r+\lambda} \leq p \leq \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$  and  $f \in M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)$ .

- (i) If  $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ , then the operator  $M_\alpha$  is bounded from the space  $M_{p,r;\lambda}^{\text{loc}}$  to  $M_{q,s;\lambda}^{\text{loc}}$  if and only if  $\frac{1}{p} - \frac{1}{q} = \lambda \left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{n}$ .
- (ii) If  $p = \frac{r}{r+\lambda}$ , then the operator  $M_\alpha$  is bounded from the space  $M_{p,r;\lambda}^{\text{loc}}$  to  $WM_{q,s;\lambda}^{\text{loc}}$  if and only if  $1 - \frac{1}{q} = \frac{\alpha}{n} - \frac{\lambda}{s}$ .
- (iii) If  $p = \left(\frac{\lambda}{r} + \frac{\alpha}{n}\right)^{-1}$ , then the operator  $M_\alpha$  is bounded from the space  $M_{p,r;\lambda}^{\text{loc}}$  to  $M_{q,s;\lambda}^{\text{loc}}$ .

### 3. $M_{p,r;\lambda}^{\text{loc}}$ -boundedness of the fractional maximal commutator operator $M_b$

In this section we obtain a new characterization for the Lipschitz space  $\dot{\Lambda}_\beta(\mathbb{R}^n)$ .

**Definition 3.2:** Let  $0 < \beta < 1$ , we say a function  $b$  belongs to the Lipschitz space  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  if there exists a constant  $C$  such that for all  $x, y \in \mathbb{R}^n$ ,

$$|b(x) - b(y)| \leq C|x - y|^\beta.$$

The smallest such constant  $C$  is called the  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  norm of  $b$  and is denoted by  $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$ .

To prove the following theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [30] and Janson, Taibleson and Weiss [31] (see also Paluszynski [32]).

**Lemma 3.3:** Let  $0 < \beta < 1$ , we have

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| dx,$$

where  $f_B = \frac{1}{|B|} \int_B f(y) dy$ .

Denote by  $M_B f$  the local maximal function of  $f$ :

$$M_B f(x) := \sup_{B' \ni x: B' \subset B} \frac{1}{|B'|} \int_{B'} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

From the proof of Theorem 1.4 in [33], we have the following characterization of non-negative Lipschitz functions.

**Lemma 3.4:** Let  $0 < \beta < 1$  and  $b$  be a locally integrable function. Then  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$  if and only if there exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |b(x) - M_B(b)(x)| dx \leq C.$$

The following lemma is valid.

**Lemma 3.5:** Let  $0 \leq \alpha < n$ ,  $0 < \beta < 1$  and  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ , then the following pointwise estimate holds:

$$M_{b,\alpha} f(x) \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_{\alpha+\beta} f(x).$$

**Proof:** If  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ , then

$$\begin{aligned} M_{b,\alpha}(f)(x) &\approx \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x) - b(y)| |f(y)| dy \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{B \ni x} |B|^{-1+\frac{\alpha+\beta}{n}} \int_B |f(y)| dy \end{aligned}$$

$$\approx \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_{\alpha+\beta} f(x).$$

■

In the following theorem we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator  $M_{b,\alpha}$  from the space  $M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)$ , which is one of the main theorems of this paper.

**Theorem 3.3:** *Let  $0 \leq \alpha < n$ ,  $0 \leq \lambda < 1$ ,  $0 < \beta < 1$ ,  $1 \leq r \leq s \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha+\beta}{n}\right)^{-1}$  and  $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha+\beta}{n}$ . The following assertions are equivalent:*

- (i)  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ .
- (ii) The operator  $M_{b,\alpha}$  is bounded from the space  $M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)$ .
- (iii) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)}} \leq C. \quad (3.1)$$

- (iv) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C. \quad (3.2)$$

**Proof:** (i)  $\Rightarrow$  (ii). Suppose that  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ . Combining Theorem 2.2 and Lemma 3.5, we get

$$\begin{aligned} \|M_{b,\alpha} f\|_{M_{q,s;\lambda}^{\text{loc}}} &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_{\alpha+\beta} f\|_{M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{M_{p,q;\lambda}^{\text{loc}}(\mathbb{R}^n)}. \end{aligned}$$

