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Novel characterizations of rough soft sets: Equivalent soft sets in Pawlak approximation space with applications

Akın Osman Atagün

Department of Mathematics, Kırşehir Ahi Evran University, Kırşehir, 40100, Turkey

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ABSTRACT

Rough soft sets include both a partition set of alternatives and the parameters that classify the alternatives. Therefore, the decision-making models proposed on the rough soft set structure should be able to select the appropriate partition set and the appropriate alternative. In this paper, a novel concept of equivalent soft sets in an approximation space is introduced, and various characterizations of rough soft sets are given. Parametric and alternative equivalence numbers of a soft set in a Pawlak approximation space are assigned, and using these, three functions, namely, the set determinant parametric function, the alternative determinant parametric function, and the common determinant parametric function, are defined, and then a group decision-making method, *SetAltDM* on rough soft sets, is presented. The effectiveness of *SetAltDM* is supported by many comparative examples. Two novel similarity measures, P_R and weighted-P_R similarity methods, are proposed, and a similarity-based decision-making method with a real-life application about benchmarking is given.

1. Introduction

In the 1980s, Pawlak (1982) proposed rough set theory, which deals with problems that have inherent partitioning and then has become one of the important tools for modeling uncertainties. Today, machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, knowledge discovery, decision making, expert systems, artificial intelligence, cognitive sciences, and benchmarking research are areas where rough set theory is effectively used since the papers (Pawlak, 1981, 1982, 2002; Pawlak & Skowron, 2007a,b). Algebraic structures (Bonikowaski, 1995; Iwinski, 1987; Pomykala & Pomykala, 1998), graph theory (Park et al., 2019; Shah et al., 2018) and topology (Bloch, 2000; Lashin et al., 2005) related to rough sets have been studied. In (Atagün & Güneş, 2008), all components of a Pawlak approximation space were expressed with matrices, and the novel concept *equivalent matrices* was defined; then, using this concept, similarity and decision-making algorithms were proposed. In (Ma & Li, 2025a), covering similarity in a covering approximation space was introduced to study the rule of quantity characteristics of rough set theory, and then novel covering rough set models, novel fuzzy rough set models, and novel soft rough set models were defined.

From a completely different perspective from fuzzy set theory (Zadeh, 1965) and rough set theory (Pawlak, 1982), soft set theory, which is based on parameterized sets to model uncertainties, was put forward by Molodtsov (1999) in 1999. This approach, in short, provides a model that determines parameterized subsets of a set without any restrictions. Since it is expressed on a broad concept such as set structure and with an unrestricted function, soft set theory continues to be applied in many different areas such as the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, measurement theory and decision-making.

In (Aktaş & Çagman, 2007,?), soft sets were compared to fuzzy sets and rough sets, and the concept of a soft group was defined. Also, in Ali et al. (2009); Atagün and Aygün (2016); Maji et al. (2003); Pei and Miao (2005); Sezer et al. (2014, 2015); Sezgin and Atagün (2011), several effective operations and their algebraic properties on soft sets were introduced. In (Acar et al., 2010; Feng et al., 2008; Kazancıet al., 2010; Sezgin et al., 2011), algebraic structures and operations on soft sets were examined in detail. With the introduction of soft matrix theory to the literature by Çagman and Enginoglu (2010a), comprehensive similarity and decision-making models were produced (Atagün & Aygün, 2016; Atagün et al., 2018; Çagman & Enginoglu, 2010a; Kamacıet al., 2018; Maji et al., 2002; Petchimuthu et al., 2020). Soft sets and new structures based on soft sets are still used effectively in solving various decision-making problems (Kamacı et al., 2025; Khan et al., 2025a,b; Musa et al., 2025).

In real life, problems that involve uncertainty and are naturally decomposable fall within the scope of rough set theory. In solving these types of problems, using the advantages of soft set theory will, of course, increase the solution possibilities. Considering soft sets in a Pawlak

E-mail address: aosman.atagun@ahievran.edu.tr

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approximation space, the approximation of a soft set was proposed to obtain a hybrid model called rough soft sets in Feng et al. (2010). Motivated by Dubois and Prade's original idea about rough fuzzy sets (Dubois & Prade, 1990), the lower and upper approximations of a soft set in a Pawlak approximation space were designed, and also several properties of rough soft sets were given. Using both theories of bipolar soft set and rough set, in Shabir and Gul (2020), the modified rough bipolar soft set *MRBS*-set was defined. In (Sarwar & Akram, 2023), the notions of rough approximations, type-2 soft sets and fuzzy sets were integrated, and the concepts of rough fuzzy soft set, rough type-2 soft set and rough fuzzy type-2 soft set were introduced. Algebraic studies based on rough soft sets have also been done. In (Ali, 2011), concept of an approximation space associated with each parameter in a soft set, and an approximation space associated with the soft set were considered. An extended notion of a rough hemiring and a soft hemiring, rough soft hemiring, was presented in Zhan et al. (2015). The algebraic structures of rough soft rings and rough idealistic soft rings were introduced in Zhan and Davvaz (2016).

There are few decision-making and similarity methods proposed in the literature using only rough soft sets. In (Sarwar et al., 2023), the concept of rough approximations was related to other algebraic structures under a soft environment, and rough soft relations, rough soft graphs, and rough soft hypergraphs were introduced. Also, the importance of rough soft information was described with a group decision-making problem. In (Ma et al., 2017), two kinds of decision-making methods were illustrated for rough soft sets and provided some relevant applied examples, respectively. In (Liu et al., 2018), it was mentioned that the two decision-making methods proposed in Ma et al. (2017) on rough soft sets were limited; then their improved versions and also group decision-making were studied. In both of the papers Liu et al. (2018) and Ma et al. (2017), in fact, inverse soft sets are used due to the nature of the problems chosen. In the present study, soft sets are used in the given group decision-making method, so the partition set consists of alternatives. *Therefore, it becomes meaningless to make comparisons between all these methods.*

Also, unlike the methods given in Liu et al. (2018) and (Ma et al., 2017), the group decision-making method proposed in the present study is created by defining three functions that give the effective set of the partition, the alternative, and their joint selection. Furthermore, these results are presented in a ranking form, thus widening the choice ranges of decision-makers.

Expressing uncertain information in numbers is important because it brings them into a comparable form. The most effective method for this is similarity measurement. In particular, thanks to the similarity measure that enables complex data to be analyzed and compared with measurements such as how close to perfect and how far from the worst, it is possible to provide effective solutions to problems in many application areas such as market forecasting, pattern recognition, decision-making, machine learning, and medical diagnosis. Similarity measures defined solely on rough set theory are not frequently seen in the literature. Similarity measures are usually defined on hybrid structures containing rough sets, such as strait rough sets (Atagün & Kamacı, 2023b) and strait fuzzy rough sets (Atagün & Kamacı, 2023a). In fuzzy set theory, thanks to the many different similarity measures proposed, comparative studies between pairs of fuzzy sets have taken their place in the literature. In (Wang, 1997; Wang et al., 1995), similarity measures of fuzzy sets were presented with their basic properties. In (Beg & Ashraf, 2009), some axioms were laid down that give the similarity between two fuzzy subsets of an initial universe as a ratio or measure having a value in the closed interval [0, 1]. Some operations, such as summation and multiplication on the similarity measures of fuzzy sets, were investigated in Omran and Hassaballah (2007). Additionally, many effective similarity measures have been defined on hybrid structures, including fuzzy sets, rough sets, soft sets and their various versions such as on intuitionistic fuzzy sets (Kang et al., 2018), on fuzzy rough sets (Qi & Chengyi, 2008; Ye & Wu, 2009), and on interval-valued fuzzy sets (Deng et al., 2017; Li et al., 2015). In (Khan et al., 2025c), circular intuitionistic fuzzy preference relations (*C-IFPRs*), using circular intuitionistic fuzzy sets and preference relations, were defined; several entropy measures for *C-IFPRs* were given, and also new similarity measures were proposed to measure the resemblance between two *C-IFPRs*. In (Li et al., 2025), as an uncertainty measure, fuzzy neighborhood entropy-based information gain was proposed, and the similarity between samples in the same fuzzy neighborhood was considered to estimate neighborhood consistency. Since determining the similarity between datasets with distance/similarity measures is important in most clustering algorithms, in Zhang and Huang (2025) two clustering algorithms with improved distance measures, namely weighted probabilistic Euclidean distance and dynamic time warping-based probabilistic Euclidean distance, were characterized. In (Ma & Li, 2025b), novel covering rough set models, novel fuzzy rough set models and novel soft rough set models were defined by using the concept covering similarity in a covering approximation space. In (Atagün & Güneş, 2008), similarity methods and decision-making algorithms on rough sets without the need for hybrid structures were proposed. In (Liu et al., 2018), decision making on rough soft sets was studied using similarity measures defined on soft sets.

In this study, two new similarity methods, namely P_R and weighted-P_R similarity, defined on rough soft sets are proposed and a similarity-based decision-making process is presented using them.

1.1. The motivation and contribution

The main purpose of constructing a partition set of a set is to determine the elements of this set that are equivalent to each other according to a certain relation. The match of this in rough soft set theory is to determine the soft sets that are equivalent to each other. Therefore, in this study, firstly, the concept of equivalent soft sets in an approximation space $(U; R)$ is introduced. Especially in decision-making problems, using a partition of alternatives according to the importance of the findings to be investigated is an effective way to reach a more accurate result and also facilitates the analysis of the solution. For this reason, creating a decision-making mechanism with rough soft sets provides great convenience. In order to use all the advantages of rough soft set theory, a set determinant parametric function that gives the effective partition set, an alternative determinant parametric function that lists the feasible alternatives, and a common determinant parametric function that gives the feasible partition and the alternative together are defined. By means of these functions, a decision-making method *SetAltDM*, which expands the solution range of decision makers and includes three sequential decisions, is proposed. Moreover, the results obtained for both the partition set, the alternatives, and the common solution are given in order. This allows decision-makers to see the best choices or to determine the alternatives to be eliminated. In rough soft set theory, the similarity measures proposed so far have generally used similarity measures already defined on soft sets. In this study, P_R and weighted-P_R similarity measures are defined in accordance with the rough soft set structure, and thus a similarity-based decision-making method is proposed.

1.2. The authentic life motivation about rough soft sets and equivalent soft sets

In order to explain the advantages of using rough soft set theory and to provide motivation, a real-life example can be given to carry out search and rescue operations in the fastest and most effective way in a natural disaster. According to the data announced by AFAD (Disaster and Emergency Management Presidency), two earthquakes with magnitudes of Mw 7.7 (8.6 km deep) and Mw 7.6 (7.6 km deep) occurred on February 6, 2023, at 04.17 and 13.24 local time, with epicenters in Pazarcık (Kahramanmaraş) and Elbistan (Kahramanmaraş) (Türkiye). The 11 provinces directly affected by the earthquake are as follows:

Adana, Adıyaman, Diyarbakır, Elazığ, Gaziantep, Hatay, Kahramanmaraş, Kilis, Malatya, Osmaniye, Şanlıurfa. However, Batman, Bingöl, Kayseri, Mardin, Niğde, Sivas, Tunceli, which were also affected by the earthquakes as of June 2023, have been added to these provinces, increasing the number of damaged provinces (AFAD, 2023a). While the emergency response teams were being formed, professionals and volunteers who do the same job in many fields, such as health personnel, construction equipment operators, professional diggers, food service personnel, carrier personnel, etc., took part. Let U be all the search and rescue personnel. If $R \subseteq U \times U$ is a relation defined as $aRb \Leftrightarrow a$ and b are in the same work division, then R is an equivalence relation on U and $(U;R)$ be a Pawlak approximation space. Let the partition set $U/R = \{C_i | i = 1, 2, \dots, r\}$. If C_h is the set of all health personnel and $h_1, h_2 \in C_h$, then $h_1 R h_2$ and $h \in \{1, 2, \dots, r\}$. Now, it is necessary to form teams that will work in shifts on a 24-hour basis and to determine which team will respond to which type of debris. Many different rough soft sets can be created depending on need. For example, parameters can be determined for planning the team that will respond to the debris in neighborhood X, considering shifts, as: e_1 : Wreckage of a single-storey reinforced concrete building; e_2 : Wreckage of two or three storey reinforced concrete building; e_3 : Wreckage of four or six-storey reinforced concrete building; e_4 : Wreckage of reinforced concrete building with seven floors and higher. As a result of the examinations, a soft set is determined for the X neighborhood by determining the minimum required personnel and areas of expertise:

$$f_X = \{(e_1, \{h_1, d_1, v_1\}), (e_2, \{h_2, d_2, o_1, v_2\}), (e_3, \{h_3, d_3, d_4, o_2, v_3\}), (e_4, \{h_4, d_5, d_6, d_7, o_3, v_4, v_5\})\},$$

here h_i denotes health personnel, d_i denotes professional diggers, o_i denotes construction equipment operators, and v_i denotes volunteers. After this team has done its job for 8 hours, the shift of this team begins, and the work continues uninterrupted. The equivalent soft set structure defined in this study is a very useful method to determine the team that has a shift. If there is no shift with exactly the same number of members as the specified team, the team that is most similar to this team is determined as the shift using the P_R and weighted-P_R similarity methods suggested in this study.

The rest of this study is arranged as follows. Section 2 has brief information on the preliminaries related to soft sets, rough sets, and rough soft sets. Section 3 is dedicated to introducing a novel concept of equivalent soft sets in an approximation space $(U;R)$ and to giving various characterizations of rough soft sets. Section 4 presents parametric and alternative equivalence numbers of a soft set in a Pawlak approximation space and lays the foundation for the applications to be given in other sections. In Section 5, three functions namely, the set determinant parametric function, the alternative determinant parametric function, and the common determinant parametric function are defined, and then a decision-making method *SetAltDM* on rough soft sets is proposed with applications, using these functions. In Section 6, two novel similarity measures are proposed, namely P_R and weighted-P_R similarity methods, and a similarity-based decision-making method is given with an application. Section 7 presents some conclusions.

