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# TRANSLATION SURFACES IN PSEUDO-GALILEAN SPACE WITH PRESCRIBED MEAN AND GAUSSIAN CURVATURES

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**Abstract.** We study the translation surfaces in the pseudo–Galilean space with the condition that one of generating curves is planar. We classify these surfaces whose mean and Gaussian curvatures are functions of one variable.

## 1. Introduction

We concern with a special class among the family of surfaces in differential geometry, called *translation surfaces.* Let $u \mapsto \phi(u)$ and $v \mapsto \varphi(v)$ be two parametric curves in the Euclidean 3-space $\mathbb{R}^3$, $u \in I \subset \mathbb{R}$, $v \in J \subset \mathbb{R}$. A translation surface $S$ is locally a sum of $\phi(u)$ and $\varphi(v)$, i.e. $\mathbf{r}(u,v) = \phi(u) + \varphi(v)$ [6]. The curves $\phi := \phi(u)$ and $\varphi := \varphi(v)$ are called the *generating curves* of $S$. When the generating curves are planar and the planes containing them are orthogonal to each other, the surface $S$ is the graph $z = f(x) + g(y)$, for smooth functions $f, g$. In such a case $S$ is so-called *translation surface of planar type*, see [10], [11], [12]. A minimal translation surface of planar type (i.e. the mean curvature vanishes identically) is the *Scherk surface* $z = c^{-1}\log|\cos(cy) - \cos(cx)|$, $(x,y) \in (-\pi/2, \pi/2)$, $c \in \mathbb{R} - \{0\}$, [35].

A common problem in differential geometry of surfaces is to determine the ones with constant Gaussian ($K$) and mean ($H$) curvatures and, specific to the translation surfaces, has been studied from 1800s to present. Besides the cited studies in previous paragraph, in 3-dimensional (pseudo-)Euclidean setting, we refer for this long process to [8], [13], [16], [18], [19], [20], [22], [23], [37], and, in higher dimensional case, to [7], [9], [17], [28], [29], [36], [38], [40].

Most recently, Ruiz-Hernández [33] generalized the mentioned problem to finding the translation hypersurfaces in $\mathbb{R}^n$ whose mean and Gauss-Kronecker curvatures depend on its first $p$ (or second $q$) variables, $p+q=n$, and solved it, obtaining that those are the cylinders. This is indeed, in 3-dimensional setting, a well-known framework for surfaces of revolution or, more generally, helicoidal

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surfaces due to the fact the mean and Gaussian curvatures only depend on the parameter of the profile curve, see [3], [15].

In this study, following Ruiz-Hernández [33], we are interested in a special real Cayley-Klein geometry equipped with the projective metric of signature $(0,0,+,-)$, the *pseudo-Galilean 3-space* $G_3^1$ (see for details [26], [27], [31], [32], [39]), and tackle the problem of *determining the translation surfaces in* $G_3^1$ *whose mean and Gaussian curvatures are functions of one variable.* The reason why we study in this special ambient space may be explained as follows: let $(x,y,z)$ be the affine coordinates in $G_3^1$ and $S\subset G_3^1$ the translation surface $\mathbf{r}(u,v)=\phi(u)+\varphi(v)$ such that $\varphi$ (or symmetrically $\phi$) is Lorentzian planar, namely lie in the plane $x=const$. Then its mean curvature only depends on the variable $v$; in particular, equals to negative of the half of signed Frenet curvature $k_\varphi$ of $\varphi$, yielding that $S$ is completely determined by $k_\varphi$ via Fundamental Theorem of the Lorentzian plane curves [5], [21], [30]. Therefore our problem directly finds meaning in $G_3^1$. Furthermore, by the absolute figure of $G_3^1$, there are five different types of translation surfaces:

- type 1. $\phi$ is isotropic planar and $\varphi$ Lorentzian planar.
- type 2. $\phi$ and $\varphi$ are both isotropic planar.
- type 3. $\phi$ is spatial and $\varphi$ Lorentzian planar.
- type 4. $\phi$ is spatial and $\varphi$ isotropic planar.
- type 5. $\phi$ and $\varphi$ are both spatial.

As can be seen, the translation surfaces in $G_3^1$ have a wider categorization than the counterparts from (pseudo-)Euclidean space and worth to study from differential geometric point of view. Such surfaces in $G_3^1$ also have an interest from different geometric theories, see e.g. [1], [4], [14], [42].

The above surfaces in $G_3^1$, excepting type 5, with constant mean and Gaussian curvatures were determined in [2], [24], [25], [41]. In the present paper, we deal with the surfaces of types 1-4 whose mean and Gauss curvatures are non-constant functions of one variable. We obtain that the only translation surface with $K(u)$ (or $K(v)$) is of type 1; that is, $K$ cannot be a function of one variable for the other types. As emphasized before, for the surfaces of types 1 and 3, $2H(v)=-k_\varphi(v)$ occurs and we prove that the surface of type 2 with $H(u)$ (or $H(v)$) is a cylinder with non-isotropic rulings. We also show that, for a surface of type 4, $H$ cannot depend on the variable $v$ and the one with $H(u)$ is a cylinder with non-isotropic rulings. We notice that the present study does not speak about the surfaces of type 5.