(ii)  $\Rightarrow$  (i). Assume that  $M_{b,\alpha}$  is bounded from the space  $M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)$ . Let  $B = B(x, r)$  be a fixed ball. We consider  $f = \chi_B$ . It is easy to compute that

$$\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}} \approx |B|^{\frac{1}{q} - \frac{\lambda}{s}}. \quad (3.3)$$

On the other hand, for all  $x \in B$  we have

$$\begin{aligned} |B|^{\frac{\alpha}{n}} |b(x) - b_B| &\leq \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(x) - b(y)| dy \\ &= \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(x) - b(y)| \chi_B(y) dy \\ &\leq M_{b,\alpha}(\chi_B)(x). \end{aligned}$$

Since  $M_{b,\alpha}$  is bounded from the space  $M_{p,r;\lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q,s;\lambda}^{\text{loc}}(\mathbb{R}^n)$ , then by (3.3) we obtain

$$\begin{aligned} \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - b_B)\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}} &\leq \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|M_{b,\alpha}(\chi_B)\|_{M_{q,s;\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|\chi_B\|_{M_{p,r;\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}} \approx |B|^{-\frac{\alpha+\beta}{n} + \frac{1}{p} - \frac{\lambda}{r} - \frac{1}{q} + \frac{\lambda}{s}} = 1, \end{aligned} \quad (3.4)$$

which implies that (3.1) holds since the ball  $B \subset \mathbb{R}^n$  is arbitrary.

(iii)  $\Rightarrow$  (iv). Assume that (3.1) holds, we will prove (3.2). For any fixed ball  $B$ , by Corollary 2.1, inequalities (3.1) and (3.3), it is easy to see

$$\begin{aligned} \frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_B| \, dy &\lesssim \frac{1}{|B|^{1+\frac{\beta}{n}}} \|(b - b_B)\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}} |B|^{\frac{1}{q'} + \frac{\lambda}{s}} \\ &\approx \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - b_B)\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}} \\ &\leq \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|M_{b,\alpha}(\chi_B)\|_{M_{q,s;\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|\chi_B\|_{M_{p,r;\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s;\lambda}^{\text{loc}}}} \approx |B|^{-\frac{\alpha+\beta}{n} + \frac{1}{p} - \frac{\lambda}{r} - \frac{1}{q} + \frac{\lambda}{s}} = 1. \end{aligned}$$

(iv)  $\Rightarrow$  (i). For any fixed ball  $B$ , we have

$$\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_B| \, dy = \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - b_B)\chi_B\|_{L_1}}{|B|} \lesssim 1,$$

which implies that  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  by using Lemma 3.3. Thus the proof of the theorem is completed. ■

In the case  $r = p, s = q$  from Theorem 3.3 we get the following new corollary.

**Corollary 3.2:** Let  $0 \leq \alpha < n, 0 \leq \lambda < 1, 0 < \beta < 1, 1 < p < \frac{n(1-\lambda)}{\alpha+\beta}, 1 \leq q \leq \infty$  and  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n(1-\lambda)}$ . The following assertions are equivalent:

- (i)  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ .
- (ii) The operator  $M_{b,\alpha}$  is bounded from the space  $WL_{\frac{1-\lambda}{p}}(\mathbb{R}^n)$  to  $WL_{\frac{1-\lambda}{q}}(\mathbb{R}^n)$ .
- (iii) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{WL_{\frac{1-\lambda}{q}}(\mathbb{R}^n)}}{\|\chi_B\|_{WL_{\frac{1-\lambda}{p}}(\mathbb{R}^n)}} \leq C.$$

(iv) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C.$$

In the case  $\lambda = 0$  from Theorem 3.3 we get the following corollary.