2. Preliminaries

Definition 1 (Çagman & Enginoglu, 2010b). Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$. A soft set (F, A) or simply f_A on the universe U is defined by the ordered pairs

$$(F, A) = \{(x, F(x)) | x \in E, F(x) \in P(U)\},$$

where $F : E \to P(U)$ such that $F(x) = \emptyset$ if $x \notin A$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2 (Pei & Miao, 2005). Let (F, A) and (G, B) be soft sets over U.

a) If $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$, then (F, A) is a soft subset of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$.
b) If $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$, then (F, A) and (G, B) is said to be soft equal and denoted by $(F, A) = (G, B)$.

Definition 3 (Ali et al., 2009). The relative complement of a soft set (F, A) is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^r, A)$, where $F^r : A \to P(U)$ is a mapping given by $F^r(x) = U - F(x)$, for all $x \in A$.

Definition 4. Let (F, A) and (G, B) be soft sets over U

a) The restricted intersection of (F, A) and (G, B) is a soft set denoted by $(F, A) \cap_R (G, B)$ and defined as

$$\{(x, F(x) \cap G(x)) | x \in A \cap B, F(x), G(x) \in P(U)\}.$$

(cf. Pei & Miao, 2005)
b) The restricted union of (F, A) and (G, B) is a soft set denoted by $(F, A) \cup_R (G, B)$ and defined as

$$\{(x, F(x) \cup G(x)) | x \in A \cap B, F(x), G(x) \in P(U)\}.$$

(cf. Ali et al., 2009)
c) Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The restricted difference of (F, A) and (G, B) is denoted by $(F, A) \sim_R (G, B)$, and is defined as $(F, A) \sim_R (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) - G(c)$. Ali et al. (2009)
d) The restricted symmetric difference of (F, A) and (G, B) is a soft set denoted by $(F, A)\widetilde{\triangle}(G, B) = [(F, A) \cup_R (G, B)] \sim_R [(F, A) \cap_R (G, B)]$. Sezgin and Atagün (2011)

Definition 5 (Feng et al., 2008). Let (F, A) be soft set over U. Then the set

$$supp(F, A) = \{x \in A | F(x) \neq \emptyset\}$$

is called the support of the soft set (F, A). The null soft set is a soft set with empty support and we denote it by $\emptyset_E$. A soft set (F, A) is called non-null if $supp(F, A) \neq \emptyset$.

Definition 6 (Gong et al., 2010). Let (F, A) be soft set over U such that A is a nonempty parameter set. We say that (F, A) is a bijective soft set, if (F, A) such that

a) $\bigcup_{e \in A} F(e) = U$ (i.e. (F, A) is a full soft set),
b) $F(e_i) \cap F(e_j) = \emptyset$ for all e_i, where $e_i \in A$ such that $e_i \neq e_j$.

Definition 7 (Pawlak, 1982). Let R be an equivalence relation on the universe U, then $(U;R)$ is a Pawlak approximation space. Let $X \subseteq U$. Two operations are defined as

$$R^*X = \{x \in U \mid [x]_R \cap X \neq \emptyset\} and R_*X = \{x \in U \mid [x]_R \subseteq X\},$$

where $[x]_R$ is an equivalence class of x. Then a subset X of U is called definable if $R^*X = R_*X$; in the opposite case, i.e., if $R^*X \setminus R_*X \neq \emptyset$, X is said to be a rough set.

Definition 8 (Feng et al., 2010). Let (U; R) be a Pawlak approximation space and $f_A = (F, A)$ be a soft set over U. The lower and upper rough approximations of f_A with respect to (U; R) are denoted by $R_*(f_A) = (F_*, A)$ and $R^*(f_A) = (F^*, A)$, respectively; which are soft sets over U with the set-valued mappings given by $F_*(x) = R_*(F(x))$ and $F^*(x) = R^*(F(x))$, where $x \in A$. The operators R_* and R^* are called the lower and upper rough approximation operators on soft sets. If $R_*(f_A) \neq R^*(f_A)$, then the soft set f_A is said to be a rough soft set; otherwise it is called a definable soft set in $(U;R)$.

3. Equivalent soft sets and characterizations of rough soft sets

Let $U = \{u_1, u_2, \ldots, u_m\}$ be an m-element initial universe set, R be an equivalence relation on the universe U, $[x]_R$ is an equivalence class of $x \in U$ and let $(U; R)$ be a Pawlak approximation space. Also, let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of all parameters and $A, B \subseteq E$.

Definition 9. Let $(U; R)$ be a Pawlak approximation space and f_A, f_B be soft sets over U. If $R_*(f_A) = R_*(f_B)$ and $R^*(f_A) = R^*(f_B)$, then f_A and f_B are called **equivalent soft sets** in the approximation space $(U; R)$, denoted by $f_A \equiv_R f_B$.

The null soft set is equivalent only to itself, according to the relation R.

Definition 9 leads directly to the following inference:

Proposition 1. *Let (U; R) be a Pawlak approximation space and f_A, f_B be soft sets over U and let $f_A \equiv_R f_B$. Then f_A is a rough soft set in (U; R) if and only if f_B is a rough soft set in (U; R).*

A different characterization on the equivalence of soft sets in a Pawlak approximation space is as follows:

Theorem 1. *Let $(U; R)$ be a Pawlak approximation space, E be a set of all parameters and f_A and f_B be soft sets over U.*

a) If $f_A = f_B$, then $f_A \equiv_R f_B$.
b) $f_A \neq f_B$ and $f_A \equiv_R f_B$ if and only if there exists at least an element $u_j \in (f_B(e_k) \cap [u_i]_R)$ such that $u_i \in f_A(e_k)$ and $u_i \notin f_B(e_k)$ for at least a parameter $e_k \in E$ such that $f_A(e_k) \neq \emptyset$ and $f_A(e_t) = f_B(e_t)$ for each $t \neq k$.

Proof. a) If $f_A = f_B$, then it is clear that $R_*(f_A) = R_*(f_B)$ and $R^*(f_A) = R^*(f_B)$. Then, $f_A \equiv_R f_B$, by Definition 9.

b) Assume that $f_A \neq f_B$ and $f_A \equiv_R f_B$. Without loss of generality, let us assume that f_A and f_B break the equality for the parameter $e_k \in E$ and that all other pairs are same. Since $R_*(f_A) = R_*(f_B)$ and $R^*(f_A) = R^*(f_B)$, then there exist $u_i, u_j \in U$ such that $u_j \in (f_B(e_k) \cap [u_i]_R)$, where $u_i \in f_A(e_k)$ and $u_i \notin f_B(e_k)$.

Now, assume that there exists at least an element $u_j \in (f_B(e_k) \cap [u_i]_R)$ such that $u_i \in f_A(e_k)$ and $u_i \notin f_B(e_k)$ for at least a parameter $e_k \in E$ such that $f_A(e_k) \neq \emptyset$ and $f_A(e_t) = f_B(e_t)$ for each $t \neq k$. Then, clearly $f_A \neq f_B$. If $(e_k, [u_i]_R) \in R_*(f_A)$ for the parameter $e_k \in E$, then $(e_k, [u_i]_R) \in R_*(f_B)$. Hence, $R_*(f_A) = R_*(f_B)$. Similar argument also shows that $R^*(f_A) = R^*(f_B)$. Therefore, $f_A \equiv_R f_B$, by Definition 9. □

Theorem 2. *Let $(U; R)$ be a Pawlak approximation space and f_A and f_B be soft sets over U. If $|f_A(e_i) \cap [u_j]_R| = |f_B(e_i) \cap [u_j]_R|$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, then $f_A \equiv_R f_B$. The converse is not true, in general.*

Proof. If $f_A = f_B$, it is clear that $|f_A(e_i) \cap [u_j]_R| = |f_B(e_i) \cap [u_j]_R|$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ and $f_A \equiv_R f_B$. Assume that $f_A \neq f_B$ and $|f_A(e_i) \cap [u_j]_R| = |f_B(e_i) \cap [u_j]_R|$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Then, there exists at least an element $u_j \in (f_B(e_k) \cap [u_i]_R)$ such that $u_i \in f_A(e_k)$ and $u_i \notin f_B(e_k)$ for at least a parameter $e_k \in E$. Therefore, $f_A \equiv_R f_B$ by Theorem 1 (b). As can be seen in the Example 3, $f_C \equiv_R f_D$, but $|f_C(e_1) \cap [u_2]_R| = 1 \neq 2 = |f_D(e_1) \cap [u_2]_R|$, therefore, the converse is not true, in general. □

For two integers a and b, if there exists an integer c such that $b = ac$, then a is said to divide b and is denoted by $a|b$. Let a, b, m be integers with $m > 0$. Then, a is congruent to b modulo m, denoted by $a \equiv b(mod m)$ and defined as $a \equiv b(mod m)$ if and only if $m|a - b$. It is well-known that the notion of congruence is an equivalence relation on the set of integers.

To clarify the following example, if a relation R is defined on a subset of integers $A = \{1, 2, \ldots, 8\}$ by $aRb \Leftrightarrow 3|a - b$, then for each $a \in A$, since $3|a - a$, then aRa, that's mean the reflection property is satisfied, and aRb, i.e. $3|a - b$, which implies $3| - (a - b) = b - a$, i.e. bRa, that's mean the symmetry property is satisfied. Finally, if aRb and bRc, i.e. $3|a - b$ and $3|b - c$, then $3|a - b + b - c = a - c$, i.e. aRc, that's mean the transitivity property is satisfied. Therefore, R is an equivalence relation on A. Accordingly, if the equivalence class of $1 \in A$ according to the equivalence relation R is denoted by $[1]_R$, then $[1]_R = \{1, 4, 7\}$ is obtained.

Example 1. Let $U = \{u_1, u_2, \ldots, u_8\}$ and $R \subseteq U \times U$ such that $u_i R u_j \iff 3|i - j$ for all $u_i, u_j \in U$. Then R is an equivalence relation on U. Let the Pawlak approximation space $(U; R)$. All the equivalence classes are $[u_1]_R = \{u_1, u_4, u_7\}$, $[u_2]_R = \{u_2, u_5, u_8\}$ and $[u_3]_R = \{u_3, u_6\}$. Let the set of all parameters $E = \{e_1, e_2, e_3, e_4\}$, $A = B = E$ and let the soft sets

$$f_A = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_1, u_3, u_6, u_7\}), (e_3, \{u_4, u_7, u_8\}), (e_4, \emptyset)\}$$

and

$$f_B = \{(e_1, \{u_2, u_3, u_4\}), (e_2, \{u_3, u_4, u_6, u_7\}), (e_3, \{u_1, u_4, u_5\}), (e_4, \emptyset)\}$$

over U. Then,
$R_*(f_A) = \{(e_1, \emptyset), (e_2, \{u_3, u_6\}), (e_3, \emptyset), (e_4, \emptyset)\} = R_*(f_B)$ and
$R^*(f_A) = \{(e_1, U), (e_2, \{u_1, u_3, u_4, u_6, u_7\}), (e_3, \{u_1, u_2, u_4, u_5, u_7, u_8\}),$
$(e_4, \emptyset)\} = R^*(f_B)$ are obtained. Therefore $f_A \equiv_R f_B$ in $(U; R)$.
The following calculations are given to support the Theorems 1 and 2.

The parts marked with different colors in Fig. 1 represent the sets that constitute the given partition of the set U. Then, it is seen that

$$|f_A(e_1) \cap [u_1]_R| = 1 = |f_B(e_1) \cap [u_1]_R|$$
$$|f_A(e_2) \cap [u_1]_R| = 2 = |f_B(e_2) \cap [u_1]_R|$$
$$|f_A(e_3) \cap [u_1]_R| = 2 = |f_B(e_3) \cap [u_1]_R|$$
$$|f_A(e_4) \cap [u_1]_R| = 0 = |f_B(e_4) \cap [u_1]_R|,$$

$$|f_A(e_1) \cap [u_2]_R| = 1 = |f_B(e_1) \cap [u_2]_R|$$
$$|f_A(e_2) \cap [u_2]_R| = 0 = |f_B(e_2) \cap [u_2]_R|$$
$$|f_A(e_3) \cap [u_2]_R| = 1 = |f_B(e_3) \cap [u_2]_R|$$
$$|f_A(e_4) \cap [u_2]_R| = 0 = |f_B(e_4) \cap [u_2]_R|$$

and

$$|f_A(e_1) \cap [u_3]_R| = 1 = |f_B(e_1) \cap [u_3]_R|$$
$$|f_A(e_2) \cap [u_3]_R| = 2 = |f_B(e_2) \cap [u_3]_R|$$
$$|f_A(e_3) \cap [u_3]_R| = 0 = |f_B(e_3) \cap [u_3]_R|$$
$$|f_A(e_4) \cap [u_3]_R| = 0 = |f_B(e_4) \cap [u_3]_R|.$$

It is seen that $|f_A(e_i) \cap [u_j]_R| = |f_B(e_i) \cap [u_j]_R|$ for all $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, 8\}$.
Also, since $f_A \neq f_B$ and there exists $u_4 \in (f_B(e_1) \cap [u_1]_R)$ such that $u_1 \in f_A(e_1)$ and $u_1 \notin f_B(e_1)$; there exists $u_4 \in (f_B(e_2) \cap [u_1]_R)$ such that $u_1 \in f_A(e_2)$ and $u_1 \notin f_B(e_2)$; there exists $u_1 \in (f_B(e_3) \cap [u_7]_R)$ such that $u_7 \in f_A(e_3)$ and $u_7 \notin f_B(e_3)$; there exists $u_5 \in (f_B(e_3) \cap [u_8]_R)$ such that $u_8 \in f_A(e_3)$ and $u_8 \notin f_B(e_3)$, then $f_A \equiv_R f_B$ in $(U; R)$ as Theorem 1(b).

In Example 1, the fact that $f_A \equiv_R f_B$ in $(U; R)$ means that in practice these soft sets can be used interchangeably, in other words, they will do the same job. Depending on the type of problem, this can be used to determine soft sets that will do the same job or to determine soft sets that are different from each other.

Using the notion of equivalence soft set, the following theorem provides a new characterization that, depending on the equivalence relation R on U, there is no need to use lower and upper approximations to determine whether a soft set f_A is a rough soft set in $(U; R)$.

