DE GRUYTER Demonstratio Mathematica 2025; 58: 20250133

Research Article

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Orthogonalizing *q*-Bernoulli polynomials

https://doi.org/10.1515/dema-2025-0133
received October 17, 2024; accepted April 5, 2025

Abstract: In this study, we utilize the Gram-Schmidt orthogonalization method to construct a new set of orthogonal polynomials called $\mathrm{OB}_n(x, q)$ from the q-Bernoulli polynomials. We demonstrate the relationship between polynomials $\mathrm{OB}_n(x, q)$ and the little q-Legendre polynomials, and derive a generalized formula for $\mathrm{OB}_n(x, q)$ by leveraging the little q-Legendre polynomials. Furthermore, we present some properties of polynomials $\mathrm{OB}_n(x, q)$. Finally, we introduce a hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$ and examine various properties of these polynomials.

Keywords: q-Bernoulli polynomials, orthonormal polynomials, block-pulse functions

MSC 2020: 33C45, 11B68

1 Introduction

Orthogonal polynomials have been the focus of many mathematicians, initially appearing in mathematical analysis. However, in terms of functionality, they have become an indispensable part of many fields of mathematics today, including numerical analysis, data analysis, physics, and engineering. From a historical perspective, we can see that their first systematic use began with Legendre polynomials and Chebyshev polynomials [1,2]. The objectives and benefits of using orthogonal polynomials can be listed as follows: minimizing error in approximation theory, improving numerical stability in operations involving higher-degree polynomials in numerical analysis, enabling the analysis of functions through their independent components, facilitating easier results in differentiation and integration processes. They are widely used in numerical analysis, particularly in approximation theory and solving differential equations, as they provide efficient and accurate solutions. In quantum mechanics, they play a crucial role as solutions to the Schrödinger equation, helping to describe the behavior of quantum systems. In statistics, orthogonal polynomials are applied in regression analysis to model complex relationships between variables, reducing multicollinearity and improving model accuracy. Additionally, they are fundamental in signal processing and image compression techniques, such as the use of Legendre and Chebyshev polynomials, enhancing data representation and transformation. Moreover, orthogonal polynomials are extensively utilized in spectral methods for solving partial differential equations in engineering and physics, making computations more efficient and precise. Their broad applications and mathematical significance make them a powerful tool in both theoretical and applied sciences.

On the other hand, with the increasing attention given to q-calculus in recent times, we observe that certain special q-polynomials and q-analogues of orthogonal polynomials are frequently appearing. q-Bernoulli polynomials and q-Legendre polynomials are among these polynomials [3–7]. Koekoek et al. has studied various hypergeometric orthogonal polynomials and written their q-hypergeometric representations

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[8]. Mboutngam has defined q-hypergeometric Bernoulli polynomials with one and two real parameters and obtained various recurrence relations [9].

Hybrid functions simplify computational processes in areas such as numerical analysis, function approximation, and integral equations by combining various mathematical structures. Their integration with orthogonal polynomials enhances numerical stability and improves approximation accuracy. In particular, the hybridization of orthogonal polynomials with block-pulse functions simplifies computations and offers flexibility in representing functions over specific intervals. Thus, in function approximation, this integration enables more efficient and precise numerical analysis methods. As a result, these functions have become an important research area in both theoretical and applied mathematics. Over time, hybrid functions have been explored by various researchers in different forms, with an increasing number of numerical examples. Recent works have highlighted the utility of such hybrid approaches in diverse mathematical and applied contexts [10–14]. These developments align closely with the objectives of this study, reinforcing the significance of hybrid functions for improved performance in numerical analysis and approximation theory.

The primary motivation of our study is to render non-orthogonal q-Bernoulli polynomials orthogonal through the Gram-Schmidt orthogonalization process and then combine them with block-pulse functions to construct a hybrid function. This approach aims to make these polynomials more practical in mathematical fields such as approximation theory and numerical analysis, contributing to the simplification of integral solutions for complex functions.

The fundamental concepts we use throughout our study are presented to you with the following properties.