**Corollary 3.3 ([23, Theorem 4.1]):** Let  $0 \leq \alpha < n$ ,  $0 < \beta < 1$ ,  $1 \leq r \leq s \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $1 < p < \frac{n}{\alpha + \beta}$  and  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha + \beta}{n}$ . The following assertions are equivalent:

- (i)  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ .
- (ii) The operator  $M_{b,\alpha}$  is bounded from  $L_{p,r}(\mathbb{R}^n)$  to  $L_{q,s}(\mathbb{R}^n)$ .
- (iii) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{L_{q,s}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{q,s}(\mathbb{R}^n)}} \leq C.$$

(iv) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b(\cdot) - b_B)\chi_B\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C.$$

#### 4. $M_{p,q;\lambda}^{\text{loc}}$ -boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$

In this section, we obtain a new characterization for some subclasses of the Lipschitz space  $\dot{\Lambda}_\beta(\mathbb{R}^n)$ .

For a function  $b$  defined on  $\mathbb{R}^n$ , we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and  $b^+(x) := |b(x)| - b^-(x)$ . Obviously,  $b^+(x) - b^-(x) = b(x)$ .

The following relations between  $[b, M_\alpha]$  and  $M_{b,\alpha}$  are valid:

Let  $b$  be any non-negative locally integrable function. Then for all  $f \in L_1^{\text{loc}}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$  the following inequality is valid

$$\begin{aligned} |[b, M_\alpha]f(x)| &= |b(x)M_\alpha f(x) - M_\alpha(bf)(x)| \\ &= |M_\alpha(b(x)f)(x) - M_\alpha(bf)(x)| \leq M_\alpha(|b(x) - bf(x)|) = M_{b,\alpha}f(x). \end{aligned}$$

Denote by  $M_{\alpha,B}f$  the local fractional maximal function of  $f$ :

$$M_{\alpha,B}f(x) := \sup_{B' \ni x: B' \subset B} \frac{1}{|B'|^{1-\frac{\alpha}{n}}} \int_{B'} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

When  $\alpha = 0$ , we simply denote by  $M_B = M_{0,B}$ .

**Lemma 4.6:** Let  $0 \leq \alpha < n$ ,  $b \in L_1^{\text{loc}}(\mathbb{R}^n)$  and  $B$  be a ball in  $\mathbb{R}^n$ . Then for any  $x \in B$ ,

$$M_{\alpha,B}(b)(x) = M_\alpha(b\chi_B)(x) \quad (4.1)$$

and

$$M_{\alpha,B}(\chi_B)(x) = M_\alpha(\chi_B)(x) = |B|^{\frac{\alpha}{n}}. \quad (4.2)$$

**Proof:** Let us prove (4.1) first. For any  $x \in B$ , obviously we have

$$M_{\alpha,B}(b)(x) \leq M_\alpha(b\chi_B)(x).$$

On the other hand, for any  $x \in B$ , it follows from the definition of  $M_\alpha$  that

$$\begin{aligned} M_\alpha(b\chi_B)(x) &= \sup_{B' \ni x} |B'|^{-1+\frac{\alpha}{n}} \int_{B'} |f(y)| \chi_B(y) \, dy \\ &= \sup_{B' \ni x} |B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy. \end{aligned}$$

Then, to finish the proof, there remains to check that, for any  $x \in B$

$$\sup_{B' \ni x} |B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy \leq M_{\alpha,B}(b)(x).$$

This needs to verify that for any  $x \in B$  and any ball  $B' \ni x$ ,

$$|B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy \leq M_{\alpha,B}(b)(x). \quad (4.3)$$

If  $|B' \cap B| = 0$ , then (4.3) obviously holds.

If  $|B' \cap B| \neq 0$  and  $B' \cap B$  is a ball, then, then there are three possible cases:  $B = B'$  or  $B \subset B'$  or  $B' \subset B$ .

In the case of  $B = B'$  we get

$$|B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy = |B|^{-1+\frac{\alpha}{n}} \int_B |f(y)| \, dy \leq M_{\alpha,B}(b)(x).$$

In the case of  $B' \subset B$  we get

$$|B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy = |B'|^{-1+\frac{\alpha}{n}} \int_{B'} |f(y)| \, dy \leq M_{\alpha,B}(b)(x).$$

In the case of  $B \subset B'$  we get

$$\begin{aligned} |B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy &= |B'|^{-1+\frac{\alpha}{n}} \int_B |f(y)| \, dy \\ &\leq |B|^{-1+\frac{\alpha}{n}} \int_B |f(y)| \, dy \leq M_{\alpha,B}(b)(x). \end{aligned}$$