Theorem 3. *Let $f_A = (F, A)$ be a soft set over U and R be an equivalence relation on U. Then, there exists at least one soft set $f_B = (G, B)$ over U such that $f_A \neq f_B$ and $f_A \equiv_R f_B$ if and only if f_A is a rough soft set in $(U; R)$.*

Proof. Let E be a set of all parameters. Assume that, there exists a soft set f_B over U such that $f_A \neq f_B$ and $f_A \equiv_R f_B$. Then, by Theorem 1, there exists at least an element $u_j \in (f_B(e_k) \cap [u_i]_R)$ such that $u_i \in f_A(e_k)$ and $u_i \notin f_B(e_k)$ for each $e_k \in E$ such that $f_A(e_k) \neq \emptyset$.

By Definition 8, since $R_*(f_A) = (F_*, A)$, then either $R_*(f_A) = \emptyset_E$ or $R_*(f_A) \neq \emptyset_E$.

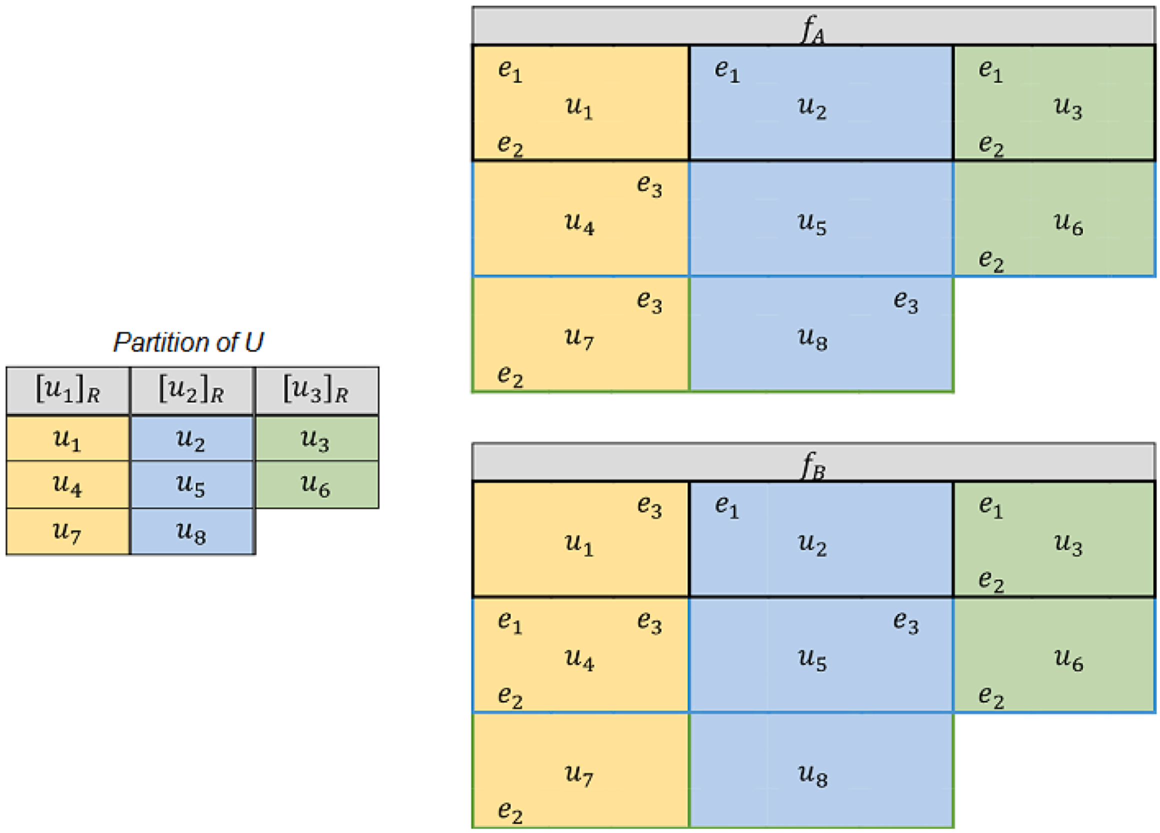


Fig. 1. Distribution of parameters on soft sets over partitions of the set U.

If $R_*(f_A) = (F_*, A) = \emptyset_E$, then f_A is a rough soft set, since $R_*(f_A)\widetilde{\subset} f_A \widetilde{\subseteq} R^*(f_A)$.

Assume that $R_*(f_A) \neq \emptyset_E$. Then, there exist elements $x \in A$ and $u_i \in U$ such that $[u_i]_R \subseteq F(x)$. Since $f_A \equiv_R f_B$, $[u_i]_R \subseteq G(x)$ and since $f_A \neq f_B$, there exists at least an element $u_j \in (G(y) \cap [u_i]_R)$ such that $u_i \in F(y)$, $u_j \notin F(y)$ and $u_i \notin G(y)$ for each $y \in E$ such that $F(y) \neq \emptyset$. Since, $u_j \notin F(y)$ and $G(y) \cap [u_i]_R \neq \emptyset$, then $(y, u_j) \notin R_*(f_A)$ and $(y, u_j) \in R^*(f_A) = R^*(f_B)$. Hence $R_*(f_A) \neq R^*(f_A)$, therefore f_A is a rough soft set in $(U; R)$.

Now, assume that f_A is a rough soft set in $(U; R)$. Then, there exist elements $x \in A$ and $u_i \in U$ such that $(x, u_i) \in R^*(f_A)$ and $(x, u_i) \notin R_*(f_A)$. If we construct a soft set $f_B = (G, B)$ over U such as:

- Take $A = B$,
- Write an element (x, u_s), $u_s \in [u_i]_R \setminus F(x)$, instead of the element (x, u_j), where $u_j \in F(x) \cap [u_i]_R$ and $u_s \neq u_j$,
- Take the other elements of f_A and f_B are the same.

Then clearly $f_A \neq f_B$, $R_*(f_A) = R_*(f_B)$ and $R^*(f_A) = R^*(f_B)$. Therefore, $f_A \equiv_R f_B$ by Definition 9. □

Example 2. Let the soft sets

$$f_A = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_1, u_3, u_6, u_7\}), (e_3, \{u_4, u_7, u_8\}), (e_4, \emptyset)\}$$

and

$$f_B = \{(e_1, \{u_2, u_3, u_4\}), (e_2, \{u_3, u_4, u_6, u_7\}), (e_3, \{u_1, u_4, u_5\}), (e_4, \emptyset)\}$$

from Example 1. Then, it is seen that $f_A \neq f_B$, $f_A \equiv_R f_B$ and f_A is a rough soft set in $(U; R)$. Also, f_B is a rough soft set in $(U; R)$ by Proposition 1.

Theorem 4. *Let f_A and f_B be a soft sets over U and R be an equivalence relation on U. Then, $f_A \equiv_R f_B$ if and only if $f_A^r \equiv_R f_B^r$.*

Proof. Let $f_A \equiv_R f_B$. By Definition 3, if $f_A = f_B$, then $f_A^r = f_B^r$ implies that $f_A^r \equiv_R f_B^r$, by Theorem 1 (a). Assume that $f_A \neq f_B$, then there exists at least an element $u_j \in (f_B(e_k) \cap [u_i]_R)$ such that $u_i \in f_A(e_k)$ and $u_i \notin f_B(e_k)$ for at least a parameter $e_k \in E$ such that $\emptyset \neq f_A(e_k) \subsetneq U$ by Theorem 1 (b). Then, $f_A^r \neq f_B^r$ and there exists at least an element $u_j \in (f_B^r(e_k) \cap [u_i]_R)$ such that $u_i \in f_A^r(e_k)$ and $u_i \notin f_B^r(e_k)$ for at least a parameter $e_k \in E$ such that $\emptyset \neq f_A^r(e_k) \subsetneq U$. Here, the equality of the lower and upper approximations is easily seen for the cases $f_A(e_k) = \emptyset$ and $f_A(e_k) = U$, where $e_k \in E$. Therefore, $f_A^r \equiv_R f_B^r$, by Theorem 1 (b). The reverse entailment of the proof is obtained in the same way. □

4. Parametric and alternative equivalence numbers of a soft set in the Pawlak approximation space

Let $U = \{u_1, u_2, \ldots, u_m\}$ be an m-element initial universe set, R be an equivalence relation on the universe U, $E = \{e_1, e_2, \ldots, e_n\}$ be the set of all parameters, $A \subseteq E$ and let $U/R = \{C_i | i = 1, 2, \ldots, r\}$ is a partition of U. $|A|$ denotes the cardinality of the set A. Then, we have following:

Definition 10. Let $f_A = (F, A)$ be a soft set over U and let $(U; R)$ be a Pawlak approximation space.

a. The set $P^i(f_A) = \{x \in A | C_i \cap F(x) \neq \emptyset\} \subseteq A$ is called the i.th **upper parametric set** of f_A in $(U; R)$.
b. The set $P_i(f_A) = \{x \in A | C_i \subseteq F(x)\} \subseteq A$ is called the i.th **lower parametric set** of f_A in $(U; R)$.
c. The numbers $|P^i(f_A)|$ and $|P_i(f_A)|$ are called the i.th **upper** and **lower parametric equivalence numbers** of f_A in $(U; R)$, respectively.

d. The set $A^i(f_A) = \{C_i \cap F(x) | x \in A\} \subseteq C_i$ is called the i.th **alternative set** of f_A in $(U;R)$.
e. The number $|A^i(f_A)|$ is called the i.th **alternative equivalence number** of f_A in $(U;R)$.

Lemma 1. *Let $f_A = (F,A)$ and $f_B = (G,B)$ be soft sets over U, $A,B \subseteq E = \{e_1,e_2,\dots,e_n\}$, $(U;R)$ be a Pawlak approximation space and let $U/R = \{C_i | i = 1,2,\dots,r\}$.*

a. If $f_A \equiv_R f_B$, then $P^i(f_A) = P^i(f_B)$ for each $i = 1,2,\dots,r$. The converse is not true, in general.
b. If $f_A \equiv_R f_B$, then $P_i(f_A) = P_i(f_B)$ for each $i = 1,2,\dots,r$. The converse is not true, in general.
c. If $|A^i(f_A)| = |A^i(f_B)|$ for each $i = 1,2,\dots,r$, then $f_A \equiv_R f_B$. The converse is not true, in general.
d. $A^k(f_A) = \bigcup_{i=1}^n (f_A(e_i) \cap C_k)$ for each $k = 1,2,\dots,r$.
e. If $f_A \tilde{\subseteq} f_B$, then $P^i(f_A) \subseteq P^i(f_B)$ for each $i = 1,2,\dots,r$.
f. If $f_A \tilde{\subseteq} f_B$, then $P_i(f_A) \subseteq P_i(f_B)$ for each $i = 1,2,\dots,r$.
g. If $f_A \tilde{\subseteq} f_B$, then $|A^i(f_A)| \leq |A^i(f_B)|$ for each $i = 1,2,\dots,r$.
h. $|P^i(f_A)| \leq |A|$ for each $i = 1,2,\dots,r$ and then $\sum_{i=1}^r |P^i(f_A)| \leq r|A|$.
i. $|P_i(f_A)| \leq |P^i(f_A)| \leq |A|$ for each $i = 1,2,\dots,r$ and then $\sum_{i=1}^r |P_i(f_A)| \leq \sum_{i=1}^r |P^i(f_A)| \leq r|A|$.
j. For absolute soft set $U_E = \{(e_i,U) : i = 1,2,\dots,r\}$, $P^i(U_E) = P_i(U_E) = E$ for each $i = 1,2,\dots,r$.
k. For null soft set $\emptyset_E = \{(e_i,\emptyset) : i = 1,2,\dots,r\}$, $P^i(\emptyset_E) = P_i(\emptyset_E) = \emptyset$ for each $i = 1,2,\dots,r$.
l. $|A^i(f_A)| \leq |C_i|$ for each $i = 1,2,\dots,r$ and then $\sum_{i=1}^r |A^i(f_A)| \leq |U|$.

Proof. a) Since $f_A \equiv_R f_B$, then $R_*(f_A) = R_*(f_B)$ and $R^*(f_A) = R^*(f_B)$, by Definition 9. Assume that for a $k \in \{1,2,\dots,r\}$, $P^k(f_A) \neq P^k(f_B)$. Then, the first situation is $C_k \subseteq F(x)$ and $P^k(f_B) = \emptyset$, for $\exists x \in A$. Then, $P^k(f_A) \neq P^k(f_B)$ and $R_*(f_A) \neq R_*(f_B)$ is a contradiction. The other situation is $x \in P^k(f_B) \setminus P^k(f_A)$. Then $R^*(f_A) \subsetneq R^*(f_B)$ and it is a contradiction of $R^*(f_A) = R^*(f_B)$. Therefore $P^i(f_A) = P^i(f_B)$ for each $i = 1,2,\dots,r$. For the converse, let

$$f_A = \{(e_1,\{u_1,u_2,u_3\}),(e_2,\{u_1,u_3,u_6,u_7\}),(e_3,\{u_4,u_7,u_8\}),(e_4,\emptyset)\}$$

in Example 1 and let another soft set

$$f_G = \{(e_1,\{u_1,u_2,u_3\}),(e_2,\{u_1,u_6,u_7\}),(e_3,\{u_4,u_7,u_8\}),(e_4,\emptyset)\}$$

in $(U;R)$. Then, it is seen that $P^i(f_A) = P^i(f_G)$ for each $i = 1,2,3$, but f_A is not equivalent to f_G in $(U;R)$.
Similarly, to prove the converse of (b) is not true, in general, we take the soft set

$$f_H = \{(e_1,\{u_1,u_2,u_3\}),(e_2,\{u_3,u_6\}),(e_3,\{u_4,u_7,u_8\}),(e_4,\emptyset)\}$$

in $(U;R)$. Then, it is seen that $P_i(f_A) = P_i(f_H)$ for each $i = 1,2,3$, but f_A is not equivalent to f_H in $(U;R)$.
The example 3 proves the converse of (c) is not true, in general. The remaining part of the proof is left to the reader since it is done in a similar manner. □