## 2. Preliminaries

Denote $(u_0:u_1:u_2:u_3)$ the homogeneous coordinates of the real projective 3-space $P_3(\mathbb{R})$. The pseudo-Galilean 3-space $G_3^1$ is a Cayley-Klein space $P_3(\mathbb{R})$ with the absolute figure $\{\omega,f,I\}$ such that $\omega$ is the absolute plane $u_0=0$, $f$ the line $u_0=u_1=0$ and $I$ the fixed hyperbolic involution of points of $f$. The hyperbolic involution is $(0:0:u_2:u_3) \mapsto (0:0:u_3:u_2)$ and $u_2^2 - u_3^2 = 0$ is the absolute conic.

Let us introduce the affine coordinates by $(u_0 : u_1 : u_2 : u_3) = (1 : x : y : z)$. Up to the absolute figure, the *pseudo-Galilean distance* between the points $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ is

$$d(\mathbf{p},\mathbf{q}) = \begin{cases} |q_1 - p_1|, & \text{if } p_1 \neq q_1, \\ \sqrt{\left|(q_2-p_2)^2 - (q_3-p_3)^2\right|}, & \text{if } p_1 = q_1. \end{cases}$$

Let $a_1, \ldots, a_5, \varphi$ be some constants. Then the six-parameter group of motions of $G_3^1$ which leaves invariant the absolute figure and pseudo-Galilean distance is given in terms of affine coordinates by

$$\begin{aligned} \overline{x} &= a_1 + x \\ \overline{y} &= a_2 + a_3 x + y\cosh\varphi + z\sinh\varphi \\ \overline{z} &= a_4 + a_5 x + y\sinh\varphi + z\cosh\varphi. \end{aligned}$$

A line in $G_3^1$ is said to be *isotropic* if its intersection with the absolute line $f$ is non-empty and *non-isotropic* otherwise. A plane is said to be *isotropic* if it does not involve $f$ and *non-isotropic* otherwise. The non-isotropic planes are in the form $x = const.$ and so-called *Lorentzian* since its induced geometry is Lorentzian.

A vector $\mathbf{v} = (v_1, v_2, v_3)$ is said to be *isotropic* (*non-isotropic*) if $v_1 = 0$ ($\neq 0$). Let $\mathbf{w} = (w_1, w_2, w_3)$ and $\langle \cdot, \cdot \rangle_G$ denote the *pseudo-Galilean dot product* such that

$$\langle v, w \rangle = \begin{cases} v_1 w_1, & \text{if } v_1 \neq 0 \text{ or } w_1 \neq 0, \\ v_2 w_2 - v_3 w_3, & \text{if } v_1 = 0 \text{ and } w_1 = 0. \end{cases}$$

Then $\langle \mathbf{v}, \mathbf{w} \rangle_G$ is the Lorentzian scalar product if both $\mathbf{v}$ and $\mathbf{w}$ are isotropic. Otherwise, $v_1^2 + w_1^2 \neq 0$, it is defined by $\langle \mathbf{v}, \mathbf{w} \rangle_G = v_1 w_1$. The pseudo-Galilean angle between $\mathbf{v}$ and $\mathbf{w}$ is defined as the Lorentzian angle if $\mathbf{v}$ and $\mathbf{w}$ are isotropic. Otherwise, it is given by the pseudo-Galilean distance. We call $\mathbf{v}$ and $\mathbf{w}$ *orthogonal* if $\langle \mathbf{v}, \mathbf{w} \rangle_G = 0$.

An isotropic vector $\mathbf{v}$ is called *spacelike* if $\langle \mathbf{v}, \mathbf{v} \rangle_L > 0$; *timelike* if $\langle \mathbf{v}, \mathbf{v} \rangle_L < 0$ and *lightlike* if $\langle \mathbf{v}, \mathbf{v} \rangle_L = 0$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be standard basis vectors and $\mathbf{v}$ and $\mathbf{w}$ no both isotropic vectors. Then the *pseudo-Galilean cross-product* is

$$\mathbf{v} \times_G \mathbf{w} = \begin{vmatrix} 0 & -\mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Then $\langle \mathbf{v} \times_G \mathbf{w}, \mathbf{z} \rangle_G = -\det(\mathbf{v}, \mathbf{w}, \tilde{\mathbf{z}})$, where $\tilde{\mathbf{z}}$ is the projection of $\mathbf{z}$ onto the $yz-$plane. Note that the vector $\mathbf{v} \times_G \mathbf{w}$ is orthogonal to the vectors $\mathbf{v}$ and $\mathbf{w}$.

Let $C$ be a curve given in parametric form

$$s \mapsto \phi(s) = (x(s), y(s), z(s)), \; s \in I \subset \mathbb{R}.$$

The curve $C$ is said to be *admissible* if the following conditions hold: for each $s \in I$,

1. $\phi' = \frac{d\phi}{ds}$ is non-isotropic;
2. nowhere $C$ has no inflection points, i.e. $\phi'$ and $\phi'' = \frac{d^2\phi}{ds^2}$ are linearly independent;
3. $\tilde{\phi}'$ and $\tilde{\phi}''$ are non-lightlike.