If  $|B' \cap B| \neq 0$  and  $B' \cap B$  is not a ball, then we can construct a ball  $B_1$  such that  $B' \cap B \supseteq B_1 \ni x$  and  $|B_1| \leq |B|$  and  $|B_1| \leq |B'|$ . Then

$$\begin{aligned} |B'|^{-1+\frac{\alpha}{n}} \int_{B' \cap B} |f(y)| \, dy &\leq |B'|^{-1+\frac{\alpha}{n}} \int_{B_1} |f(y)| \, dy \\ &\leq |B_1|^{-1+\frac{\alpha}{n}} \int_{B_1} |f(y)| \, dy \leq M_{\alpha, B}(b)(x). \end{aligned}$$

We have proved (4.3) for any  $x \in B$  and any ball  $B' \ni x$ . Then (4.1) follows.

Finally, (4.2) is obvious. Indeed, for any  $x \in B$ , by the definition of  $M_{\alpha, B}$  we have

$$\begin{aligned} M_{\alpha, B}(\chi_B)(x) &= \sup_{B' \ni x: B' \subset B} \frac{1}{|B'|^{1-\frac{\alpha}{n}}} \int_{B'} \chi_B(y) \, dy \\ &= \sup_{B' \ni x: B' \subset B} \frac{|B' \cap B|}{|B'|^{1-\frac{\alpha}{n}}} = |B|^{\frac{\alpha}{n}}, \end{aligned}$$

and the first equality in (4.2) follows from (4.1) readily.

Now, we finish the proof of Lemma 4.6. ■

Applying Theorem 3.3, we obtain the following main result. We give necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator  $[b, M_\alpha]$  from the space  $M_{p, r; \lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q, s; \lambda}^{\text{loc}}(\mathbb{R}^n)$ , which is one of the main theorems of this paper.

**Theorem 4.4:** *Let  $0 \leq \alpha < n$ ,  $0 \leq \lambda < 1$ ,  $0 < \beta < 1$ ,  $1 \leq r \leq s \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha+\beta}{n}\right)^{-1}$  and  $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha+\beta}{n}$ . The following assertions are equivalent:*

- (i)  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$ .
- (ii) The operator  $[b, M_\alpha]$  is bounded from the space  $M_{p, r; \lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q, s; \lambda}^{\text{loc}}(\mathbb{R}^n)$ .
- (iii) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\left\| \left( b(\cdot) - |B|^{-\frac{\alpha}{n}} M_B(b)(\cdot) \right) \chi_B \right\|_{M_{q, s; \lambda}^{\text{loc}}(\mathbb{R}^n)}}{\|\chi_B\|_{M_{p, r; \lambda}^{\text{loc}}(\mathbb{R}^n)}} \leq C. \quad (4.4)$$

- (iv) There exists a constant  $C > 0$  such that

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\left\| \left( b(\cdot) - |B|^{-\frac{\alpha}{n}} M_B(b)(\cdot) \right) \chi_B \right\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C. \quad (4.5)$$

**Proof:** (i)  $\Rightarrow$  (ii). Suppose that  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$ . Combining Lemma 3.5 and Theorem 3.3, we get

$$\|[b, M_\alpha]f\|_{M_{q, s; \lambda}^{\text{loc}}(\mathbb{R}^n)} \leq \|M_\alpha b f\|_{M_{q, s; \lambda}^{\text{loc}}(\mathbb{R}^n)}$$

$$\begin{aligned} &\leq \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_{\alpha+\beta} f\|_{M_{q,s,\lambda}^{\text{loc}}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we obtain that  $[b, M_\alpha]$  is bounded from the space  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q,s,\lambda}^{\text{loc}}(\mathbb{R}^n)$ .