Example 3. From Example 1, the partition of U is the set $U/R = \{C_i | i = 1,2,3\}$, where $C_1 = \{u_1,u_4,u_7\}$, $C_2 = \{u_2,u_5,u_8\}$ and $C_3 = \{u_3,u_6\}$. For the soft sets

$$f_A = \{(e_1,\{u_1,u_2,u_3\}),(e_2,\{u_1,u_3,u_6,u_7\}),(e_3,\{u_4,u_7,u_8\}),(e_4,\emptyset)\}$$

and

$$f_B = \{(e_1,\{u_2,u_3,u_4\}),(e_2,\{u_3,u_4,u_6,u_7\}),(e_3,\{u_1,u_4,u_5\}),(e_4,\emptyset)\}$$

over U then,

i	1	2	3
$P^i(f_A)$	$\{e_1,e_2,e_3\}$	$\{e_1,e_3\}$	$\{e_1,e_2\}$
$\|P^i(f_A)\|$	3	2	2
$P_i(f_A)$	$\emptyset$	$\emptyset$	$\{e_2\}$
$\|P_i(f_A)\|$	0	0	1
$P^i(f_B)$	$\{e_1,e_2,e_3\}$	$\{e_1,e_3\}$	$\{e_1,e_2\}$
$\|P^i(f_B)\|$	3	2	2
$P_i(f_B)$	$\emptyset$	$\emptyset$	$\{e_2\}$
$\|P_i(f_B)\|$	0	0	1
$A^i(f_A)$	$\{u_1,u_4,u_7\}$	$\{u_2,u_8\}$	$\{u_3,u_6\}$
$\|A^i(f_A)\|$	3	2	2
$A^i(f_B)$	$\{u_1,u_4,u_7\}$	$\{u_2,u_5\}$	$\{u_3,u_6\}$
$\|A^i(f_B)\|$	3	2	2

Then, it is seen that $P^i(f_A) = P^i(f_B)$, $|A^i(f_A)| = |A^i(f_B)|$ for each $i = 1,2,\dots,r$ and $f_A \equiv_R f_B$. To show that the converse of the implication given by Lemma 1 (c) is not true, in general, let us take the following soft sets:

$$f_C = \{(e_1,\{u_1,u_2,u_4,u_7\}),(e_2,\{u_2,u_3,u_6\}),(e_3,\{u_1\}),(e_4,\{u_3,u_5,u_8\})\}$$

and

$$f_D = \{(e_1,\{u_1,u_4,u_5,u_7,u_8\}),(e_2,\{u_3,u_5,u_6,u_8\}),(e_3,\{u_4,u_7\}),(e_4,\{u_5,u_6\})\}$$

over U, we have

i	1	2	3
$P^i(f_C)$	$\{e_1,e_3\}$	$\{e_1,e_2,e_4\}$	$\{e_2,e_4\}$
$\|P^i(f_C)\|$	2	3	2
$P_i(f_C)$	$\{e_1\}$	$\emptyset$	$\{e_2\}$
$\|P_i(f_C)\|$	1	0	1
$P^i(f_D)$	$\{e_1,e_3\}$	$\{e_1,e_2,e_4\}$	$\{e_2,e_4\}$
$\|P^i(f_D)\|$	3	2	2
$P_i(f_D)$	$\{e_1\}$	$\emptyset$	$\{e_2\}$
$\|P_i(f_D)\|$	1	0	1
$A^i(f_C)$	$\{u_1,u_4,u_7\}$	$\{u_2,u_5,u_8\}$	$\{u_3,u_6\}$
$\|A^i(f_C)\|$	3	3	2
$A^i(f_D)$	$\{u_1,u_4,u_7\}$	$\{u_5,u_8\}$	$\{u_3,u_6\}$
$\|A^i(f_D)\|$	3	2	2

Since

$$R_*(f_C) = \{(e_1,\{u_1,u_4,u_7\}),(e_2,\{u_3,u_6\}),(e_3,\emptyset),(e_4,\emptyset)\} = R_*(f_D)$$

and

$$R^*(f_C) = \{(e_1,\{u_1,u_2,u_4,u_5,u_7,u_8\}),(e_2,\{u_2,u_3,u_5,u_6,u_8\}),(e_3,\{u_1,u_4,u_7\}),(e_4,\{u_2,u_3,u_5,u_6,u_8\})\} = R^*(f_D),$$

then $f_C \equiv_R f_D$, but $|A^2(f_C)| = 3 \neq 2 = |A^2(f_D)|$. Hence, the converse of Lemma 1 (c) is not true, in general.

5. A group decision making method: Set and alternative determinant parametric functions

There are several advantages to constructing the decision-making method on rough soft sets. Since rough soft sets indicate that a soft set belongs to a Pawlak approximation space, decision makers can choose both the set that creates the partition and the appropriate alternative by means of the parameters.

In this section, firstly, two functions are defined that will help us to determine the appropriate equivalence class and suitable alternatives and rank them, depending on the parameters. Using these functions, a new function is defined that shows which alternative in which equivalence class is appropriate to choose. Also, this implication is supported with a real life example.

Let $U = \{u_1, u_2, \dots, u_m\}$ be an m-element initial universe set, R be an equivalence relation on the universe U, $E = \{e_1, e_2, \dots, e_n\}$ be the set of all parameters, $A_1, A_2, \dots, A_k \subseteq E$ and let $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U.

The set of all soft sets in Pawlak approximation space $(U; R)$ will be denoted by $S(U; R)$.

The order of the sets forming the partition in the set $U/R = \{C_i | i = 1, 2, \dots, r\}$ is not important. However, during the process of solving the decision problem, the initially chosen order should not be changed. The following definition provides a function that helps us choose which of the sets forming the partition is effective in making a decision.

Definition 11. Let $U/R = \{C_i | i = 1, 2, \dots, r\}$ and let $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$. Then, the ith **set determinant parametric function** $Sdpf(i) : \prod_k S(U; R) \to [0, 1]$ is defined as

$$Sdpf(i)(f_{A_1}, f_{A_2}, \dots, f_{A_k}) = 1 - \frac{\sum_{s<t;s,t=1}^{k} ||P^i(f_{A_s})| - |P^i(f_{A_t})|| \cdot (|A^i(f_{A_s})| + |A^i(f_{A_t})|)}{2|C_i||E||U/R|},$$

for all $i \in \{1, 2, \dots, r\}$. The number $Sdpf(i)(f_{A_1}, f_{A_2}, \dots, f_{A_k})$ is called the ith **set determinant parametric number**.

Definition 12. Let $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$ and $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U. Then, the **optimum set of U/R** is the set,

$$Opt(U/R) = \{C_i \in U/R : Sdpf(i)(x) \geq Sdpf(j)(x) \quad \text{for all} i \neq j\},$$

where $x = (f_{A_1}, f_{A_2}, \dots, f_{A_k})$.

Also, we can get ranking order of sets in U/R, as

$$C_{i_1} \geq C_{i_2} \geq \dots \geq C_{i_r} \text{if} Sdpf(i_1)(x) \geq Sdpf(i_2)(x) \geq \dots \geq Sdpf(i_r)(x).$$

With the functions $Sdpf(i)$, k-decision makers can determine which set of partitions they use is more effective and a ranking of the sets in U/R is obtained according to these effects. While most decision-making methods determine alternatives or capabilities, this function enables a two-stage decision-making process to be carried out.

Definition 13. Let $U = \{u_1, u_2, \dots, u_m\}$ be an m-element initial universe set, $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$ and $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U. Then, for all $u_i \in U$, the subset

$$E_{A_j}(u_i) = \{x \in A_j : u_i \in f_{A_j}(x)\}$$

of A_j for each $j \in \{1, 2, \dots, k\}$, is called **the parameter set of the i-th alternative**.

With the help of the function below, it is possible to choose the appropriate one among the alternatives.

Definition 14. Let $U = \{u_1, u_2, \dots, u_m\}$ be an initial universe set, $E = \{e_1, e_2, \dots, e_n\}$ be the set of all parameters, $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$ and $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U.Then, the **alternative determinant parametric function** $Adpf : U \to [0, 1]$ is defined as

$$Adpf(u_i) = \frac{\sum_{j=1}^{k} |E_{A_j}(u_i)|}{|E|k}.$$

The number $Adpf(u_i)$ is called the **alternative determinant parametric number** for $u_i \in U$.

Definition 15. Let $U = \{u_1, u_2, \dots, u_m\}$ be an m-element initial universe set, $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$ and $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U. Then, the **optimum set of U** is the set,

$$Opt(U) = \{u_i \in U : Adpf(u_i) \geq Adpf(u_j) \quad \text{for} \quad \text{all} \quad i \neq j\}.$$

Also, we can get ranking order of sets in U, as

$$u_{i_1} \geq u_{i_2} \geq \dots \geq u_{i_m} \text{if} Adpf(u_{i_1}) \geq Adpf(u_{i_2}) \geq \dots \geq Adpf(u_{i_m}).$$

The following function allows us to evaluate the optimum partition set and the optimum alternative set together. Therefore, this function provides the final decision determining which alternative in which equivalence class is suitable. At the same time, since it also provides a ranking, alternative solutions are also suggested to decision makers.

Definition 16. Let $U = \{u_1, u_2, \dots, u_m\}$ be an initial universe set, $E = \{e_1, e_2, \dots, e_n\}$ be the set of all parameters, $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$ and $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U. Then, the **common determinant parametric function** $Cdpf : U \to [0, 1]$ is defined as

$$Cdpf(u_j) = Adpf(u_j) \cdot Sdpf(i)(f_{A_1}, f_{A_2}, \dots, f_{A_k}),$$

where $u_j \in C_i$.

The number $Cdpf(u_i)$ is called the **common determinant parametric number** for $u_i \in U$.

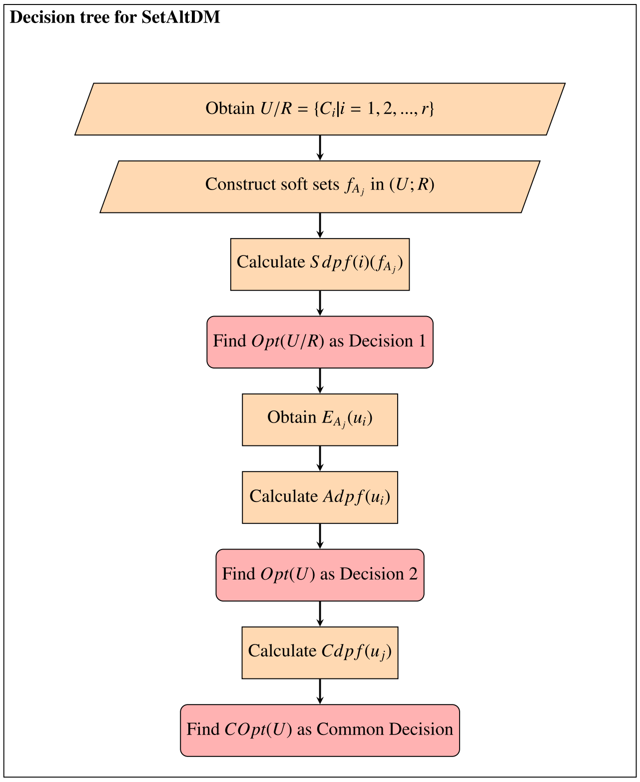
Definition 17. Let $U = \{u_1, u_2, \dots, u_m\}$ be an m-element initial universe set, $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ be soft sets in $(U; R)$ and $U/R = \{C_i | i = 1, 2, \dots, r\}$ is a partition of U. Then, the **common optimum set of U** is the set,

$$COpt(U) = \{u_j \in U : Cdpf(u_j) \geq Cdpf(u_i) \quad \text{for} \quad \text{all} \quad i \neq j\}.$$

Also, we can get ranking order of sets in U, as

$$u_{i_1} \geq u_{i_2} \geq \dots \geq u_{i_m} \text{if} Cdpf(u_{i_1}) \geq Cdpf(u_{i_2}) \geq \dots \geq Cdpf(u_{i_m}).$$

The method: Using set and alternative determinant parametric functions, we construct the **set and alternative deterministic decision making method**(*SetAltDM*) as: k decision makers form their soft sets: $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ in $(U; R)$. Then, ith set determinant parametric number $Sdpf(i)(f_{A_1}, f_{A_2}, \dots, f_{A_k})$ is calculated for each $i \in \{1, 2, \dots, r\}$. Then, the optimum set of U/R and the ranking of $C_i \in U/R$ are obtained. Then, for all $u_i \in U$, the parameter set of the ith alternative $E_{A_j}(u_i)$, for each $j \in \{1, 2, \dots, k\}$ is obtained. For all $u_i \in U$, using alternative determinant parametric number $Adpf(u_i)$, the optimum set of U and the ranking of $u_i \in U$ are obtained. Then, common determinant parametric number $Cdpf(u_i)$, for all $u_i \in U$, is calculated and finally, common optimum set *COpt(U)* of U and the ranking of $u_i \in U$ are obtained.



Decision Making Algorithm for ***SetAltDM***

Step 1. k decision makers obtain a partition $U/R = \{C_i|i = 1, 2, \ldots, r\}$ of U to select optimum set of U/R and optimum alternatives of the set of all alternatives $U = \{u_1, u_2, \ldots, u_m\}$.
Step 2. k decision makers form their soft sets: $f_{A_1}, f_{A_2}, \ldots, f_{A_k}$ in $(U; R)$.
Step 3. ith set determinant parametric number $Sdpf(i)(f_{A_1}, f_{A_2}, \ldots, f_{A_k})$ is calculated for each $i \in \{1, 2, \ldots, r\}$.
Step 4. The optimum set $Opt(U/R)$ of U/R and the ranking of $C_i \in U/R$ are obtained.(*Decision on sets*).
Step 5. For all $u_i \in U$, the parameter set of the ith alternative $E_{A_j}(u_i)$, for each $j \in \{1, 2, \ldots, k\}$ is obtained.
Step 6. For all $u_i \in U$, alternative determinant parametric number $Adpf(u_i)$ is calculated.
Step 7. The optimum set $Opt(U)$ of U and the ranking of $u_i \in U$ are obtained. (*Decision on alternatives*).
Step 8. For all $u_j \in U$, common determinant parametric number $Cdpf(u_j)$ is calculated.
Step 9. The common optimum set $COpt(U)$ of U and the ranking of $u_i \in U$ are obtained. (*Common decision*).