In this article, we give some definition for a real number $q \in (0, 1)$. The quantum integer is defined by

$$[n]_q = \frac{q^n - 1}{q - 1},$$

for any positive integer n [15]. The q-analogue of $n!$ is

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [n]_q[n-1]_q[n-2]_q \dots [1]_q, & \text{if } n = 1, 2, \dots. \end{cases}$$

q-Shifted factorial $(a; q)_n$ is

$$(a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n = 1, 2, \dots. \end{cases}$$

The q-binomial coefficient is defined as

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \quad k = 0, 1, 2, \dots, n \tag{1}$$

with $\binom{n}{0}_q = 1$ and $\binom{n}{k}_q = 0$ for $n < k$. q-Binomial coefficients have the following property:

$$\binom{n}{k}_q = \binom{n}{n-k}_q.$$

Gauss's binomial formula is given by

$$(x + a)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} a^k x^{n-k},$$

for $n \geq 1$. If $yx = qxy$, where q is a number that varies by both x and y, then we have

$$(x + y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}.$$

The Jackson integral is defined as

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

where

$$\int_0^a f(x)d_qx = a(1-q)\sum_{i=0}^{\infty} q^i f(aq^i),$$

for $0 < a < b$ [15]. Specifically, using the description of the q-integral, we can obtain

$$\int_a^b x^n d_qx = \frac{b^{n+1} - a^{n+1}}{[n+1]_q},$$

for $n > -1$, and

$$\int_0^1 x^n d_qx = \frac{1}{[n+1]_q}.$$

Let $\mathbb{P}_n = \{f(x) : \deg(f(x)) = n\}$ with inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)d_qx,$$

where $f(x), g(x) \in \mathbb{P}_n$ [7].

In this research, inspired by the functionality of orthogonal polynomials and hybrid functions, we structured our article as follows: In Section 2 of our study, we present the definitions and various properties of q-Bernoulli polynomials and little q-Legendre polynomials. By orthogonalizing the q-Bernoulli polynomials, we establish connections with the little q-Legendre polynomials. In Section 3, we introduce hybrid functions and define a hybrid of the orthogonalized q-Bernoulli polynomials with block-pulse functions. We represent this function defined for the first time by $\mathrm{OBH}_{nm}(t,q)$. Moreover, by obtaining the operational matrix of integration for the $\mathrm{OBH}_{nm}(t,q)$, we exemplify the solution of the q-variational problem we defined with the help of the hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x,q)$. Section 4 discusses the contributions and significance of our study.

2 Orthogonalizing *q*-Bernoulli polynomials

Orthogonal polynomials constitute a specialized class of polynomials widely employed in various fields such as mathematics, applied mathematics, statistics, analysis, quantum mechanics, physics, and engineering. They serve the purpose of organizing and enhancing the comprehensibility of mathematical calculations. Examples of orthogonal polynomials include Hermite, Laguerre, Chebyshev, and Legendre polynomials [16]. Bernoulli polynomials have also gained particular significance among orthogonal polynomials, especially when rendered orthogonal [17].

In this section, first, we mention q- Bernoulli polynomials and generalized q-Bernoulli polynomials. Then, we give the definition of little q-Legendre polynomials and their some properties. Finally, we apply the Gram-Schmidt orthogonalization method to q-Bernoulli polynomials.

The q-Bernoulli polynomials $B_n(x,q)$ are defined by the following generating function:

$$\frac{te_q(xt)}{e_q(t)-1} = \sum_{n=0}^{\infty} B_n(x,q)\frac{t^n}{[n]_q!}.$$

It is clear that when $x = 0$, $B_n(0, q) = B_n(q)$ are q-Bernoulli numbers [3,5]. Specifically, with the help of q-Bernoulli numbers, q-Bernoulli polynomials can be expressed as

$$B_n(x, q) = \sum_{k=0}^{n} \binom{n}{k}_q B_k(q) x^{n-k}.$$

The first few q-Bernoulli polynomials are as follows:

n	$B_n(x, q)$
0	1
1	$x - \frac{1}{2}$
2	$x^2 - \frac{[2]_q}{2}x + \frac{[2]_q}{12}$
3	$x^3 - \frac{[3]_q}{2}x^2 + \frac{[3]_q[2]_q}{12}x$
4	$x^4 - \frac{[4]_q}{2}x^3 + \frac{[4]_q[3]_q}{12}x^2 - \frac{[4]_q!}{720}$
5	$x^5 - \frac{[5]_q}{2}x^4 + \frac{[5]_q[4]_q}{12}x^3 - \frac{[5]_q!}{720}x$