An admissible curve $C$ is said to be *parameterized by arc length* if the function $x$ is the identity, up to a translation of $G_3^1$. Let $C$ be such a curve. We call $\mathbf{t} = \phi'$ unit *tangent* to $C$ and $\kappa = \sqrt{|\langle \phi'', \phi'' \rangle_L|}$ *curvature* of $C$. The *normal* and *binormal* to $C$ are defined by

$$\mathbf{n} = \frac{1}{\kappa(s)} (0, y'', z'') \text{ and } \mathbf{b} = \frac{1}{\kappa(s)} (0, z'', y'').$$

The *torsion* of $C$ is introduced by

$$\tau = \frac{\det(\phi', \phi'', \phi''')}{\kappa^2}.$$

We call the admissible curve $C$ *spatial* provided $\tau \neq 0$ for each $s \in I$. We call an admissible curve *isotropic planar* if it fully lies in an isotropic plane and in such case $\tau$ vanishes identically. We also call a curve *Lorentzian planar* if it fully lies in a Lorentzian plane. For a Lorentzian planar curve the Frenet apparatus are well known.

Let $S$ be a surface in $G_3^1$ locally given by a regular map

$$(u_1, u_2) \mapsto \mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)), \quad (u_1, u_2) \in D \subset \mathbb{R}^2.$$

Denote $x_{,i} = \frac{\partial x}{\partial u_i}$ and $x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}$ and etc., $1 \leq i, j \leq 2$. Then $S$ is said to be *admissible* if $x_{,i} \neq 0$ for some $i = 1, 2$. For such an admissible surface $S$, the *first fundamental form* is

$$\langle d\mathbf{x}, d\mathbf{x} \rangle_G = E du_1^2 + 2F du_1 du_2 + G du_2^2,$$

where $E = (x_{,1})^2$, $F = x_{,1} x_{,2}$, $G = (x_{,2})^2$. Since nowhere an admissible surface has Lorentzian tangent plane, up to the absolute figure, the isotropic vector $\mathbf{x}_{,1} \times_G \mathbf{x}_{,2}$ is *normal* to $S$. Let

$$W = \langle \mathbf{x}_{,1} \times_G \mathbf{x}_{,2}, \mathbf{x}_{,1} \times_G \mathbf{x}_{,2} \rangle_G.$$

Then the surface $S$ is called *spacelike* if $W < 0$; *timelike* if $W > 0$; and *lightlike* if $W = 0$. The spacelike and timelike surfaces are so-called *non-degenerate* and, throughout this study, we deal with the only non-degenerate admissible surfaces. The *unit normal vector* to the non-degenerate surface $S$ is

$$\mathbf{N} = \frac{\mathbf{x}_{,1} \times_G \mathbf{x}_{,2}}{\sqrt{|W|}}.$$

Let $\epsilon = \langle \mathbf{N}, \mathbf{N} \rangle_L = \pm 1$ and

$$L_{ij} = \epsilon \frac{1}{x_{,1}} \left\langle x_{,1} \tilde{\mathbf{x}}_{,ij} - (x_{,i})_{,j} \tilde{\mathbf{x}}_{,1}, \mathbf{N} \right\rangle_G, \text{ or}$$
$$L_{ij} = \epsilon \frac{1}{x_{,2}} \left\langle x_{,2} \tilde{\mathbf{x}}_{,ij} - (x_{,i})_{,j} \tilde{\mathbf{x}}_{,2}, \mathbf{N} \right\rangle_G,$$

in which $\tilde{\mathbf{x}}_{,ij}$ is the projection of vectors $\mathbf{x}_{,ij}$ onto the $yz$-plane and one of $x_{,1}$ and $x_{,2}$ is always nonzero due to the admissibility. Then the *second fundamental form* of $S$ is

$$II = Ldu_1^2 + 2Mdu_1du_2 + Ndu_2^2,$$

where $L = L_{11}$, $M = L_{12}$, $N = L_{22}$. Thereby, *the Gaussian and mean curvatures* are defined by

$$K = -\epsilon \frac{LN - M^2}{W} \text{ and } H = -\epsilon \frac{GL - 2FM + EN}{2W}.$$

A surface is said to be *minimal* if $H$ vanishes identically (see [25, 24, 34]).