Example 4. Mr. X and Mrs. Y want to buy a house on 417th, 418th, 419th, or 420th streets because of its proximity to their workplaces. There are a total of 10 houses for sale on these streets. The set of houses for sale on 417th street is $C_1 = \{h_1, h_2, h_3\}$, on 418th street is $C_2 = \{h_4, h_5\}$, on 419th street is $C_3 = \{h_6, h_7\}$ and on 420th street is $C_4 = \{h_8, h_9, h_{10}\}$. Then, **(Step 1)** $U/R = \{C_i|i = 1, 2, 3, 4\}$ is a partition of $U = \{h_1, h_2, \ldots, h_{10}\}$ and $(U; R)$ is a Pawlak approximation space.

They determine *all their common parameters* for the house they want as follows: e_1 : Made in the last 5 years; e_2 : Wide; e_3 : South facing; e_4 : It has a big kitchen; e_5 : With en-suite bathroom; e_6 : With a garden and e_7 : Close to shopping malls and then, the set of all parameters is $E = \{e_1, e_2, \ldots, e_7\}$. Since the common parameters are selected, Mr. X and Mrs. Y determine the soft sets $f_A = (F, A)$ and $f_B = (G, B)$, respectively, according to the houses that they think provide these parameters, where $A = B = E$.

Mr. X and Mrs. Y examine all the houses and create the soft sets f_A and f_B, respectively, according to the condition that their parameters are met, as follows **(Step 2)**:

$$f_A = \{(e_1, \{h_1, h_2, h_5, h_7, h_9\}), (e_2, \{h_1, h_3, h_4, h_6, h_8, h_9\}), (e_3, \{h_2, h_4, h_7, h_8, h_{10}\}), (e_4, \{h_1, h_2, h_4, h_5, h_7, h_8, h_{10}\}), (e_5, \{h_4, h_6, h_7, h_8, h_9\}), (e_6, \{h_3, h_5, h_6, h_7, h_8\}), (e_7, \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\})\} \text{ and}$$

$$f_B = \{(e_1, \{h_1, h_2, h_5, h_7, h_9\}), (e_2, \{h_2, h_3, h_4, h_5, h_7, h_8, h_{10}\}), (e_3, \{h_2, h_7, h_9\}), (e_4, \{h_2, h_5, h_8, h_9, h_{10}\}), (e_5, \{h_4, h_6, h_7, h_8, h_9\}), (e_6, \{h_2, h_5, h_7, h_8, h_9\}), (e_7, \{h_1, h_2, h_3, h_6, h_7\})\} \text{ over } U.$$

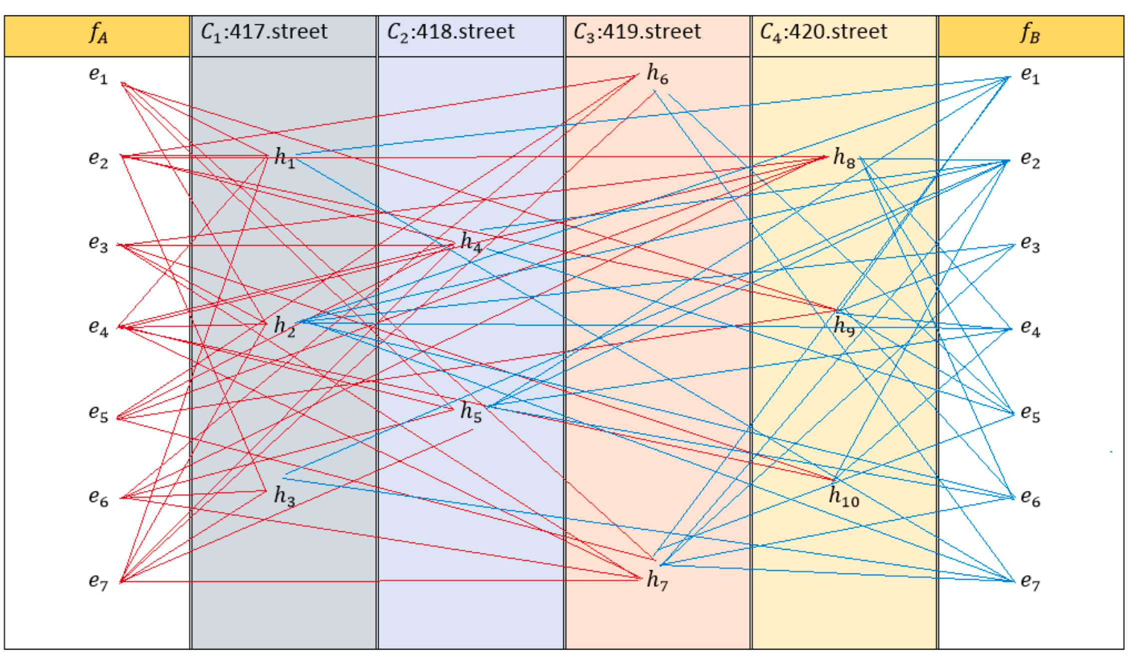


Fig. 2. Distribution of parameters on soft sets over partitions of the set U.

Fig. 2 also states why both the alternatives, the appropriate set of partitions, and a common solution that encompasses them should be proposed. Accordingly, Mr. X and Mrs. Y have to choose which street is more suitable for buying a house and also the house that meets their common criteria to the maximum extent.

i	1	2	3	4
$P^i(f_A)$	$\{e_1,e_2,e_3,e_4,e_6,e_7\}$	$\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$	$\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$	$\{e_1,e_2,e_3,e_4,e_5,e_6\}$
$\lvert P^i(f_A)\rvert$	6	7	7	6
$P^i(f_B)$	$\{e_1,e_2,e_3,e_4,e_6,e_7\}$	$\{e_1,e_2,e_4,e_5,e_6\}$	$\{e_1,e_2,e_3,e_5,e_6,e_7\}$	$\{e_1,e_2,e_3,e_4,e_5,e_6\}$
$\lvert P^i(f_B)\rvert$	6	5	6	6
$\lvert A^i(f_A)\rvert$	3	2	2	3
$\lvert A^i(f_B)\rvert$	3	2	2	3

(Step 3) ith set determinant parametric number $Sdpf(i)(f_A,f_B) = 1 - \frac{||P^i(f_A)|-|P^i(f_B)||(|A^i(f_A)|+|A^i(f_B)|)}{2\cdot|C_i|\cdot 7\cdot 4}$ is calculated for each $i \in \{1,2,3,4\}$ as:

i	1	2	3	4
$Sdpf(i)(f_A,f_B)$	1	0.92	0.96	1

Then, we have the decision on sets **(Step 4)**, the optimum set
$Opt(U/R) = \{C_i \in U/R : Sdpf(i)(x) \geq Sdpf(j)(x) \quad \text{for} \quad \text{all} \quad i \neq j\} = \{C_1 = C_4, C_3, C_2\}$
of U/R and the ranking of $C_i \in U/R$ is $C_1 = C_4 > C_3 > C_2$ are obtained. This means that the first streets to be chosen are 417th and 420th streets, then 419th street and finally 418th street.
Then, **(Step 5)**, the element numbers $|E_A(h_i)|$ and $|E_B(h_i)|$ of the parameter sets of the ith alternative are determined as follows:

i	1	2	3	4	5	6	7	8	9	10
$\lvert E_A(h_i)\rvert$	4	4	3	5	4	4	6	5	3	2
$\lvert E_B(h_i)\rvert$	2	2	2	2	4	2	6	4	5	2

(Step 6) For all $h_i \in U$, alternative determinant parametric number $Adpf(h_i) = \frac{|E_A(h_i)|+|E_B(h_i)|}{7\cdot 2}$ is calculated as follows:

i	1	2	3	4	5	6	7	8	9	10
$Adpf(h_i)$	0.42	0.42	0.35	0.5	0.57	0.42	0.85	0.64	0.57	0.28

Then, we have the decision on alternatives **(Step 7)**, the optimum set is
$Opt(U) = \{h_i \in U : Adpf(h_i) \geq Adpf(h_j) \quad \text{for} \quad \text{all} \quad i \neq j\} = \{h_7, h_8, h_5 = h_9, h_4, h_1 = h_2 = h_6, h_3, h_{10}\}$
and the ranking of $h_i \in U$ is $h_7 > h_8 > h_5 = h_9 > h_4 > h_1 = h_2 = h_6 > h_3 > h_{10}$.
Since $C_1 = \{h_1,h_2,h_3\}$, $C_2 = \{h_4,h_5\}$, $C_3 = \{h_6,h_7\}$ and $C_4 = \{h_8,h_9,h_{10}\}$, then **(Step 8)** for all $h_j \in U$, common determinant parametric number $Cdpf(h_j) = Adpf(h_j)Sdpf(i)(f_A,f_B)$, where $h_j \in C_i$, as

i	1	2	3	4	5	6	7	8	9	10
$Cdpf(h_i)$	0.42	0.42	0.35	0.46	0.52	0.4	0.81	0.64	0.57	0.28

Then, we have the common decision on alternatives **(Step 9)**, the common optimum set is
$COpt(U) = \{h_j \in U : Cdpf(h_j) \geq Cdpf(h_i) \quad \text{for} \quad \text{all} \quad i \neq j\} = \{h_7, h_8, h_5 = h_9, h_4, h_1 = h_2 = h_6, h_3, h_{10}\}$
and the ranking of $h_i \in U$ is $h_7 > h_8 > h_5 = h_9 > h_4 > h_1 = h_2 = h_6 > h_3 > h_{10}$.

Fig. 3 also indicates which house on which street is more suitable.
As a result, a decision is made for the set and alternatives, and it is seen that h_7, located on 419th street, is the most suitable house to buy.

Notes on comparisons: As noted in the Introduction, comparing *SetAltDM* with the methods given in Liu et al. (2018) and Ma et al. (2017) is logically meaningless. Because Liu et al. (2018) and Ma et al. (2017) used inverse soft sets in their methods, but soft sets are used in *SetAltDM*. This means that the partition set is chosen from the parameters (capabilities) in the methods of both Liu et al. (2018) and Ma et al. (2017) but from the alternatives (candidates) in *SetAltDM*.

A comparative example

To discuss the validity, advantages and disadvantages of *SetAltDM*, the following example is obtained by adapting Example 6 and Example 7 from the paper (Atagün & Kamacı, 2023b).

Example 5 (adapted from Atagün and Kamacı (2023b), Examples 6. and 7.). Let $U = \{u_1, u_2, \dots, u_8\}$ be a set of some smartphones released in 2021. The sets of best-selling smartphones of these smartphones in the countries C_1, C_2 and C_3 are, respectively, X_1, X_2 and X_3 as:
$X_1 = \{u_1,u_3,u_7,u_8\}, X_2 = \{u_1,u_5,u_6,u_8,\}, X_3 = \{u_2,u_5,u_7\}$.

An expert group will first determine some attributes or parameters for the smartphones in the set U and then decide which attributes can be recommended for the smartphones that will be manufactured in the future, considering the best sellers of these smartphones in the countries C_1, C_2 and C_3.
The sets of attributes for different colors, screen technologies, materials and screen types are, respectively,
$E_1 = \{e^1_1 : \text{light color}, e^1_2 : \text{dark color}, e^1_3 : \text{golden silver}, e^1_4 : \text{mixed color}\}$,
$E_2 = \{e^2_1 : \text{OLED}, e^2_2 : \text{AMOLED}, e^2_3 : \text{Super AMOLED}, e^2_4 : \text{TFT LCD}, e^2_5 : \text{IPS LCD}, e^2_6 : \text{PLS LCD}\}$,
$E_3 = \{e^3_1 : \text{plastic glass}, e^3_2 : \text{plastic aluminum}, e^3_3 : \text{polycarbonate glass}, e^3_4 : \text{aluminum glass}, e^3_5 : \text{aluminum polycarbonate}\}$,
$E_4 = \{e^4_1 : \text{dual screen}, e^4_2 : \text{classic single}, e^4_3 : \text{expandable single}, e^4_4 : \text{rollable/sliding screen}, e^4_5 : \text{foldable screen}\}$.
The expert group determines subsets from these disjoint attribute sets as follows: $\beta_1 = E_1$, $\beta_2 = E_2 \setminus \{e^2_6\}$, $\beta_3 = E_3$ and $\beta_4 = E_4$.

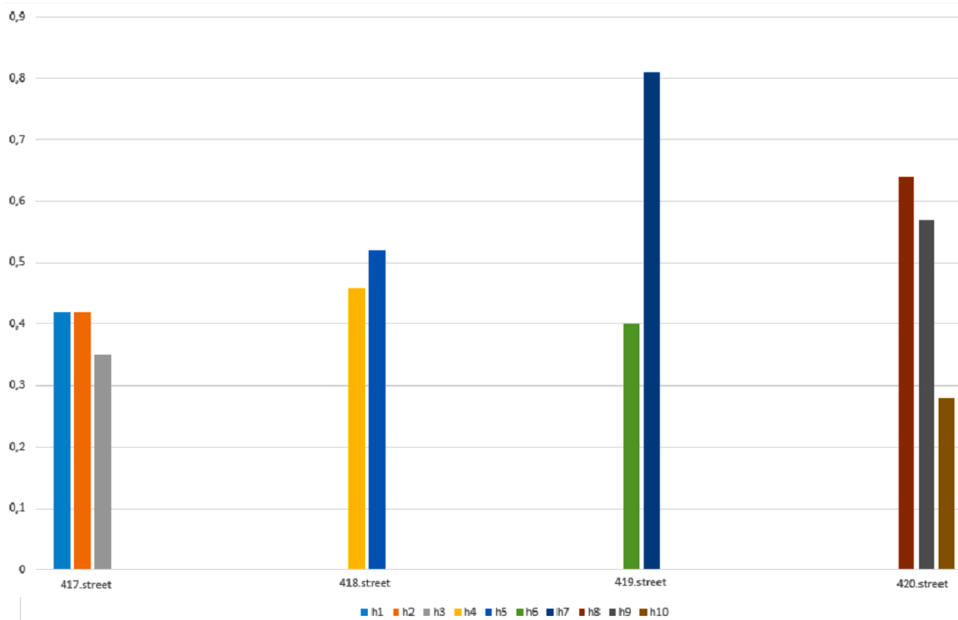


Fig. 3. The ranking of alternatives according to partition sets.