(ii)  $\Rightarrow$  (iii). Assume that  $[b, M_\alpha]$  is bounded from the space  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  to  $M_{q,s,\lambda}^{\text{loc}}(\mathbb{R}^n)$ . Let  $B = B(x, r)$  be a fixed ball. Since from the Lemma 4.6

$$M_\alpha(b\chi_B) = M_{\alpha,B}(b) \quad \text{and} \quad M_\alpha(\chi_B) = M_{\alpha,B}(\chi_B) = |B|^{\frac{\alpha}{n}},$$

we have

$$|M_{\alpha,B}(b) - b|B|^{\frac{\alpha}{n}}| = |M_\alpha(b\chi_B) - bM_\alpha(\chi_B)| = |[b, M_\alpha]\chi_B|.$$

Hence

$$\| |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b) - b\chi_B \|_{M_{q,s,\lambda}^{\text{loc}}(\mathbb{R}^n)} = |B|^{-\frac{\alpha}{n}} \|[b, M_\alpha]\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}(\mathbb{R}^n)}.$$

Thus we get

$$\begin{aligned} &\frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b))\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}} \leq \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|[b, M_\alpha](\chi_B)\|_{M_{q,s,\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|\chi_B\|_{M_{p,r,\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}} \approx |B|^{-\frac{\alpha+\beta}{n} + \frac{1}{p} - \frac{\lambda}{r} - \frac{1}{q} + \frac{\lambda}{s}} = 1, \end{aligned}$$

which deduces that (iii).

(iii)  $\Rightarrow$  (iv). Assume that (4.4) holds, then for any fixed ball  $B$ , by Corollary 2.1, we conclude that

$$\begin{aligned} &\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(x)| \, dx \\ &\lesssim \frac{1}{|B|^{1+\frac{\beta}{n}}} \|(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b))\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}} |B|^{\frac{1}{q} + \frac{\lambda}{s}} \\ &\approx \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\|(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b))\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}} \\ &\leq \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|[b, M_\alpha](\chi_B)\|_{M_{q,s,\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\|\chi_B\|_{M_{p,r,\lambda}^{\text{loc}}}}{\|\chi_B\|_{M_{q,s,\lambda}^{\text{loc}}}} \approx |B|^{-\frac{\alpha+\beta}{n} + \frac{1}{p} - \frac{\lambda}{r} - \frac{1}{q} + \frac{\lambda}{s}} = 1. \end{aligned}$$

(iv)  $\Rightarrow$  (i). Assume that (4.5) holds, we will prove  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$ .

Denote by

$$E := \{x \in B : b(x) \leq b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B| dt = \int_F |b(t) - b_B| dt,$$

in view of the inequality  $b(x) \leq b_B \leq |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)$ ,  $x \in E$ , we get

$$\begin{aligned} \frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b - b_B| &= \frac{2}{|B|^{1+\frac{\beta}{n}}} \int_E |b - b_B| \\ &\leq \frac{2}{|B|^{1+\frac{\beta}{n}}} \int_E |b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)| \\ &\leq \frac{2}{|B|^{1+\frac{\beta}{n}}} \int_B |b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)| \lesssim 1. \end{aligned}$$

Consequently, by Lemma 3.3  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ . In order to show that  $b \geq 0$  a.e. in  $\mathbb{R}^n$ , it suffices to show  $b^- = 0$  a.e. in  $\mathbb{R}^n$ . Observe that  $0 \leq b^-(y) \leq |b(y)| \leq M_B(b)(y)$  for  $y \in B$ , therefore, for any  $y \in B$ , there holds

$$0 \leq b^-(y) = |b(y)| - b^+(y) \leq M_B(b)(y) - b^+(y) + b^-(y) = M_B(b)(y) - b(y).$$

Since  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ , then for any ball  $B$  from Lemma 3.4 we have

$$\begin{aligned} \frac{1}{|B|} \int_B b^-(y) dy &\leq \frac{1}{|B|} \int_B (M_B(b)(y) - b(y)) dy \\ &= \frac{1}{|B|} \int_B |b(y) - M_B(b)(y)| dy \\ &\leq \frac{|B|^{\frac{\beta}{n}}}{|B|^{1+\frac{\beta}{n}}} \int_B |b(y) - M_B(b)(y)| dy \leq C |B|^{\frac{\beta}{n}}. \end{aligned}$$

Let  $|B| \rightarrow 0$  with  $x \in B$ . Lebesgue's differentiation theorem assures that

$$0 \leq b^-(x) = \lim_{|B| \rightarrow 0} \frac{1}{|B|} \int_B b^-(y) dy = 0.$$

Thus  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$ . Thus the proof of the theorem is completed.  $\blacksquare$

In the case  $r = p, s = q$  from Theorem 4.4 we get the following new corollary.