## 3. The surfaces of type 1

Let $u \mapsto \phi(u)$ and $v \mapsto \varphi(v)$ be two parametric curves in $G_3^1$ such that $\phi := \phi(u)$ is admissible isotropic planar and $\varphi := \varphi(v)$ Lorentzian planar. If the planes containing $\phi$ and $\varphi$ are orthogonal in Euclidean setting, then we may assume that $\phi(u) = (u, 0, f(u))$ and $\varphi(v) = (0, g(v), h(v))$, where $f$, $g$ and $h$ are smooth functions and $u \in I \subset \mathbb{R}$ and $v \in J \subset \mathbb{R}$. Therefore the generated translation surface by $\phi$ and $\varphi$ is locally

$$\mathbf{r}(u, v) = \phi(u) + \varphi(v) = (u, g(v), f(u) + h(v)). \tag{3.1}$$

Assume $\beta(v)$ is parameterized by arc-length. Since the induced metric is the Lorentzian we deduce

$$g'^2 - h'^2 = \delta = \pm 1,$$

where $g' = \frac{dg}{dv}$ and so on. Notice that $\delta = -\epsilon$. Denote by $k_\varphi$ and $\mathbf{n}_\varphi$ the signed Frenet curvature and principal normal of $\varphi$, respectively. Then it follows $\mathbf{N} = \mathbf{n}_\varphi$ and $2H = -k_\varphi$, providing

**Proposition 3.1.** *Twice mean curvature of a translation surface of type 1 is the negative of the signed Frenet curvature of $\varphi$.*

Via the Fundamental Theorem of the Lorentzian plane curves, the surface (3.1) can be completely determined by its mean curvature, namely

$$\mathbf{r}(u, v) = \phi(u) + \left(0, \int^v \cosh\left(\int^v -2H(t)\,dt\right) dt, \int^v \sinh\left(\int^v -2H(t)\,dt\right) dt\right),$$

in which $\varphi$ is assumed to be spacelike and the positions of the integrals change otherwise. Therefore the minimal surface (3.1) is a cylinder whose rulings correspond to the geodesics of the $yz-$plane. Analogously, for the surface (3.1) with $H = const. \neq 0$, the generating curve $\varphi$ is the hyperbolas in the $yz-$plane.

The Gaussian curvature of the surface (3.1) is obtained by

$$K = f''h''. \tag{3.2}$$

Unlike the mean curvature, $K$ depends on the parameters of both $\phi$ and $\varphi$. If $K$ only depends on the variable $v$, say $K = K(v)$, then $f$ is a quadratic. Letting $f'' = a \in \mathbb{R} - \{0\}$ for each $u \in I$, the curve $\phi$ is a parabolic circle in the $xz-$plane with $\kappa = |a|$ and $\varphi$ is of the form

$$\varphi(v) = \left(0, \pm \int^{v} \left(1 + \left(\int^{v} q(t)\, dt\right)^{2}\right)^{1/2} dt, \int^{v} \left(\int^{v} q(t)\, dt\right) dv\right), \tag{3.3}$$

where $q(v) = \frac{1}{a} K(v)$. Analogously, if $K$ only depends on the variable $u$, then $h$ is a quadratic, and up to suitable transformations of the Lorentzian $yz-$plane, we may take $h(v) = bv^2$, $b \in \mathbb{R} - \{0\}$ and hence

$$\varphi(v) = \left(0, \pm \frac{v}{2}\sqrt{1 + (2bv)^{2}} \pm \frac{1}{4b} \sinh^{-1}(2bv), bv^{2}\right). \tag{3.4}$$

Then the curve $\phi$ becomes

$$\phi(u) = \left(u, 0, \frac{-\delta}{2b} \int^{u} \left(\int^{u} K(t)\, dt\right) dt\right), \tag{3.5}$$

proving the following result:

**Theorem 3.2.** *Let $S$ be the translation surface of type 1 given by (3.1). If its Gaussian curvature $K$ is a non-constant function which only depends on the variable $v$ then $\phi$ is a parabolic circle and $\varphi$ of the form (3.3). If $K$ only depends on the variable $u$ then $\phi$ and $\varphi$ are given by (3.4) and (3.5).*

**Example 3.3.** *Consider a spacelike translation surface of type 1 with the Gaussian curvature $K(v) = \cosh v$. We will find its parametrization. Then the generating curve $\phi$ is a parabolic circle of constant curvature $\kappa(u)$. Letting $\kappa(u)$=1 yields $\phi(u) = (u, 0, \frac{u^2}{2} + bv + c)$, $b, c \in \mathbb{R}$. Up to constants and translations, we may assume $b = 0 = c$. (3.3) is now $\varphi(v) = (0, \pm \sinh v, \cosh v)$. Fix the sign of $\pm$ as plus. The surface that we want to find is,*

$$\mathbf{r}(u, v) = (u, \sinh v, \cosh v, \frac{u^2}{2}).$$

*This surface can be drawn as in Fig. 1.*

**Example 3.4.** *Take a spacelike translation surface of type 1 with the Gaussian curvature $K(u) = \cos u$. As in the previous example, we will get its* *parametrization. Up to constants and translations, by (3.2) we have* $h(v) = \frac{v^2}{2}$ *and* $f(u) = -\cos u$. *Then*

$$\mathbf{r}\,(u,v) = \left( u, \frac{v}{2}\sqrt{1+v^2} + \frac{1}{2}\sinh^{-1}(v)\,, \frac{1}{2}v^2 - \cos u \right).$$