Considering the sets X_1, X_2 and X_3, the expert group can construct partition sets such as best-selling smartphones and others **(Step 1)**:
$U_1 = \{C_1^1 = \{u_1, u_3, u_7, u_8\}, C_2^1 = \{u_2, u_4, u_5, u_6\}\}$,
$U_2 = \{C_1^2 = \{u_1, u_5, u_6, u_8\}, C_2^2 = \{u_2, u_3, u_4, u_7\}\}$ and,
$U_3 = \{C_1^3 = \{u_2, u_5, u_7\}, C_2^3 = \{u_1, u_3, u_4, u_6, u_8\}\}$.
The expert group evaluates the smartphones and then creates the soft sets as the following **(Step 2)**:

$$f_{\beta_1} = \{(e_1^1, \{u_7, u_8\}), (e_2^1, \{u_1\}), (e_3^1, \{u_2, u_6\}), (e_4^1, \{u_3, u_4, u_5\})\},$$

$$f_{\beta_2} = \{(e_1^2, \{u_2\}), (e_2^2, \{u_3, u_8\}), (e_3^2, \{u_5, u_7\}), (e_4^2, \{u_1, u_6\}), (e_5^2, \{u_4\})\},$$

$$f_{\beta_3} = \{(e_1^3, \{u_3, u_4\}), (e_2^3, \{u_1, u_6\}), (e_3^3, \{u_2, u_5, u_7\}), (e_4^3, \{u_8\})\} and$$

$$f_{\beta_4} = \{(e_1^4, \{u_1, u_5\}), (e_2^4, \{u_2, u_6, u_7\}), (e_3^4, \{u_3, u_4, u_8\}), (e_4^4, \{u_3, u_4, u_8\}), (e_5^4, \{u_1, u_5\})\}.$$

Now, depending on U_1, U_2 and U_3, we have,

U_1	i	1	2
	$\lvert P^i(f_{\beta_1})\rvert$	3	2
	$\lvert A^i(f_{\beta_1})\rvert$	4	4

,

U_2	i	1	2
	$\lvert P^i(f_{\beta_1})\rvert$	4	3
	$\lvert A^i(f_{\beta_1})\rvert$	4	4

,

U_3	i	1	2
	$\lvert P^i(f_{\beta_1})\rvert$	3	4
	$\lvert A^i(f_{\beta_1})\rvert$	3	5

U_1	i	1	2
	$\lvert P^i(f_{\beta_2})\rvert$	3	4
	$\lvert A^i(f_{\beta_2})\rvert$	4	4

,

U_2	i	1	2
	$\lvert P^i(f_{\beta_2})\rvert$	3	4
	$\lvert A^i(f_{\beta_2})\rvert$	4	4

,

U_3	i	1	2
	$\lvert P^i(f_{\beta_2})\rvert$	2	3
	$\lvert A^i(f_{\beta_2})\rvert$	3	5

U_1	i	1	2
	$\lvert P^i(f_{\beta_3})\rvert$	4	3
	$\lvert A^i(f_{\beta_3})\rvert$	4	4

,

U_2	i	1	2
	$\lvert P^i(f_{\beta_3})\rvert$	3	2
	$\lvert A^i(f_{\beta_3})\rvert$	4	4

,

U_3	i	1	2
	$\lvert P^i(f_{\beta_3})\rvert$	1	3
	$\lvert A^i(f_{\beta_3})\rvert$	3	5

and

U_1	i	1	2
	$\lvert P^i(f_{\beta_4})\rvert$	5	5
	$\lvert A^i(f_{\beta_4})\rvert$	4	4

,

U_2	i	1	2
	$\lvert P^i(f_{\beta_4})\rvert$	5	3
	$\lvert A^i(f_{\beta_4})\rvert$	4	4

,

U_3	i	1	2
	$\lvert P^i(f_{\beta_4})\rvert$	3	5
	$\lvert A^i(f_{\beta_4})\rvert$	3	5

.

Now, for the soft set f_{β_1}, ith set determinant parametric number $Sdpf(i)(f_{\beta_1}) = 1 - \frac{||P^i(f_{\beta_1})||(|A^i(f_{\beta_1})|)}{2\cdot|C_i^1|\cdot|E_1|\cdot|U_1|}$ is calculated for each U_1, U_2 and U_3 as **(Step 3)**:

	i	1	2
U_1	$Sdpf(i)(f_{\beta_1})$	0.81	0.75
U_2	$Sdpf(i)(f_{\beta_1})$	0.75	0.81
U_3	$Sdpf(i)(f_{\beta_1})$	0.81	0.58

By Definition 12, according to f_{β_1}, the rates at which smartphone sales in cities C_1, C_2 and C_3 are affected by the colors of these smartphones are as follows**(Step 4)**:
For the city C_1: $C_2^1 < C_1^1$, that's mean colors have a high impact on sales;
For the city C_2: $C_1^2 < C_2^2$, that's mean colors have little impact on sales;
For the city C_3: $C_2^3 < C_1^3$, that's mean colors have a high impact on sales.

For the soft set f_{β_2},

	i	1	2
U_1	$Sdpf(i)(f_{\beta_2})$	0.87	0.83
U_2	$Sdpf(i)(f_{\beta_2})$	0.87	0.83
U_3	$Sdpf(i)(f_{\beta_2})$	0.91	0.87

According to f_{β_2}, the rates at which smartphone sales in cities C_1, C_2 and C_3 are affected by the screen technologies of these smartphones are as follows:
For the city C_1: $C_2^1 < C_1^1$, that's mean screen technologies have a high impact on sales;
For the city C_2: $C_2^2 < C_1^2$, that's mean screen technologies have a high impact on sales;
For the city C_3: $C_2^3 < C_1^3$, that's mean screen technologies have a high impact on sales.

For the soft set f_{β_3},

	i	1	2
U_1	$Sdpf(i)(f_{\beta_3})$	0.8	0.85
U_2	$Sdpf(i)(f_{\beta_3})$	0.85	0.9
U_3	$Sdpf(i)(f_{\beta_3})$	0.92	0.85

According to f_{β_3}, the rates at which smartphone sales in cities C_1, C_2 and C_3 are affected by the materials of these smartphones are as follows:
For the city C_1: $C_1^1 < C_2^1$, that's mean materials have little impact on sales;
For the city C_2: $C_1^2 < C_2^2$, that's mean materials have little impact on sales;
For the city C_3: $C_2^3 < C_1^3$, that's mean materials have a high impact on sales.

For the soft set f_{β_4},

	i	1	2
U_1	$Sdpf(i)(f_{\beta_4})$	0.75	0.75
U_2	$Sdpf(i)(f_{\beta_4})$	0.75	0.85
U_3	$Sdpf(i)(f_{\beta_4})$	0.77	0.75

According to f_{β_4}, the rates at which smartphone sales in cities C_1, C_2 and C_3 are affected by the screen types of these smartphones are as follows:
For the city C_1: $C_1^1 = C_2^1$, that's mean screen types have no impact on sales;
For the city C_2: $C_1^2 < C_2^2$, that's mean screen types have little impact on sales;
For the city C_3: $C_2^3 < C_1^3$, that's mean screen types have a high impact on sales.

For each soft set f_{β_j}, $j = 1,2,3,4$, to obtain alternative determinant parametric functions $Adpf_j$, firstly we need to calculate the parameter sets of the ith alternative $E_{\beta_j}(u_i)$ for all $u_i \in U$ and $j = 1,2,3,4$ **(Step 5)**.

i	1	2	3	4	5	6	7	8
$\lvert E_{\beta_1}(u_i)\rvert$	1	1	1	1	1	1	1	1
$\lvert E_{\beta_2}(u_i)\rvert$	1	1	1	1	1	1	1	1
$\lvert E_{\beta_3}(u_i)\rvert$	1	1	1	1	1	1	1	1
$\lvert E_{\beta_4}(u_i)\rvert$	2	1	2	2	2	1	1	2

Since there exists one decision maker, then the alternative determinant parametric functions $Adpf_j(u_i) = \frac{|E_{\beta_j}(u_i)|}{|E_j|}$, for $j = 1,2,3,4$ are **(Step 6)**:

i	1	2	3	4	5	6	7	8
$Adpf_1(u_i)$	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
$Adpf_2(u_i)$	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16
$Adpf_3(u_i)$	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$Adpf_4(u_i)$	0.4	0.2	0.4	0.4	0.4	0.2	0.2	0.4

Now, we have the decision on alternatives **(Step 7)** for each soft set f_{β_j}, $j = 1,2,3,4$, then the rankings of alternatives depend on optimum sets are:

	Ranking
$Opt_1(U)$	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = u_8$
$Opt_2(U)$	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = u_8$
$Opt_3(U)$	$u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = u_8$
$Opt_4(U)$	$u_2 = u_6 = u_7 < u_1 = u_3 = u_4 = u_5 = u_8$

(Step 8 and 9) Since $U_1 = \{C_1^1 = \{u_1, u_3, u_7, u_8\}, C_2^1 = \{u_2, u_4, u_5, u_6\}\}$, for all $u_j \in U$, $Cdpf(u_j) = Adpf(u_j)Sdpf(i)(f_{\beta_k})$, where $u_j \in C^i$ and $k = 1,2,3,4$. For instance, since $u_1 \in C_1^1$, then $Cdpf(u_1) = Adpf(u_1)Sdpf(1)(f_{\beta_1}) = 0.25 \cdot 0.81 = 0.202$. By similar calculation, we have;

U_1	i	1	2	3	4	5	6	7	8	*Ranking*
f_{β_1}	$Cdpf(u_i)$	0.2	0.18	0.2	0.18	0.18	0.18	0.2	0.2	$u_2 = u_4 = u_5 = u_6 < u_1 = u_3 = u_7 = u_8$
f_{β_2}	$Cdpf(u_i)$	0.14	0.13	0.14	0.13	0.13	0.13	0.14	0.14	$u_2 = u_4 = u_5 = u_6 < u_1 = u_3 = u_7 = u_8$
f_{β_3}	$Cdpf(u_i)$	0.16	0.17	0.16	0.17	0.17	0.17	0.16	0.16	$u_1 = u_3 = u_7 = u_8 < u_2 = u_4 = u_5 = u_6$
f_{β_4}	$Cdpf(u_i)$	0.3	0.15	0.3	0.3	0.3	0.15	0.15	0.3	$u_2 = u_6 = u_7 < u_1 = u_3 = u_4 = u_5 = u_8$

U_2	i	1	2	3	4	5	6	7	8	*Ranking*
f_{β_1}	$Cdpf(u_i)$	0.18	0.2	0.2	0.2	0.18	0.18	0.2	0.18	$u_1 = u_5 = u_6 = u_8 < u_2 = u_3 = u_4 = u_7$
f_{β_2}	$Cdpf(u_i)$	0.14	0.13	0.13	0.13	0.14	0.14	0.13	0.14	$u_2 = u_3 = u_4 = u_7 < u_1 = u_5 = u_6 = u_8$
f_{β_3}	$Cdpf(u_i)$	0.17	0.18	0.18	0.18	0.17	0.17	0.18	0.17	$u_1 = u_5 = u_6 = u_8 < u_2 = u_3 = u_4 = u_7$
f_{β_4}	$Cdpf(u_i)$	0.3	0.17	0.34	0.34	0.3	0.15	0.17	0.3	$u_6 < u_2 = u_7 < u_1 = u_5 = u_8 < u_3 = u_4$

,

and

U_3	i	1	2	3	4	5	6	7	8	*Ranking*
f_{β_1}	$Cdpf(u_i)$	0.14	0.2	0.14	0.14	0.2	0.14	0.2	0.14	$u_1 = u_3 = u_4 = u_6 = u_8 < u_2 = u_5 = u_7$
f_{β_2}	$Cdpf(u_i)$	0.13	0.14	0.13	0.13	0.14	0.13	0.14	0.13	$u_1 = u_3 = u_4 = u_6 = u_8 < u_2 = u_5 = u_7$
f_{β_3}	$Cdpf(u_i)$	0.17	0.18	0.17	0.17	0.18	0.17	0.18	0.17	$u_1 = u_3 = u_4 = u_6 = u_8 < u_2 = u_5 = u_7$
f_{β_4}	$Cdpf(u_i)$	0.3	0.154	0.3	0.3	0.31	0.15	0.154	0.3	$u_6 < u_2 = u_7 < u_1 = u_3 = u_4 = u_8 < u_5$

Accordingly, the factors affecting the sales of best-selling smartphones in the city C_1 are as follows: The color of the smartphone has an effect on sales, and light, dark and mixed color are more preferred. The screen technology has an effect on sales, and TFT-LCD, AMOLED and Super AMOLED are more preferred. The material of the smartphone has no effect on sales. The screen type has an effect on sales, and dual/foldable, expandable-single, rollable/sliding screens are more preferred.
As a result, the solution of Atagün and Kamacı (2023b), Examples 6. and 7. with *SetAltDM* is the following:

	Color	Impact	Screen technology	Impact
C_1	light, dark, mixed color	*Yes*	TFT-LCD, AMOLED, Super AMOLED	*Yes*
C_2	light, golden-silver, mixed color	*No*	TFT-LCD, AMOLED	*Yes*
C_3	light, golden-silver, mixed color	*Yes*	OLED, Super AMOLED	*Yes*

	Material	Impact	Screen type	Impact
C_1	polycarbonate-glass, plastic-glass/aluminum	*No*	dual-foldable-rollable/sliding-expandable	*Yes*
C_2	polycarbonate-glass, plastic-aluminum	*Yes*	dual-foldable-rollable/sliding-expandable	*Yes*
C_3	polycarbonate-glass	*Yes*	dual-foldable	*Yes*

The solution suggested in Atagün and Kamacı (2023b), Examples 6. and 7. is as follows:

The real numbers $\alpha_k \geq 1$	Bestsellers in C_1, C_2 and C_3
$\alpha_1 = \alpha_2 = \alpha_3 = 2$	−light, Super AMOLED, polycarbonate-glass, dual-foldable −light, Super AMOLED, aluminum-glass, dual-foldable −dark, Super AMOLED, polycarbonate-glass, dual-foldable −dark, Super AMOLED, aluminum-glass, dual-foldable
$\alpha_1 = \alpha_2 = \alpha_3 = 5$	dark, Super AMOLED, aluminum-glass, dual-foldable
$\alpha_1 = \alpha_2 = \alpha_3 = 1$	−light, Super AMOLED, polycarbonate-glass, dual-foldable −mixed, Super AMOLED, polycarbonate-glass, dual-foldable
$\alpha_1 = 4, \alpha_2 = 3, \alpha_3 = 4$	dark, Super AMOLED, aluminum-glass, dual-foldable

Comparisons, advantages and disadvantages:

The comparison of *SetAltDM* with the *DM* algorithm proposed in Atagün and Kamacı (2023b) is as follows:

c_1. In the *DM* algorithm proposed in Atagün and Kamacı (2023b), the soft sets produced by the decision makers must be strait soft sets, while there is no such requirement in *SetAltDM*.
c_2. In the *DM* algorithm proposed in Atagün and Kamacı (2023b), decision makers must specify a constant $\alpha_k \geq 1$, whereas there is no such requirement in *SetAltDM*.
c_3. In the *DM* algorithm proposed in Atagün and Kamacı (2023b), decisions change according to the specified value α_k. In *SetAltDM*, a three-stage decision is obtained that gives the ranking of the alternatives, the appropriate partition set and their joint result.
c_4. In the *DM* algorithm proposed in Atagün and Kamacı (2023b), decisions are taken jointly for all cities. *SetAltDM* offers different production suggestions according to cities C_1, C_2 and C_3.