Ismail and Mansour [18] studied q-pair of analogues of the Bernoulli polynomials using the generating functions:

$$\frac{te_q(xt)}{e_q\left(\frac{t}{2}\right)E_q\left(\frac{t}{2}\right) - 1} = \sum_{n=0}^{\infty} b_n(x, q)\frac{t^n}{[n]_q!},$$

$$\frac{tE_q(xt)}{e_q\left(\frac{t}{2}\right)E_q\left(\frac{t}{2}\right) - 1} = \sum_{n=0}^{\infty} B_n(x, q)\frac{t^n}{[n]_q!},$$

and the q-Bernoulli numbers are defined by

$$\beta_n(q) = B_n(0;\, q).$$

Moreover, the q-Bernoulli polynomials $B_n(x, q)$ and $b_n(x, q)$ are given by $B_0(x;\, q) = b_0(x;\, q) = 1$,

$$B_n(x;\, q) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} \beta_{n-k}(q) x^k,$$

$$b_n(x;\, q) = \sum_{k=0}^{n} \binom{n}{k}_q \beta_{n-k}(q) x^k,$$

for $n \in \mathbb{N}$ [18]. Ismail and Mansour showed that

$$B_n(x;\, q) = q^{\frac{n(n-1)}{2}} b_n\left(x;\, \frac{1}{q}\right),$$

$$\beta_n(q) = q^{\frac{n(n-1)}{2}} \beta_n\left(\frac{1}{q}\right),$$

for $q \neq 0$.

Eweis and Mansour studied three q-analogues of the generalized Bernoulli polynomials that were presented by Frappier in his previous works [7,19–21]. The generalized q-Bernoulli polynomials $B_{n,\alpha}^{(s)}(x;\, q)$ are defined by

$$B_{n,\alpha}^{(1)}(x;\, q) = \sum_{k=0}^{n} \binom{n}{k}_q \beta_{n-k,\alpha}(q) x^k,$$

$$B_{n,\alpha}^{(2)}(x;\,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}\beta_{n-k,\alpha}(q)x^k,$$
$$B_{n,\alpha}^{(3)}(x;\,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{k(k-1)}{4}}\beta_{n-k,\alpha}^{(3)}(q)x^k$$

with $B_{0,\alpha}^{(s)}(x;\,q)=1$, for $s=1,2,3$, and $n\in\mathbb{N}$.

Specifically, writing $\alpha=\pm\frac{1}{2}$ and $s=1$ in the generalized q-Bernoulli polynomials $B_{n,\alpha}^{(s)}(x;\,q)$, they showed that the q-Bernoulli polynomials are given by $B_n(x;\,q)$ and $b_n(x;\,q)$ in [18].

The little q-Legendre polynomials $P_n(x|q)$ are defined by

$$P_n(x|q)={}_2\varphi_1\left(\begin{matrix}q^{-n},q^{n+1}|q,qx\\q\end{matrix}\right)=\sum_{k=0}^{n}\frac{(q^{-n};\,q)_k(q^{n+1};\,q)_k}{(q;\,q)_k}\frac{x^kq^k}{(q;\,q)_k},$$

for $n\in\mathbb{N}$ [4,6]. By means of (1), the little q-Legendre polynomials $P_n(x|q)$ can be written as

$$P_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{-nk+\frac{k(k+1)}{2}}(-x)^k. \tag{2}$$

The little q-Legendre polynomials provide an orthogonality relation given in the following:

$$\int_0^1 P_n(x|q)P_m(x|q)d_q(x)=\frac{q^n(1-q)}{(1-q^{2n+1})}\delta_{nm}, \tag{3}$$

for $m,n\in\mathbb{Z}^+$, where δ_{nm} is the Kronecker delta function [4]. Some little q-Legendre polynomials are as follows:

$$\begin{aligned}
P_0(x|q)&=1,\\
P_1(x|q)&=-[2]_qx+1,\\
P_2(x|q)&=\frac{[4]_q[3]_q}{q[2]_q}x^2-\frac{[2]_q[3]_q}{q}x+1,\\
P_3(x|q)&=-\frac{[6]_q[5]_q[4]_q}{q^3[3]_q[2]_q}x^3+\frac{[5]_q[4]_q[3]_q}{q^3[2]_q}x^2-\frac{[4]_q[3]_q}{q^2}x+1.
\end{aligned}$$