*which can be drawn as in Fig. 2.*

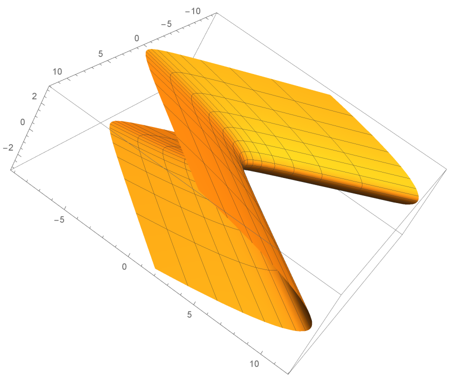

FIGURE 1. A translation surface of type 1 with $K(v) = \cosh v$

## 4. The surfaces of type 2

Let $\phi$ and $\varphi$ denote two admissible isotropic planar parametric curves in $G_3^1$. If the planes containing $\phi$ and $\varphi$ are orthogonal in Euclidean setting, then the generated translation surface by $\phi$ and $\varphi$ is locally

$$\mathbf{r}\,(u,v) = \phi\,(u) + \varphi\,(v) = (u+v, g\,(v)\,, f\,(u))\,, \tag{4.1}$$

where $f$ and $g$ are some smooth functions. The mean curvature is

$$H = \frac{f''g' + f'g''}{2\left|f'^2 - g'^2\right|^{\frac{3}{2}}}, \tag{4.2}$$

where $f' = \frac{df}{du}$ and $g' = \frac{dg}{dv}$ and then we state the following:

**Theorem 4.1.** *A translation surface of type 2 with* $H\,(u)$ *(or* $H\,(v)$*), where* $H(u)$ *is a non-constant function of* $u$*, is a cylinder with non-isotropic rulings, namely* $K \equiv 0$.

*Proof.* Notice that the roles of $f$ and $g$ in (4.2) are symmetric and so we concentrate for the function $f$, assuming that $H$ is non-constant function of $u$, say $H = H\,(u)$. Without loss of generality suppose $f'^2 > g'^2$. If $f''\,(u_0) = 0$ for some $u_0 \in I$ and then we may assume that $f$ is linear around $u_0$. In such

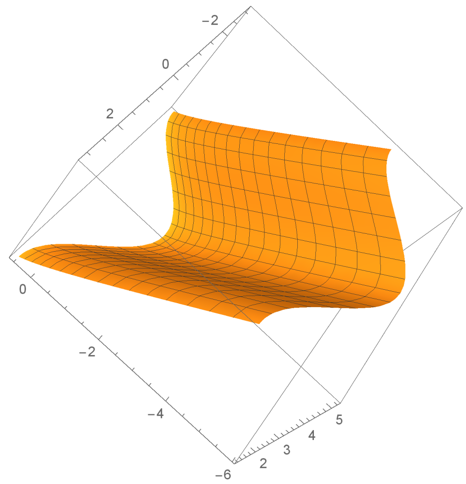


FIGURE 2. A translation surface of type 1 with $K(u) = \cos u$

a case, the left-hand side of (4.2) is a function of $u$ while the other side is a function of $v$, which is not possible. This discussion implies $f'' \neq 0$ for each $u \in I$. Assume next that $g'' = 0$. Then we may take $g(v) = av + b$, $a, b \in \mathbb{R}$ and hence the surface is a cylinder with non-isotropic rulings as follows

$$\mathbf{r}(u, v) = (u, 0, f(u) + b) + v(1, a, 0).$$

Here we remark $a \neq 0$ because $H(u)$ is non-vanishing. In order to complete proof we have to show that (4.2) has no solution provided that $f''g'' \neq 0$ for each $(u, v)$. By contradiction assume that $f''g'' \neq 0$. Without loss of generality we may consider the timelike case, i.e. $f'^2 > g'^2$. Then the derivative of (4.2) with respect to $v$ deduces

$$(f''g'' - f'g''')\left(f'^2 - g'^2\right) + 3\left(f''g' - f'g''\right)g'g'' = 0,$$

or equivalently

$$\frac{f''}{f'} = \frac{-\left(g'''/g''\right)\left(f'^2 - g'^2\right) + 3g'g''}{f'^2 + 2g'^2}. \tag{4.3}$$

The derivative with respect to $v$ yields

$$\sum_{n=0}^{4} P_n \left(f'\right)^n = 0,$$

where

$$\begin{aligned} P_0 &= 2g'^4 \left(g'''/g''\right)' + 14g'^3 g''' - 6\left(g'g''\right)^2, \\ P_1 &= P_3 = 0, \\ P_2 &= g'^2 \left(g'''/g''\right)' + g'g''' + 3g''^2, \\ P_4 &= -\left(g'''/g''\right)'. \end{aligned}$$

Because $f'$ is a non-constant function of $u$, the coefficients $P_0, \ldots, P_4$ must vanish, immediately implying $\left(g'''/g''\right)' = 0$, or equivalently $g''' = ag''$, $a \in \mathbb{R}$. If $a = 0$ then follows from $P_2 = 0$ that $3g''^2 = 0$: contradiction. Otherwise, $a \neq 0$, we have from $P_2 = 0$ that $ag' + 3g'' = 0$. Therefore from $P_2 = 0$ we get $7ag' - 3g'' = 0$, or equivalently $8ag' = 0$, which is a contradiction. □