**Corollary 4.4:** *Let  $0 \leq \alpha < n, 0 \leq \lambda < 1, 0 < \beta < 1, 1 < p < \frac{n(1-\lambda)}{\alpha+\beta}, 1 \leq q \leq \infty$  and  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n(1-\lambda)}$ . The following assertions are equivalent:*

- (i)  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$ .
- (ii) The operator  $M_{b,\alpha}$  is bounded from the space  $WL_{\frac{1-\lambda}{p}}(\mathbb{R}^n)$  to  $WL_{\frac{1-\lambda}{q}}(\mathbb{R}^n)$ .

(iii) *There exists a constant  $C > 0$  such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\left\| (b(\cdot) - |B|^{-\frac{\alpha}{n}} M_B(b)(\cdot)) \chi_B \right\|_{WL_{\frac{1-\lambda}{q}}(\mathbb{R}^n)}}{\|\chi_B\|_{WL_{\frac{1-\lambda}{q}}(\mathbb{R}^n)}} \leq C.$$

(iv) *There exists a constant  $C > 0$  such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\left\| (b(\cdot) - |B|^{-\frac{\alpha}{n}} M_B(b)(\cdot)) \chi_B \right\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C.$$

In the case  $\lambda = 0$  from Theorem 4.4 we get the following corollary.

**Corollary 4.5 ([22]):** *Let  $0 \leq \alpha < n$ ,  $0 < \beta < 1$ ,  $1 \leq r \leq s \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $1 < p < \frac{n}{\alpha+\beta}$  and  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n}$ . The following assertions are equivalent:*

- (i)  $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$  and  $b \geq 0$  a.e. in  $\mathbb{R}^n$ .
- (ii) *The operator  $[b, M_\alpha]$  is bounded from the space  $L_{p,r}(\mathbb{R}^n)$  to  $L_{q,s}(\mathbb{R}^n)$ .*
- (iii) *There exists a constant  $C > 0$  such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\left\| (b(\cdot) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(\cdot)) \chi_B \right\|_{L_{q,s}(\mathbb{R}^n)}}{\|\chi_B\|_{L_{q,s}(\mathbb{R}^n)}} \leq C.$$

(iv) *There exists a constant  $C > 0$  such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{n}}} \frac{\left\| (b(\cdot) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(\cdot)) \chi_B \right\|_{L_1(\mathbb{R}^n)}}{|B|} \leq C.$$

**Remark 1:** It should be noted that, in the case  $\alpha = 0$  Theorems 3.3 and 4.4 were already proved in [34]. Also, in the case  $\alpha = 0$  from Corollaries 3.2 and 4.4 we obtain the correct version of Corollaries 3.1 and 4.1 in [34].

## 5. Conclusion

The paper gives necessary and sufficient conditions for the boundedness of fractional maximal commutator operator  $M_{b,\alpha}$  and the commutators of the fractional maximal operator  $[b, M_\alpha]$  in the local Morrey–Lorentz spaces  $M_{p,r,\lambda}^{\text{loc}}(\mathbb{R}^n)$  when  $b$  belongs to Lipschitz spaces  $\dot{\Lambda}_\beta(\mathbb{R}^n)$ . As an application, new characterizations of some subclasses of Lipschitz spaces  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  are obtained.

## Disclosure statement

No potential conflict of interest was reported by the author.

## Funding

The author thanks the referee(s) for careful reading the paper and useful comments. The research of V. Guliyev was supported by the RUDN University Strategic Academic Leadership Program.