Eweis and Mansour [7] showed relations between the generalized q-Bernoulli polynomials $B_{n,\alpha}^{(1)}(x;\,q)$ and little q-Legendre polynomials $P_n(x|q)$ and they obtained

$$B_{n,\alpha}^{(1)}(x;\,q)=\sum_{k=0}^{n}C_{k,\alpha}(q)P_k(x|q), \tag{4}$$

where

$$C_{k,\alpha}(q)=\frac{1-q^{2k+1}}{(1-q)q^k}\int_0^1 B_{n,\alpha}^{(1)}(x;\,q)P_k(x|q)d_qx. \tag{5}$$

If we write $\alpha=\pm\frac{1}{2}$ in q-Bernoulli polynomials $B_{n,\alpha}^{(1)}(x;\,q)$, (4) and (5) reduce to

$$B_n(x;\,q)=\sum_{k=0}^{n}C_k(q)P_k(x|q)$$

and

$$C_k(q)=\frac{1-q^{2k+1}}{(1-q)q^k}\int_0^1 B_n(x;\,q)P_k(x|q)d_qx. \tag{6}$$

Now, we apply the Gram-Schmidt orthogonalization method to certain q-Bernoulli polynomials in the following example and then generalize the process using the following theorem. These newly obtained polynomials are denoted as $\mathrm{OB}_n(x,q)$ for $n \in \mathbb{N}$.

$$\begin{aligned}
\mathrm{OB}_0(x,q) &= B_0(x,q) = 1,\\
\mathrm{OB}_1(x,q) &= B_1(x,q) - \frac{\langle B_1(x,q), \mathrm{OB}_0(x,q)\rangle}{\langle \mathrm{OB}_0(x,q), \mathrm{OB}_0(x,q)\rangle}\mathrm{OB}_0(x,q)\\
&= x - \frac{1}{2} - \frac{\left\langle \left(x-\frac{1}{2}\right), 1\right\rangle}{\langle 1,1\rangle}\\
&= x - \frac{1}{[2]_q},\\
\mathrm{OB}_2(x,q) &= B_2(x,q) - \frac{\langle B_2(x,q), \mathrm{OB}_1(x,q)\rangle}{\langle \mathrm{OB}_1(x,q), \mathrm{OB}_1(x,q)\rangle}\mathrm{OB}_1(x,q) - \frac{\langle B_2(x,q), \mathrm{OB}_0(x,q)\rangle}{\langle \mathrm{OB}_0(x,q), \mathrm{OB}_0(x,q)\rangle}\mathrm{OB}_0(x,q)\\
&= x^2 - \frac{[2]_q}{q^2+1}x + \frac{q}{(q^2+q+1)(q^2+1)}\\
&= x^2 - \frac{[2]_q}{q^2+1}x + \frac{q}{(q^2+1)[3]_q}.
\end{aligned}$$

If we reconsider these polynomials, we see that the polynomials $\mathrm{OB}_n(x,q)$ can be expressed in terms of a type of little q-Legendre polynomials,

$$\begin{aligned}
\mathrm{OB}_0(x,q) &= P_0(x|q),\\
\mathrm{OB}_1(x,q) &= -\frac{1}{[2]_q}P_1(x|q),\\
\mathrm{OB}_2(x,q) &= \frac{q[2]_q}{[4]_q[3]_q}P_2(x|q),\\
\mathrm{OB}_3(x,q) &= -\frac{q^3[3]_q[2]_q}{[6]_q[5]_q[4]_q}P_3(x|q).
\end{aligned}$$