In light of these results, the following advantages of *SetAltDM* are evident:

a_1. *SetAltDM* provides clearer data to the manufacturer as it separately indicates the best-selling smartphone models in the cities C_1, C_2 and C_3.
a_2. *SetAltDM* offers different production suggestions according to cities C_1, C_2 and C_3.
a_3. *SetAltDM* also determines the features that have an impact on sales and those that do not.
a_4. *SetAltDM* also helps reach a common conclusion by considering the common features of the highest selling smartphones in these three cities.
a_5. *SetAltDM* can be applied both to group decision-making problems, as in Example 4, and to multi-attribute decision making problems, as in this example.
a_6. In this example where *SetAltDM* is applied, the search state can be customized by selecting specific partition sets. Here, a customization based on sales rate is used. For example, special solutions can be produced by creating partition sets according to the purchasing power of the people living in these cities or the average age of the city's population, etc.

A disadvantage that can be noted in this example is that since f_{β_1}, f_{β_2} and f_{β_3} are bijective soft sets, the values of the corresponding alternative determinant parametric functions $Adpf_j(u_i)$ are fixed. Therefore, for parts containing bijective soft sets, the only data determining the common solution is obtained from the set determinant parametric function.

Additionally, the comparison results of *SetAltDM* with some existing soft decision making algorithms in the literature are listed below:

[6], Example 5.1	Ranking of results	Partition set (U/R)	Result
SDmMDM	X	X	$\{u_1, u_3\}$
SetAltDM	✓	$U/R = \{C_1 = \{u_1, u_2, u_4\}, C_2 = \{u_3, u_5\}\}$	$u_2 < u_5 < u_4 < u_1 < u_3$

In the example given in this study, two decision-makers determine the appropriate common alternative. For this, the operation generalized AND product was used. In order to solve the problem given in this example with *SetAltDM*, the partition set specified as $U/R = \{C_1 = \{u_1, u_2, u_4\}, C_2 = \{u_3, u_5\}\}$ is selected. The authors found that $\{u_1, u_3\}$ are viable alternatives, but one still needs to choose among them. The result $u_2 < u_5 < u_4 < u_1 < u_3$ obtained in the solution of the same problem using *SetAltDM* presents the appropriate alternatives in order. As can be seen, the results are completely compatible. In this problem, since *SetAltDM* gives the results in order, the best alternative is determined, and decision-makers do not have to choose again.

[24], Example 4.6	Ranking	Partition set (U/R)	Result
Soft diff. max-row DM	X	X	$\{u_6\}$
SetAltDM	✓	$U/R = \{C_1 = \{u_1, u_3, u_5\}, C_2 = \{u_2, u_4, u_6\}\}$	$u_5 < u_1 = u_3 < u_4 < u_2 < u_6$

In the example given in this study, the first of the two decision makers determines the suitable candidates, while the second one selects the unsuitable candidates. This is why it is appropriate to use the difference operation. In order to solve the problem given in this example with *SetAltDM*, first the partition set specified above is selected, and the complement of the soft set (G, B) of the second decision maker is used. The authors found that the feasible alternative is u_6 and the infeasible alternative is u_5. The result $u_5 < u_1 = u_3 < u_4 < u_2 < u_6$ obtained in the solution of the same problem using *SetAltDM* presents the appropriate alternatives in order. As can be seen, the results are completely compatible. In this problem, the biggest advantage of *SetAltDM* is that it gives the results in order.

[15], Example 6	Ranking of results	Partition set (U/R)	Result
SMmDM	X	X	$\{u_1\}$
SetAltDM	✓	$U/R = \{C_1 = \{u_1, u_3, u_4\}, C_2 = \{u_2, u_5\}\}$	$u_5 < u_4 < u_3 < u_2 < u_1$

In the example given in this study, using the operation AND-product, two decision makers determine the appropriate common alternative. In order to solve the problem given in this example with *SetAltDM*, the partition set specified above is selected. The authors found that the most suitable alternative is u_1. The result $u_5 < u_4 < u_3 < u_2 < u_1$ obtained in the solution of the same problem using *SetAltDM* presents the appropriate alternatives in order. As can be seen, the results are completely compatible. The biggest advantage of *SetAltDM* here is that the solution can be narrowed down by choosing the appropriate partition set. In this particular example, clearer choices can be made by determining partition sets according to situations such as 'buying a house close to work', 'buying a house close to the child's school', 'buying a house within a certain price range', etc. In addition, *SetAltDM* also produces a solution when there are more than two decision-makers. However, the AND product operation used in *SMmDM* is defined for the multiplication of two soft matrices of the same type. Therefore, *SMmDM* is not a valid method for solving problems involving more than two decision-makers.

[11], Example 5.1	Ranking of results	Partition set (U/R)	Result
Soft matrix based DM	X	X	$\{d_2, d_3\}$
SetAltDM	✓	$U/R = \{C_1 = \{d_1, d_3\}, C_2 = \{d_2, d_4\}\}$	$d_4 < d_1 < d_2 < d_3$

In this example, the authors investigate the common solution of three decision-makers with the help of matrix multiplication and addition operations. The authors found that $\{d_2, d_3\}$ are appropriate alternatives, but one still needs to choose among them. In order to solve the problem given in this example with *SetAltDM*, the partition set specified above is selected. The result $d_4 < d_1 < d_2 < d_3$ obtained in the solution of the same problem using *SetAltDM* presents the appropriate alternatives in order. As can be seen, the results are completely compatible. In this problem, since *SetAltDM* gives the results in order, the best alternative is determined, and decision-makers do not have to choose again. Additionally, *SetAltDM* is advantageous in that it offers the opportunity to select the appropriate partitioning set for the situation and can present different solution suggestions and/or narrow down the solution.

[32], Example 31	Ranking of results	Partition set (U/R)	Result
$uni_x int_y - uni_y int_x DM$	X	X	$\{u_3\}$
SetAltDM	✓	$U/R = \{C_1 = \{u_1, u_4, u_5\}, C_2 = \{u_2, u_3\}\}$	$u_1 < u_5 < u_4 < u_2 < u_3$

In the example given in this study, using the operators $uni_x - int_y$ and $uni_y - int_x$, two decision-makers determine the appropriate common alternative. In order to solve the problem given in this example with *SetAltDM*, the partition set specified above is selected. The author found that the feasible alternative is u_3. The result $u_1 < u_5 < u_4 < u_2 < u_3$ obtained in the solution of the same problem using *SetAltDM* presents the appropriate alternatives in order. As can be seen, the results are completely compatible. In this method, it is not presented how to obtain the solution when there are more than two decision-makers. *SetAltDM* has advantages over this method in terms of being able to select the appropriate partition set, presenting the solution in order, and obtaining the solution independently of the number of decision-makers.

6. A similarity measure: parametric similarity

Definition 18. Let R be an equivalence relation on U. Then a mapping $P_R : S(U;R) \times S(U;R) \to [0,1]$ is called a **P_R-similarity measure** if its value $P_R(f_A, f_B)$ satisfies the following axioms:

s1. $0 \le P_R(f_A, f_B) \le 1$;
s2. $f_A \equiv_R f_B$ if and only if $P_R(f_A, f_B) = 1$;
s3. $P_R(f_A, f_B) = P_R(f_B, f_A)$;
s4. If $f_A \tilde{\subseteq} f_B \tilde{\subseteq} f_C$, then $P_R(f_A, f_C) \le P_R(f_A, f_B)$ and $P_R(f_A, f_C) \le P_R(f_B, f_C)$.

for all $\emptyset_E \ne f_A, f_B, f_C \in S(U;R)$.

Using the concepts upper and lower parametric equivalence sets of soft sets in $(U;R)$, equivalence of soft sets is characterized with the following:

Theorem 5. *Let $f_A = (F, A)$ and $f_B = (G, B)$ be soft sets over U, $A, B \subseteq E = \{e_1, e_2, \dots, e_n\}$, $(U;R)$ be a Pawlak approximation space and let $U/R = \{C_i | i = 1, 2, \dots, r\}$.*
$f_A \equiv_R f_B$ if and only if $P^i(f_A) = P^i(f_B)$ and $P_i(f_A) = P_i(f_B)$ for each $i = 1, 2, \dots, r$.

Proof. If $f_A \equiv_R f_B$, then by Lemma 1 (a) and (b) $P^i(f_A) = P^i(f_B)$ and $P_i(f_A) = P_i(f_B)$ for each $i = 1, 2, \dots, r$.
Assume that $P^i(f_A) = P^i(f_B)$ and $P_i(f_A) = P_i(f_B)$ for each $i = 1, 2, \dots, r$.
Since,
$\{x \in A | C_i \cap F(x) \ne \emptyset\} = \{x \in B | C_i \cap G(x) \ne \emptyset\}$ and
$\{x \in A | C_i \subseteq F(x)\} = \{x \in B | C_i \subseteq G(x)\}$ for each $i = 1, 2, \dots, r$, then $A = B$.
By Definition 8, $R_*(f_A) = (F_*, A)$ and $R^*(f_A) = (F^*, A)$; which are soft sets over U with the set-valued mappings given by
$F_*(x) = R_*(F(x)) = \{u \in U \mid [u]_R \subseteq F(x)\}$ and $F^*(x) = R^*(F(x)) = \{u \in U \mid [u]_R \cap F(x) \ne \emptyset\}$, where $x \in A$. Since,
$\bigcup_{i\in\{1,2,\dots,r\}} P_i(f_A) = \bigcup_{i\in\{1,2,\dots,r\}} P_i(f_B)$, then
$R^*(f_A) = \bigcup_{x\in\bigcup P_i(f_A)} R^*(F(x)) = \bigcup_{x\in\bigcup P_i(f_B)} R^*(G(x)) = R^*(f_B)$ and
$R_*(f_A) = \bigcup_{x\in\bigcup P_i(f_A)} R_*(F(x)) = \bigcup_{x\in\bigcup P_i(f_B)} R_*(G(x)) = R_*(f_B)$. Therefore, $f_A \equiv_R f_B$, by Definition 9. □

A P_R-similarity method:
Let $A, B \subseteq E = \{e_1, e_2, \dots, e_n\}$, $(U;R)$ be a Pawlak approximation space and let $U/R = \{C_i | i = 1, 2, \dots, r\}$.
Consider the function

$\sigma : S(U;R) \times S(U;R) \to [0,1]$,

$$\sigma(f_A, f_B) = 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} ||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)||$$

Theorem 6. *σ is an P_R-similarity measure on $S(U;R)$.*

Proof. Let $f_A, f_B, f_C \in S(U;R)$.

s1.s2. If we take absolute and null soft sets, i.e. $U_E = \{(e_i, U) : i = 1, 2, \dots, r\}$ and $\emptyset_E = \{(e_i, \emptyset) : i = 1, 2, \dots, r\}$, then, $\sigma(U_E, \emptyset_E) = 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} ||P^i(U_E)| - |P^i(\emptyset_E)|| + ||P_i(U_E)| - |P_i(\emptyset_E)|| = 1 - 1 = 0$,
by Lemma 1 (j) and (k).
By Theorem 5, $f_A \equiv_R f_B$ if and only if $P^i(f_A) = P^i(f_B)$ and $P_i(f_A) = P_i(f_B)$ for each $i = 1, 2, \dots, r$ if and only if $|P^i(f_A)| = |P^i(f_B)|$ and $|P_i(f_A)| = |P_i(f_B)|$ for each $i = 1, 2, \dots, r$ if and only if
$\frac{1}{2r|E|} \sum_{i=1}^{r} ||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)|| = 0$ if and only if $\sigma(f_A, f_B) = 1$.
For all other situations, $0 \le \sigma(f_A, f_B) < 1$, by Lemma 1 (h) and (i).

s3. Symmetry is clearly seen by definition of the function σ.

s4. Let $f_A \tilde{\subseteq} f_B \tilde{\subseteq} f_C$. Then, $|P^i(f_A)| \le |P^i(f_B)| \le |P^i(f_C)|$ and $|P_i(f_A)| \le |P_i(f_B)| \le |P_i(f_C)|$ for each $i = 1, 2, \dots, r$, by Lemma 1 (e) and (f). Then, we have

$$\begin{aligned}
\sigma(f_A, f_C) &= 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} ||P^i(f_A)| - |P^i(f_C)|| + ||P_i(f_A)| - |P_i(f_C)|| \\
&= 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} (|P^i(f_C)| - |P^i(f_A)|) + (|P_i(f_C)| - |P_i(f_A)|) \\
&= 1 - \frac{1}{2r|E|} [\sum_{i=1}^{r} (|P^i(f_C)| - |P^i(f_B)| + |P^i(f_B)| - |P^i(f_A)|) \\
&\quad + (|P_i(f_C)| - |P_i(f_B)| + |P_i(f_B)| - |P_i(f_A)|)] \\
&= 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} (|P^i(f_C)| - |P^i(f_B)|) + (|P_i(f_C)| - |P_i(f_B)|) - 1 \\
&\quad + 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} (|P^i(f_B)| - |P^i(f_A)|) + (|P_i(f_B)| - |P_i(f_A)|) \\
&= \sigma(f_B, f_C) - 1 + \sigma(f_A, f_B)
\end{aligned}$$

Then $1 + \sigma(f_A, f_C) = \sigma(f_A, f_B) + \sigma(f_B, f_C)$. Since $0 \le \sigma(f_A, f_C) \le 1$, $0 \le \sigma(f_A, f_B) \le 1$ and $0 \le \sigma(f_B, f_C) \le 1$, then $\sigma(f_A, f_C) \le \sigma(f_A, f_B)$ and $\sigma(f_A, f_C) \le \sigma(f_B, f_C)$ are satisfied.