Let us see what happen to (4.2) when $g'' = 0$. Letting $g' = a \in \mathbb{R}$ and $f'^2 > a^2$, it follows

$$H(u) = \frac{af''}{2\left(f'^2 - a^2\right)^{\frac{3}{2}}}.$$

After integrating, we write

$$2\int^u H(t)\,dt = \frac{-f'}{\sqrt{f'^2 - a^2}}. \tag{4.4}$$

Put $p = \int^u H(t)\,dt$. Then the solution is

$$f(u) = \pm \int^u \frac{2ap}{\left(-1 + 4p^2\right)^{1/2}} dt.$$

The Gaussian curvature is

$$K = \frac{f'f''g'g''}{\left(f'^2 - g'^2\right)^2}. \tag{4.5}$$

We provide the following result:

**Theorem 4.2.** *There does not exist a translation surface of type 2 with $K(u)$ (or $K(v)$).*

*Proof.* Because the roles of $f$ and $g$ are symmetric, we may concentrate for $f(u)$ without loss of generality. Assume that $K$ is a non-constant function of $u$, deducing $f'f''g'g'' \neq 0$ for each $(u, v)$. Then (4.5) may be written as

$$\frac{K(u)}{f'f''} = \frac{g'g''}{\left(f'^2 - g'^2\right)^2}. \tag{4.6}$$

Derivating of (4.6) with respect to $v$, we may obtain

$$\left(g'g''\right)'\left(f'^2 - g'^2\right) + 4\left(g'g''\right)^2 = 0.$$

The derivative with respect to $u$ implies $(g'g'')' = 0$ because $f'f'' \neq 0$ and therefore the contradiction is obtained $4(g'g'')^2 = 0$. □

## 5. The surfaces of type 3

Let $\phi$ and $\varphi$ be an admissible spatial and a Lorentzian planar curves, respectively. Hence the generated translation surface by $\phi$ and $\varphi$ is

$$\mathbf{r}(u,v) = (u, f(u) + g(v), p(u) + h(v)), \tag{5.1}$$

where $f, g, h, p$ are some smooth functions. Being $\phi$ spatial implies that $f'''p'' - f''p''' \neq 0$. As in the previous section, if we assume that $\varphi$ is parameterized by arc length then we easily deduce that $2H = -k_\varphi$, where $k_\varphi$ is the signed Frenet curvature of $\varphi$. Therefore Proposition 1 remains true for the translation surface of type 3.

Next, the Gaussian curvature is

$$K(u,v) = \pm(f''h' - p''g')\,k_\varphi. \tag{5.2}$$

Therefore we have the following non-existence result:

**Theorem 5.1.** *There does not exist a translation surface of type 3 with $K(u)$ (or $K(v)$), where $K(u)$ is a non-constant function of $u$.*

*Proof.* Let $K$ be a non-constant function. We separate two cases:
**Case i** $K = K(u)$. We divide (5.2) with $\pm f''$ and next derivative with respect to $v$, deducing

$$0 = (h'k_\varphi)' - \frac{p''}{f''}(g'k_\varphi)'.$$

Because $\alpha(u)$ is spatial, $f''$ and $p''$ are linearly independent and hence the ratio $p''/f''$ is a non-constant function of $u$. Then there are nonzero constants $a$ and $b$ such that

$$h'k_\varphi = a \text{ and } g'k_\varphi = b.$$

This implies $g' = \frac{b}{a}h'$. Because $g'^2 - h'^2 = \pm 1$, we arrive to the case $g'' = h'' = 0$, giving $K(u) = 0$, which is not our case.
**Case ii.** $K(u,v) = K(v)$. We divide (5.2) with $\pm k_\varphi g'$ and next derivative with respect to $u$, deducing

$$0 = f'''\frac{h'}{g'} - p'''.$$

Because $K$ is non-vanishing, $g'$ and $h'$ are linearly independent and hence we derive the contradiction $f''' = p''' = 0$. □

## 6. The surfaces of type 4

Let $\phi$ and $\varphi$ be an admissible spatial and an isotropic planar curves, respectively. Hence the generated translation surface by $\phi$ and $\varphi$ is

$$\mathbf{r}(u,v) = (u+v, f(u)+g(v), p(u)), \tag{6.1}$$

where $f, g, p$ are some smooth functions. Being $\phi$ spatial implies $f''p''' - f'''p'' \neq 0$.

The mean curvature of $S$ is

$$H = \frac{(f''+g'')p' + p''(f'-g')}{2\left|p'^2 - (f'-g')^2\right|^{3/2}}. \tag{6.2}$$

We introduce

$$\begin{aligned} \alpha_1 &= f''p' + p''f', \\ \alpha_2 &= -p'', \\ \alpha_3 &= p', \\ \alpha_4 &= p'^2 - f'^2, \\ \alpha_5 &= 2f'. \end{aligned}$$

Then, from (6.2) it follows

$$H = \frac{\alpha_1 + \alpha_2 g' + \alpha_3 g''}{2\left|\alpha_4 + \alpha_5 g' - g'^2\right|^{3/2}} \tag{6.3}$$

We separately observe (6.3) up to $H = H(u)$ or $H = H(v)$.