Theorem 1. *Let $\{B_n(x,q)\}$ be the family of q-Bernoulli polynomials. Then, when the Gram-Schmidt orthogonalization method is applied to the q-Bernoulli polynomials, the orthogonal set $\{\mathrm{OB}_n(x,q)\}$ is obtained and*

$$\mathrm{OB}_n(x,q) = C_n(q)P_n(x|q), \tag{7}$$

where $P_n(x|q)$ are the little q-Legendre polynomials and

$$C_n(q) = \frac{([n]_q!)^2}{[2n]_q!}(-1)^n q^{\frac{n^2-n}{2}}.$$

Proof. Let us proceed with a proof by induction. For $n = 1$, we have

$$\begin{aligned}
\mathrm{OB}_1(x,q) &= B_1(x,q) - \frac{\langle B_1(x,q), \mathrm{OB}_0(x,q)\rangle}{\langle \mathrm{OB}_0(x,q), \mathrm{OB}_0(x,q)\rangle}\mathrm{OB}_0(x,q)\\
&= \frac{([1]_q!)^2}{[2]_q!}(-1)(-[2]_q x + 1)\\
&= C_1(q)P_1(x|q).
\end{aligned}$$

Assuming that the statement is true for $n-1$, i.e.,

$$\mathrm{OB}_{n-1}(x,q) = \frac{([n-1]_q!)^2}{[2n-2]_q!}(-1)^{n-1}q^{\frac{(n-1)^2-(n-1)}{2}}P_{n-1}(x|q).$$

We show that it is true for n. First, we calculate

$$\mathrm{OB}_n(x,q) = B_n(x,q) - \sum_{k=0}^{n-1} \frac{\langle B_n(x,q), \mathrm{OB}_k(x,q)\rangle}{\langle \mathrm{OB}_k(x,q), \mathrm{OB}_k(x,q)\rangle} \mathrm{OB}_k(x,q). \tag{8}$$

Using the orthogonality relations in equation (3), we obtain following equation:

$$\langle \mathrm{OB}_k(x,q), \mathrm{OB}_k(x,q)\rangle = \int_0^1 \frac{([k]_q!)^4}{([2k]_q!)^2} (-1)^{2k} q^{k^2-k} P_k(x|q) P_k(x|q) = \frac{([k]_q!)^4}{([2k]_q!)^2} q^{k^2} \frac{(1-q)}{1-q^{2k+1}}. \tag{9}$$

If we substitute the equation for (9) into equation (8), we obtain

$$\begin{aligned}
\mathrm{OB}_n(x,q) &= B_n(x,q) - \sum_{k=0}^{n-1} \frac{\frac{([k]_q!)^4}{([2k]_q!)^2}(-1)^{2k} q^{k^2-k} \int_0^1 B_n(x,q) P_k(x|q) d_q x}{\frac{([k]_q!)^4}{([2k]_q!)^2} q^{k^2} \frac{(1-q)}{1-q^{2k+1}}} P_k(x|q) \\
&= B_n(x,q) - \sum_{k=0}^{n-1} \frac{1-q^{2k+1}}{(1-q)q^k} \left(\int_0^1 B_n(x,q) P_k(x|q) d_q x \right) P_k(x|q).
\end{aligned}$$

If we use equation (6), we obtain

$$\begin{aligned}
\mathrm{OB}_n(x,q) &= \sum_{k=0}^{n} C_k(q) P_k(x|q) - \sum_{k=0}^{n-1} C_k(q) P_k(x|q) \\
&= C_n(q) P_n(x|q) + \sum_{k=0}^{n-1} C_k(q) P_k(x|q) - \sum_{k=0}^{n-1} C_k(q) P_k(x|q) \\
&= C_n(q) P_n(x|q).
\end{aligned}$$

□

If we utilize the definition of little q-Legendre polynomials as presented in equation (2) and make necessary adjustments, we arrive at the following expression:

$$\begin{aligned}
\mathrm{OB}_n(x,q) &= \frac{([n]_q!)^2}{[2n]_q!} (-1)^n q^{\frac{n^2-n}{2}} P_n(x|q) \\
&= \frac{([n]_q!)^2}{[2n]_q!} (-1)^n q^{\frac{n^2-n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q \binom{n+k}{k}_q q^{-nk+\frac{k(k+1)}{2}} (-x)^k \\
&= \frac{([n]_q!)^2}{[2n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q \binom{n+k}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k.
\end{aligned}$$

If the last equation is rearranged according to decreasing powers of x,

$$\mathrm{OB}_n(x,q) = \frac{([n]_q!)^2}{[2n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q \binom{2n-k}{n}_q (-1)^k q^{\binom{k}{2}} x^{n-k}$$

is obtained.