## ORCID

V. S. Guliyev  <http://orcid.org/0000-0001-7486-0298>

## References

- [1] Aykol C, Guliyev VS, Serbetci A. Boundedness of the maximal operator in the local Morrey–Lorentz spaces. *J Inequal Appl*. 2013;2013: Article number 346. doi: [10.1186/1029-242X-2013-346](https://doi.org/10.1186/1029-242X-2013-346)
- [2] Aykol C, Guliyev VS, Kucukaslan A, et al. The boundedness of Hilbert transform in the local Morrey–Lorentz spaces. *Integral Transforms Spec Funct*. 2016;27(4):318–330. doi: [10.1080/10652469.2015.1121483](https://doi.org/10.1080/10652469.2015.1121483)
- [3] Guliyev VS, Aykol C, Kucukaslan A, et al. Maximal operator and Calderón–Zygmund operators in local Morrey–Lorentz spaces. *Integral Transform Spec Funct*. 2016;27(11):866–877. doi: [10.1080/10652469.2016.1227329](https://doi.org/10.1080/10652469.2016.1227329)
- [4] Guliyev VS, Aykol C, Kucukaslan A, et al. Fractional maximal operator in the local Morrey–Lorentz spaces and some applications. *Afr Mat*. 2024;35: Article number 4. doi: [10.1007/s13370-023-01145-6](https://doi.org/10.1007/s13370-023-01145-6)
- [5] Guliyev VS. Integral operators on function spaces on the homogeneous groups and on domains in  $\mathbb{R}^n$  [Doctoral degree dissertation]. Moscow: Mathematical Institute Steklov; 1994. p. 329. (in Russian).
- [6] Garcia-Cuerva J, Herrero MJL. A theory of Hardy spaces associated to Herz spaces. *Proc London Math Soc*. 1994;s3-69(3):605–628. doi: [10.1112/plms/s3-69.3.605](https://doi.org/10.1112/plms/s3-69.3.605)
- [7] Alvarez J, Lakey J, Guzman-Partida M. Spaces of bounded  $\lambda$ -central mean oscillation, Morrey spaces, and  $\lambda$ -central Carleson measures. *Collect Math*. 2000;51(1):1–47.
- [8] Coifman RR, Rochberg R, Weiss G. Factorization theorems for Hardy spaces in several variables. *Ann Math*. 1976;103(3):611–635. doi: [10.2307/1970954](https://doi.org/10.2307/1970954)
- [9] Grafakos L. *Modern Fourier analysis*. 2nd ed. New York: Springer; 2009. (Graduate texts in mathematics; vol. 250).
- [10] Bonami A, Iwaniec T, Jones P, et al. On the product of functions in  $BMO$  and  $H_1$ . *Ann Inst Fourier*. 2007;57(5):1405–1439. doi: [10.5802/aif.2299](https://doi.org/10.5802/aif.2299)
- [11] Garcia-Cuerva J, Harboure E, Segovia C, et al. Weighted norm inequalities for commutators of strongly singular integrals. *Indiana Univ Math J*. 1991;40:1397–1420. doi: [10.1512/iumj.1991.40.40063](https://doi.org/10.1512/iumj.1991.40.40063)
- [12] Bastero J, Milman M, Ruiz FJ. Commutators for the maximal and sharp functions. *Proc Am Math Soc*. 2000;128(11):3329–3334. doi: [10.1090/proc/2000-128-11](https://doi.org/10.1090/proc/2000-128-11)
- [13] Celik S, Akbulut A, Omarova MN. Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in total anisotropic Morrey spaces. *Trans Natl Acad Sci Azerb Ser Phys-Tech Math Sci Math*. 2025;45(1):25–37.
- [14] Cruz-Uribe D, Fiorenza A. Endpoint estimate and weighted norm inequalities for commutators of fractional integrals. *Publ Mat*. 2003;47:103–131. doi: [10.5565/PUBLMAT\\_47103\\_05](https://doi.org/10.5565/PUBLMAT_47103_05)
- [15] Guliyev VS. Characterizations of Lipschitz functions via the commutators of fractional maximal function in Orlicz spaces on stratified Lie groups. *J Math Inequal*. 2007;19(2):421–439. doi: [10.7153/jmi-2025-19-27](https://doi.org/10.7153/jmi-2025-19-27)
- [16] Omarova MN. Commutators of parabolic fractional maximal operators on parabolic total Morrey spaces. *Math Meth Appl Sci*. 2025;48(11):11037–11044. doi: [10.1002/mma.v48.11](https://doi.org/10.1002/mma.v48.11)
- [17] Serbetci A. Boundedness of the  $B$ -maximal operator in  $B$ -total Morrey–Guliyev spaces. *Trans Natl Acad Sci Azerb Ser Phys-Tech Math Sci, Math*. 2025;45(4):145–154.