□

A weighted P_R-similarity method
Weight ratio is a useful material in many areas such as determining workloads, proximity or cost situations in the transportation sector, qualifications in personnel selection, and determining the importance of data types in the health sector. Using this method of assigning weight ratios to set C_i for each $i = 1, 2, \dots, r$ that forms $U/R = \{C_i | i = 1, 2, \dots, r\}$, yields more accurate results. The weight of the equivalence class C_i is w_i, where $0 < w_i < 1$ and $\sum_{i=1}^{r} w(i) = 1$.
Consider the function

$$\sigma_w : S(U;R) \times S(U;R) \to [0,1],$$

$$\sigma_w(f_A, f_B) = 1 - \frac{1}{2r|E|} \sum_{i=1}^{r} w_i(||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)||)$$

Theorem 7. *σ_w is an P_R-similarity measure on $S(U;R)$.*

Proof. Since $0 < w_i < 1$ and $\sum_{i=1}^{r} w_i = 1$, the proof is obtained similar to the proof of Theorem 6. □

Example 6. Let $U/R = \{C_i | i = 1, 2, 3\}$, where $C_1 = \{u_1, u_4, u_7\}$, $C_2 = \{u_2, u_5, u_8\}$ and $C_3 = \{u_3, u_6\}$ and let the soft sets
$f_A = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_1, u_3, u_6, u_7\}), (e_3, \{u_4, u_7, u_8\}), (e_4, \emptyset)\}$ and
$f_B = \{(e_1, \{u_2, u_3, u_4\}), (e_2, \{u_3, u_4, u_6, u_7\}), (e_3, \{u_1, u_4, u_5\}), (e_4, \emptyset)\}$
$f_C = \{(e_1, \{u_1, u_2, u_4, u_7\}), (e_2, \{u_2, u_3, u_6\}), (e_3, \{u_1\}), (e_4, \{u_3, u_5, u_8\})\}$ and
$f_D = \{(e_1, \{u_1, u_4, u_5, u_7, u_8\}), (e_2, \{u_3, u_5, u_6, u_8\}), (e_3, \{u_4, u_7\}), (e_4, \{u_5, u_6\})\}$
in $S(U;R)$, given in Example 3. Assume that the weight ratios of the sets $\{C_i | i = 1, 2, 3\}$ are $w_1 = 0.45$, $w_2 = 0.25$ and $w_3 = 0.3$. The applications of Theorems 6 and 7 are given with the following tables:

σ	f_A	f_B	f_C	f_D
f_A	1	1	0.875	0.875
f_B	1	1	0.875	0.875
f_C	0.875	0.875	1	1
f_D	0.875	0.875	1	1

Since $f_A \equiv_R f_B$ and $f_C \equiv_R f_D$, it is seen that $\sigma(f_A, f_B) = \sigma(f_C, f_D) = 1$ and $\sigma(f_A, f_C) = \sigma(f_B, f_C) = \sigma(f_A, f_D) = \sigma(f_B, f_D)$.

σ_w	f_A	f_B	f_C	f_D
f_A	1	1	0.952	0.952
f_B	1	1	0.952	0.952
f_C	0.952	0.952	1	1
f_D	0.952	0.952	1	1

Similarly, $\sigma_w(f_A, f_B) = \sigma_w(f_C, f_D) = 1$ and $\sigma_w(f_A, f_C) = \sigma_w(f_B, f_C) = \sigma_w(f_A, f_D) = \sigma_w(f_B, f_D)$, since $f_A \equiv_R f_B$ and $f_C \equiv_R f_D$.

Similarity-based Decision Making Algorithm:

Similarity-based decision-making methods are used in many areas such as comparisons of sales performance of similar products, employee productivity, cost or profit ratios, comparisons of success of units in educational institutions, comparisons of similar departments in different educational institutions, comparisons of success of political parties or candidates in local or general elections, comparisons of positive effects or side-effects of medical drugs, etc.

Similarity-based Decision Making Algorithm using the functions σ and σ_w

With the algorithm below, it is possible to find the closest similarity and create a ranking using similarity rates.

Step 1. $U/R = \{C_i | i = 1, 2, \dots, r\}$ of U is obtained.
Step 2. The soft set f_A in $(U; R)$ is determined for the product A that is to be compared.
Step 3. The soft sets $f_{A_1}, f_{A_2}, \dots, f_{A_k}$ in $(U; R)$ are selected for the products to be compared with f_A.
Step 4. If deemed necessary, weight values w_i for the set C_i are determined for all $i = 1, 2, \dots, r$.
Step 5. The numbers $\sigma_w(f_A, f_{A_i})$ (or $\sigma(f_A, f_{A_i})$) are calculated, for all $i = 1, 2, \dots, r$.
Step 6. The one whose value $\sigma_w(f_A, f_{A_i})$ (or $\sigma(f_A, f_{A_i})$), for $i \in \{1, 2, \dots, r\}$, is closest to 1 is the one that is most similar to f_A.

Companies can use similarity-based decision-making algorithms to benchmark themselves or against competitors.

Example 7. A ready-to-eat food manufacturer company X wants to compare its A products, which consist of 10 varieties produced in three sections as children's menu, adult menu and vegetarian menu, with the B, C and D products of rival companies offering the same products in the same sector. In this case, their common universal set will be $U = \{u_1, u_2, \dots, u_{10}\}$ and the partition of U is $U/R = \{C_i | i = 1, 2, 3\}$, where $C_1 = \{u_1, u_2, u_3, u_4\}$ for children's menu, $C_2 = \{u_5, u_6, u_7\}$ for adult menu and $C_3 = \{u_8, u_9, u_{10}\}$ for vegetarian menu. Company X examines these products according to the parameters it has determined. e_1 :Price range appealing to low income, e_2 :Price range appealing to middle-income, e_3 :Price range appealing to high income, e_4 :Products preferred by female customers, e_5 :Products preferred by male customers, e_6 :Products preferred by young customers, e_7 :Products preferred by middle-aged customers, e_8 :Products preferred by elderly customers, e_9 :Products with high customer satisfaction, e_{10} :Products with low sales rates, e_{11} :Products with high sales rates. Then, the common parameter set is $E = \{e_1, e_2, \dots, e_{11}\}$. The corresponding soft sets f_A, f_B, f_C, and f_D are obtained in the following tables.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
f_A	u_1 u_2 u_3	u_4 u_5 u_7	u_6 u_8 u_9 u_{10}	u_1 u_2 u_3 u_4 u_7 u_8 u_9 u_{10}	u_1 u_3 u_5 u_6 u_7 u_9 u_{10}	u_1 u_2 u_3 u_4 u_6 u_{10}	u_4 u_5 u_6 u_8	u_4 u_7 u_8 u_9 u_{10}	u_1 u_2 u_3 u_4 u_6 u_{10}	u_6 u_8 u_5 u_6 u_8 u_9 u_{10}	u_1 u_2 u_3 u_4 u_6 u_7 u_{10}

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
f_B	u_2 u_6 u_8	u_3 u_5 u_7 u_9 u_{10}	u_1 u_4	u_2 u_3 u_4 u_5 u_7 u_{10}	u_1 u_2 u_3 u_6 u_8 u_9	u_1 u_3 u_4 u_7 u_{10}	u_2 u_3 u_6 u_8 u_{10}	u_2 u_5 u_6 u_9	u_1 u_3 u_4 u_6 u_8	u_2 u_4 u_5 u_7 u_{10}	u_1 u_3 u_6 u_8 u_9 u_{10}

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
f_C	u_2	u_1	u_4	u_2	u_1	u_1	u_2	u_2	u_1	u_1	u_3
	u_3	u_7		u_3	u_3	u_4	u_3	u_5	u_3	u_2	u_4
	u_5	u_8		u_4	u_4	u_6	u_6	u_6	u_4	u_5	u_6
	u_6	u_9		u_5	u_6	u_7	u_8	u_7	u_6	u_7	u_9
		u_{10}		u_6	u_7	u_9	u_9	u_8		u_8	
				u_8	u_8	u_{10}		u_9		u_{10}	
				u_{10}	u_9			u_{10}			

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
f_D	u_1	u_3	u_1	u_2	u_1	u_2	u_1	u_2	u_1	u_1	u_3
	u_2	u_5	u_4	u_3	u_3	u_4	u_2	u_5	u_3	u_2	u_4
	u_4	u_6	u_7	u_5	u_4	u_5	u_3	u_7	u_4	u_5	u_6
			u_8	u_7	u_6	u_6	u_4	u_8	u_6	u_7	u_9
			u_9	u_8	u_9	u_9	u_8	u_9		u_8	
				u_9	u_{10}	u_{10}	u_{10}	u_{10}		u_{10}	

Then, we have

i	1	2	3
$\lvert P^i(f_A)\rvert$	9	10	9
$\lvert P_i(f_A)\rvert$	4	1	4
$\lvert P^i(f_B)\rvert$	11	10	10
$\lvert P_i(f_B)\rvert$	0	0	1
$\lvert P^i(f_C)\rvert$	11	10	8
$\lvert P_i(f_C)\rvert$	0	1	2
$\lvert P^i(f_D)\rvert$	11	9	8
$\lvert P_i(f_D)\rvert$	1	0	1

Accordingly, the soft set f_A has similarities with the soft sets f_B, f_C and f_D as

$$\begin{aligned}
\sigma(f_A, f_B) &= 1 - \tfrac{1}{2r|E|} \textstyle\sum_{i=1}^{r} ||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)|| \\
&= 1 - \tfrac{1}{2\cdot 3\cdot 11} \textstyle\sum_{i=1}^{3} ||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)|| \\
&= 1 - \tfrac{1}{66}[(11-9)+(4-0)] + [(10-10)+(1-0)] + [(10-9)+(4-1)] \\
&= 0.83
\end{aligned}$$

, $\sigma(f_A, f_C) = 0.86$ and $\sigma(f_A, f_D) = 0.83$. Therefore, the similarity ranking of product A with products B, C and D of the competitors is obtained as $C > B = D$.

If company X wants to assign weight ratios to menus according to production quantities and make comparisons accordingly, a different result may be obtained. Assume that the weight ratios $w_1 = 0.5$, $w_2 = 0.4$ and $w_3 = 0.1$ for children's menu, adult menu and vegetarian menu, respectively. Then,

$$\begin{aligned}
\sigma_w(f_A, f_B) &= 1 - \tfrac{1}{2r|E|} \textstyle\sum_{i=1}^{r} w_i||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)|| \\
&= 1 - \tfrac{1}{2\cdot 3\cdot 11} \textstyle\sum_{i=1}^{3} w_i||P^i(f_A)| - |P^i(f_B)|| + ||P_i(f_A)| - |P_i(f_B)|| \\
&= 1 - \tfrac{1}{66}0.5[(11-9)+(4-0)] + 0.4[(10-10)+(1-0)] + 0.1[(10-9)+(4-1)] \\
&= 0.942
\end{aligned}$$

, $\sigma_w(f_A, f_C) = 0.95$ and $\sigma_w(f_A, f_D) = 0.943$. Therefore, the weighted similarity ranking of product A with products B, C and D of the competitors is obtained as $C > D > B$.

Comparison of *SetAltDM* and Similarity-based *DM*

In general, the differences between similarity-based decision-making algorithms and classical decision-making algorithms are also valid here. In addition, a specific comparison of *SetAltDM* with the similarity-based decision-making algorithm is as follows:

c_1. Both methods present the results sequentially.
c_2. In contrast to the single-step decision-making in similarity-based decision-making, a three-step decision is presented in *SetAltDM*.
c_3. Similarity-based decision-making is effective for comparative problems, but *SetAltDM* covers a wider range of problems.
c_4. To solve the problem given in Example 4 using similarity-based decision-making, an imaginary best soft set or worst soft set needs to be generated and compared with it. But in *SetAltDM* such extra effort is not needed.

7. Conclusion

To create a product, many different business lines work; these business lines perform similar activities within themselves, and all of them form a whole that creates this product. In such cases, problems such as performance analysis, production cost, efficiency, and benchmarking arise. In this study, it is aimed to produce solutions to these types of problems belonging to rough set theory using the advantages of soft sets, that is, to present new solution methods within the rough soft set theory. In the literature, there are very few decision-making methods obtained using only rough soft set theory. In the paper (Ma et al., 2017), two decision-making methods on rough soft sets were proposed, and in the paper (Liu et al., 2018), improved versions of these and a group decision-making method are added. In the methods proposed in both studies, applications were presented where the alternatives specify the capabilities, i.e., the parameters, and the parameters specify the candidates, i.e., the alternatives, thus essentially using the inverse soft set. In this study, we proposed a completely new and effective decision-making method called *SetAltDM* by defining a function that selects the appropriate set from the given partition set in the approximation space, a function that determines the appropriate alternative, and a function that gives their joint selection. To explain the validity of *SetAltDM*, the decision-making problem given in Atagün and Kamacı (2023b) Examples 6 and 7 is adapted and solved, and the advantages and disadvantages of the proposed method are discussed. Also, the comparison results of *SetAltDM* with some existing soft decision-making algorithms in the literature are given.

In studies involving similarity on rough soft sets, similarity methods that have already been obtained on soft sets, given in Kharal (2010), have generally been used. In this study, we proposed a new similarity method that is more suitable for the rough soft set structure, where both the partitioning of the approximation space and the soft set are used together. We also presented a similarity-based decision-making method with its application using these new similarity methods.

The original group decision-making and similarity approaches presented in this paper, based on rough soft set theory, have the potential to generalize and produce more accurate results. In further research, generation models of rough soft set theory are high-level topics that can be evaluated in similarity and decision-making.

CRediT authorship contribution statement

Akın Osman Atagün: Conceptualization, Methodology, Software, Data curation, Writing – original draft, Visualization, Investigation, Supervision, Software, Validation, Writing – review & editing.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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