**Theorem 6.1.** *A translation surface of type 4 with $H(u)$, where $H(u)$ is a non-constant function of $u$, is a cylinder with non-isotropic rulings, namely $K \equiv 0$.*

*Proof.* Let $H = H(u)$ and $W = \alpha_4 + \alpha_5 g' - g'^2$. Without loss of generality we may assume that the surface is timelike, i.e. $W > 0$. Then (6.3) turns to

$$2(H/\alpha_3)W^{3/2} = (\alpha_1/\alpha_3) + (\alpha_2/\alpha_3)g' + g''. \tag{6.4}$$

The derivative of (6.4) with respect to $v$ gives

$$3(H/\alpha_3)W^{1/2}(\alpha_5 g'' - 2g'g'') = (\alpha_2/\alpha_3)g'' + g'''.$$

That $g'' = 0$ for each $v \in J$ is a trivial solution, proving the result. Hereinafter we will suppose that $g'' \neq 0$ and obtain a contradiction. Denote $H' = dH/du$ and $W_u = \partial W/\partial u$. By derivativing (6.4) with respect to $u$ and next multiplying $W^{1/2}$, we may obtain

$$2(H/\alpha_3)'W^2 + 3(H/\alpha_3)WW_u = \left[(\alpha_1/\alpha_3)' + (\alpha_2/\alpha_3)'g'\right]W^{1/2}. \tag{6.5}$$

Squaring both hand sides

$$\sum_{n=0}^{8} P_n \left(g'\right)^n = 0,$$

where $P_0, ..., P_8$ are zero all because $g'$ is a non-constant function of $v$. It follows from $P_8 = 0$ that $(H/\alpha_3)' = 0$, implying the existence of a nonzero constant $a$ such that $H = a\alpha_3$. Then (6.5) reduces to

$$3aWW_u = \left[(\alpha_1/\alpha_3)' + (\alpha_2/\alpha_3)' g'\right] W^{1/2}.$$

After performing the same argument, we may obtain from the last equality that $\sum_{n=0}^{6} Q_n \left(g'\right)^n = 0$, where $Q_6 = 9a^2\alpha_5^2$ cannot vanish: contradiction. □

In the case that $g'' = 0$, say $g(v) = bv + c$, $b, c \in \mathbb{R}$, the surface is

$$\mathbf{r}(u,v) = (u+v, f(u) + g(v), p(u)) = (u, f(u) + c, p(u)) + v(1, b, 0),$$

which is a cylinder with non-isotropic rulings parallel to $(1, b, 0)$. Then, if $H = H(u)$, (6.2) writes

$$H(u) = \frac{f''p' - p''(f' - b)}{2\left(p'^2 - (f' - b)^2\right)^{\frac{3}{2}}},$$

or equivalently

$$2H(u)(b - f') = \frac{\left(p'/(b - f')\right)'}{\left(\left(p'/(b - f')\right)^2 - 1\right)^{\frac{3}{2}}}.$$

A first integration leads to

$$\int^u 2H(t)(b - f')\,dt = \frac{p'/(b - f')}{\sqrt{\left(p'/(b - f')\right)^2 - 1}}.$$

Letting $Q(u) = \int^u 2H(t)(b - f')\,dt$ and integrating we obtain

$$p = \pm \int^u (f' - b)\, Q\left(Q^2 - 1\right)^{-1/2} dt.$$

In the following we concern the case $H = H(v)$.

**Theorem 6.2.** *There is not a translation surface of type 4 with $H(v)$, where $H(v)$ is a non-constant function of $v$.*

*Proof.* The proof is by contradiction. Then (6.3) reduces to

$$H(v) = \frac{\alpha_1 + \alpha_2 g' + \alpha_3 g''}{2\left|\alpha_4 + \alpha_5 g' - g'^2\right|^{\frac{3}{2}}}. \tag{6.6}$$

Suppose that $\alpha_4 + \alpha_5 g' - g'^2 > 0$ without loss of generality. The derivative of (6.6) with respect to $u$ writes

$$F + Gg'' = 0, \tag{6.7}$$

where $F=\sum_{n=0}^{3}\beta_n\left(g'\right)^n$ and $G=\sum_{n=0}^{2}\gamma_n\left(g'\right)^n$ such that

$$\begin{aligned}\beta_0&=\alpha_1'\alpha_4-\frac{3}{2}\alpha_1\alpha_4',\\ \beta_1&=\alpha_1'\alpha_5+\alpha_2'\alpha_4-\frac{3}{2}\left(\alpha_1\alpha_5'+\alpha_2\alpha_4'\right),\\ \beta_2&=-\alpha_1'+\alpha_2'\alpha_5-\frac{3}{2}\alpha_2\alpha_5',\\ \beta_3&=-\alpha_2',\end{aligned}$$

and

$$\begin{aligned}\gamma_0&=\alpha_3'\alpha_4-\frac{3}{2}\alpha_3\alpha_4',\\ \gamma_1&=\alpha_3'\alpha_5-\frac{3}{2}\alpha_3\alpha_5',\\ \gamma_2&=-\alpha_3'.\end{aligned}$$