Proposition 1. *The polynomials* $\mathrm{OB}_n(x,q)$ *have the following property*:

$$\mathrm{OB}_n(x,q) = x^n - \sum_{k=0}^{n-1} \frac{\binom{n}{k+1}_q \binom{n-1}{k}_q (-1)^k q^{\binom{k+1}{2}} x^{n-1-k}}{(q^n+1)\binom{2n-1}{k}_q}.$$

Proof. Let us use the definition of polynomials $\mathrm{OB}_n(x,q)$ for proof:

$$\begin{aligned}\mathrm{OB}_n(x,q) &= \frac{([n]_q!)^2}{[2n]_q!}\sum_{k=0}^{n}\binom{n}{k}_q\binom{2n-k}{n}_q(-1)^k q^{\binom{k}{2}}x^{n-k}\\ &= x^n + \frac{([n]_q!)^2}{[2n]_q!}\sum_{k=1}^{n}\binom{n}{k}_q\frac{[2n-k]_q!}{[n-k]_q![n]_q!}(-1)^k q^{\binom{k}{2}}x^{n-k}\\ &= x^n + \sum_{k=0}^{n-1}\frac{\binom{n}{k+1}_q\binom{n-1}{k}_q(-1)^{k+1}q^{\binom{k+1}{2}}x^{n-1-k}}{\binom{2n-1}{k}_q}\frac{[n]_q}{[2n]_q}\\ &= x^n - \sum_{k=0}^{n-1}\frac{\binom{n}{k+1}_q\binom{n-1}{k}_q(-1)^k q^{\binom{k+1}{2}}}{(q^n+1)\binom{2n-1}{k}_q}x^{n-1-k}.\end{aligned}$$

□

If we check the orthogonality of the polynomials $\mathrm{OB}_n(x,q)$, we find the following result:

$$\begin{aligned}\langle \mathrm{OB}_n(x,q),\mathrm{OB}_m(x,q)\rangle &= \int_0^1 \mathrm{OB}_n(x,q)\mathrm{OB}_m(x,q)d_q(x)\\ &= \int_0^1 \frac{([n]_q!)^2}{[2n]_q!}(-1)^n q^{\frac{n^2-n}{2}}P_n(x|q)\frac{([m]_q!)^2}{[2m]_q!}(-1)^m q^{\frac{m^2-m}{2}}P_m(x|q)d_q(x)\\ &= \frac{([n]_q!)^2}{[2n]_q!}\frac{([m]_q!)^2}{[2m]_q!}(-1)^{n+m}q^{\frac{n^2-n+m^2-m}{2}}\int_0^1 P_n(x|q)P_m(x|q)d_q(x)\\ &= \frac{([n]_q![m]_q!)^2}{[2n]_q![2m]_q!}(-1)^{n+m}q^{\frac{n^2-n+m^2-m}{2}}\frac{(1-q)q^n}{(1-q^{2n+1})}\delta_{nm}.\end{aligned}$$

Thus, we see that they are orthogonal for $\forall\ n,m\in\mathbb{N}$.

Theorem 2. *The orthonormal set of polynomials* $\mathrm{OB}_n(x,q)$ *is given by*

$$\mathrm{OB}_n^{'}(x,q) = q^{-n}\sqrt{[2k+1]_q}\sum_{k=0}^{n}\binom{n}{k}_q\binom{n+k}{k}_q(-1)^{n-k}q^{\binom{n-k}{2}}x^k.$$

Proof. Let us use the relationship between inner product and norm

$$\begin{aligned}\|\mathrm{OB}_n(x,q)\|_{2,q}^2 &= \langle \mathrm{OB}_n(x,q),\mathrm{OB}_m(x,q)\rangle\\ &= \frac{([n]_q!)^2}{[2n]_q!}\frac{([n]_q!)^2}{[2n]_q!}(-1)^{2n}q^{\frac{2n^2-2n}{2}}\frac{(1-q)q^n}{(1-q^{2n+1})}\delta_{nn}\\ &= \frac{([n]_q!)^4}{([2n]_q!)^2}q^{n^2}\frac{(1-q)}{1-q^{2n+1}}.\end{aligned}$$