- [18] Zhang P, Wu J. Commutators of the fractional maximal functions. *Acta Math Sinica (Chin Ser)*. 2009;52:1235–1238.
- [19] Zhang P, Wu J, Sun J. Commutators of some maximal functions with Lipschitz function on Orlicz spaces. *Mediterr J Math*. 2018;15(6): Article number 216. doi: [10.1007/s00009-018-1263-0](https://doi.org/10.1007/s00009-018-1263-0)
- [20] Akbulut A, Isayev FA, Omarova M. Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in Lorentz spaces. *Integral Transforms Spec Funct*. 2026;0:1–16, Accepted 13 Aug 2025., doi: [10.1080/10652469.2025.2544999](https://doi.org/10.1080/10652469.2025.2544999) .
- [21] Akbulut A, Isayev F, Serbetci A. Anisotropic maximal commutator and commutator of anisotropic maximal operator on Lorentz spaces. *Trans Natl Acad Sci Azerb Ser Phys-Tech Math Sci, Math*. 2024;44(4):5–12.
- [22] Guliyev V.. Maximal commutator and commutator of maximal operator on Lorentz spaces. *Trans Natl Acad Sci Azerb Ser Phys-Tech Math Sci, Math*. 2024;44(4):43–49.
- [23] Guliyev VS. Commutator of fractional maximal function on Lorentz spaces. *SOCAR Proc*. 2024(3):113–117. doi: [10.5510/OGP20240301000](https://doi.org/10.5510/OGP20240301000)
- [24] Guliyev V. Commutators of maximal operator and sharp maximal operator on local Morrey–Lorentz spaces. *Trans Natl Acad Sci Azerb Ser Phys-Tech Math Sci, Math*. 2025;45(1):38–53.
- [25] Zhang P. Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces. *Anal Math Phys*. 2019;9:1411–1427. doi: [10.1007/s13324-018-0245-5](https://doi.org/10.1007/s13324-018-0245-5)
- [26] Janson S. Mean oscillation and commutators of singular integral operators. *Ark. Mat.*. 1978;16:263–270. doi: [10.1007/BF02386000](https://doi.org/10.1007/BF02386000)
- [27] Guliyev V, Eroglu A, Abasova G. Parabolic fractional maximal commutators with *BMO* function on Orlicz spaces. *TWMS J Pure Appl Math*. 2024;15(1):3–14.
- [28] Guliyev V. Lorentz boundedness characterization of commutators of maximal operators on spaces of homogeneous type. *J Contemp Appl Math*. 2025;15(2):64–76.
- [29] Bennett C, Sharpley R. *Interpolation of operators*. Boston: Academic Press; 1988.
- [30] DeVore R, Sharpley R. *Maximal functions measuring smoothness*. *Mem Am Math Soc*. 1984;47(293):1–115.
- [31] Janson S, Taibleson M, Weiss G. Elementary characterization of the Morrey–Campanato spaces. In: Mauceri G, Ricci F, Weiss G, editors. *Harmonic Analysis. Lecture Notes in Mathematics* Vol. 992. Berlin, Heidelberg: Springer; 1983. p. 101–114. doi: [10.1007/BFb0069154](https://doi.org/10.1007/BFb0069154)
- [32] Paluszynski M. Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. *Indiana Univ Math J*. 1995;44(1):1–17. doi: [10.1512/iumj.1995.44.1976](https://doi.org/10.1512/iumj.1995.44.1976)
- [33] Zhang P. Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function. *C R Math Acad Sci Paris*. 2017;355:336–344. doi: [10.1016/j.crma.2017.01.022](https://doi.org/10.1016/j.crma.2017.01.022)
- [34] Guliyev V. Commutators of maximal operator with Lipschitz functions on local Morrey–Lorentz spaces. *Integral Transform Spec Funct*. 2025:1–12. doi: [10.1080/10652469.2025.2562285](https://doi.org/10.1080/10652469.2025.2562285) .