Notice that $G$ cannot vanish because otherwise the polynomial equation of degree 2 on $g'$ is obtained; however, nowhere $\gamma_2=-p''$ vanishes. This discussion from (6.7) allows that $F$ cannot vanish. Then $g''=-F/G$ and considering into (6.6) follows

$$4H^2\left(v\right)\left(\alpha_4+\alpha_5g'-g'^2\right)^3G^2=\left(\left(\alpha_1+\alpha_2g'\right)G-\alpha_3F\right)^2,$$

which is polynomial equation of degree 10 on $g'$ whose leading coefficient is $-4H^2\left(v\right)p''^2$ which cannot vanish: contradiction. □

The Gaussian curvature is

$$K=-\epsilon\frac{g''p'\left(f''p'+f'p''-g'p''\right)}{\left(p'^2-\left(f'-g'\right)^2\right)^2}. \tag{6.8}$$

We introduce

$$\begin{aligned}\alpha_1&=p',\\ \alpha_2&=f''p'-p''f',\\ \alpha_3&=-p'',\\ \alpha_4&=p'^2-f'^2,\\ \alpha_5&=2f'.\end{aligned}$$

Then (6.8) turns to

$$K=-\epsilon\frac{\alpha_1\left(\alpha_2+\alpha_3g'\right)g''}{\left(\alpha_4+\alpha_5g'-g'^2\right)^2}. \tag{6.9}$$

In the following we state non-existence results:

**Theorem 6.3.** *There is not a translation surface of type 4 with $K\left(u\right)$ (or $K\left(v\right)$), where $K(u)$ is a non-constant function of $u$.*

*Proof.* It is by contradiction. We concentrate for the case $K = K(u)$ because the other is similar. Then (6.9) reduces to

$$-\epsilon\left(g''\right)^{-1} = \frac{\alpha_1\alpha_2/K(u) + (\alpha_1\alpha_3/K(u))\,g'}{\left(\alpha_4 + \alpha_5 g' - g'^2\right)^2}.$$

By derivativing with respect to $u$, we conclude

$$\sum_{n=0}^{3} P_n\left(g'\right)^n = 0,$$

where

$$\begin{aligned}
P_0 &= (\alpha_1\alpha_2/K(u))'\,\alpha_4 - 2(\alpha_1\alpha_2/K(u))\,\alpha_4',\\
P_1 &= (\alpha_1\alpha_2/K(u))'\,\alpha_5 + (\alpha_1\alpha_3/K(u))'\,\alpha_4 - 2(\alpha_1/K(u))\left(\alpha_2\alpha_5' + \alpha_3\alpha_4'\right),\\
P_2 &= -(\alpha_1\alpha_2/K(u))' + (\alpha_1\alpha_3/K(u))'\,\alpha_5 - 2(\alpha_1\alpha_3/K(u))\,\alpha_5',\\
P_3 &= -(\alpha_1\alpha_3/K(u))'.
\end{aligned}$$

Notice that $P_0, \ldots, P_4$ are zero all due to $g'' \neq 0$. It follows from $P_3 = 0$ that $\alpha_1\alpha_3 = aK(u)$, $a \in \mathbb{R} - \{0\}$ and from $P_2 = 0$ that $\alpha_1\alpha_2/K(u) + 2a\alpha_5 = b$, $b \in \mathbb{R}$. Therefore we have $\alpha_2/\alpha_3 + 2\alpha_5 = b/a$. Considering the values of $\alpha_1$, $\alpha_2$ and $\alpha_5$ deduces that $-f''p'/p'' + 3f' = b/a$, or equivalently

$$p' = c\left(b/a - 3f'\right)^{1/3},\ c \in \mathbb{R} - \{0\}. \tag{6.10}$$

Analogously, from $P_0 = 0$, we have $\alpha_1\alpha_2/K(u) = d\alpha_4^2$, $d \in \mathbb{R} - \{0\}$ and then $a\alpha_2 = d\alpha_3\alpha_4^2$. We substitute the values of $\alpha_2, \alpha_3, \alpha_4$ and write

$$af''p' + p''f' = -dp''\left(p'^2 - f'^2\right)^2. \tag{6.11}$$

We first divide (6.11) with $p''$. Then considering $-b/a + 3f' = f''p'p''$ together with (6.10) we derive a polynomial on $p'$ of degree 12 whose leading coefficient is $-d/(81c^{12})$. This is a contradiction. □

For the case $K = K(v)$, (6.9) reduces to

$$-\epsilon K(v)\left(g''\right)^{-1} = \frac{\alpha_1\left(\alpha_2 + \alpha_3 g'\right)}{\left(\alpha_4 + \alpha_5 g' - g'^2\right)^2}.$$

If we derivative with respect to $u$ then we obtain the same arguments in the previous proof by omitting the statement $K(u)$.

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