Then, we have

$$\|\mathrm{OB}_n(x,q)\|_{2,q} = \frac{q^n([n]_q!)^2}{[2n]_q!\sqrt{[2n+1]_q}}.$$

Using this result, we can write

$$\begin{aligned}\mathrm{OB}_n'(x,q) &= \frac{\mathrm{OB}_n(x,q)}{\|\mathrm{OB}_n(x,q)\|_{2,q}} \\ &= \frac{\frac{([n]_q!)^2}{[2n]_q!}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k}{\frac{q^n([n]_q!)^2}{[2n]_q!\sqrt{[2n+1]_q}}} \\ &= q^{-n}\sqrt{[2n+1]_q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k.\end{aligned}$$

Also, if we rewrite the last equation according to the decreasing powers of x, we obtain

$$\mathrm{OB}_n'(x,q) = q^{-n}\sqrt{[2n+1]_q}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^{n-k}.$$ □

Now, let $q \to 1$, then we have

$$\lim_{q\to 1}\mathrm{OB}_n'(x,q) = \sqrt{2n+1}\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n-k}(-1)^k x^{n-k}.$$

Thus, we have shown that the orthonormal Bernoulli polynomial of degree n is given by [17,22].

In Figure 1, the graphs of some $\mathrm{OB}_n(x,q)$ polynomials are shown. When we orthogonalize the q-Bernoulli polynomials, the resulting orthogonal polynomials $\mathrm{OB}_n(x,q)$ exhibit behavior similar to the orthogonal forms of classical Bernoulli polynomials as the value approaches 1. Additionally, because q-Bernoulli polynomials are not orthogonal, the $\mathrm{OB}_n(x,q)$ orthogonal polynomials make the maximum and minimum points more pronounced, especially in higher-degree polynomials.

In the subsequent section of our study, we associate the polynomials $\mathrm{OB}_n(x,q)$ with block-pulse functions, enhancing their utility through hybrid functions. Hybrid functions are frequently employed, particularly for their ability to simplify the solution of differential equations and the use of complex functions with multiple terms. Specifically, the hybrid of orthogonal polynomials and block-pulse functions combines the advantages of both types, facilitating numerical analysis and various derivative computations.

3 Hybrid of block-pulse function and orthogonal polynomials

Hybrid functions find applications in various applied fields such as function approximation, numerical analysis, integral equations, and mathematical modeling. They prove to be highly useful, particularly in

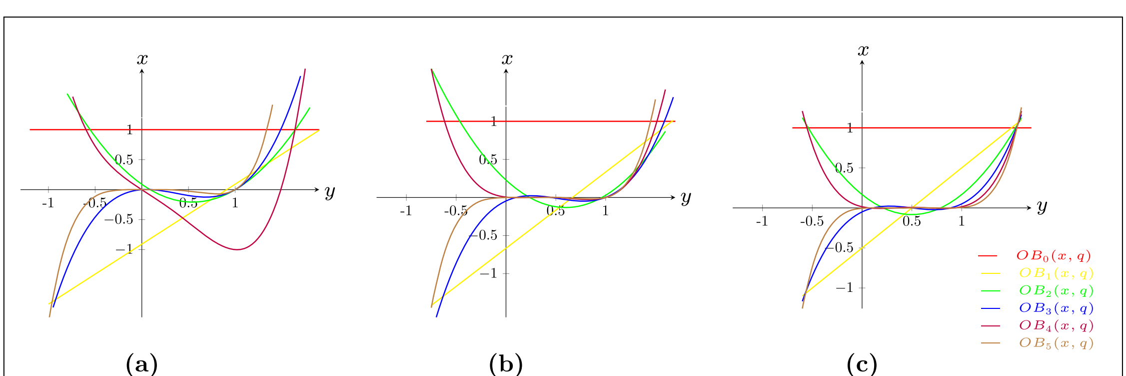

Figure 1: $\mathrm{OB}_n(x,q)$ for $k = 0, 1, 2, 3, 4, 5$. (a) $q = 0.1$, (b) $q = 0.5$, and (c) $q = 0.999$.

simplifying and augmenting the functionality of a function in function approximation. In the context of function approximation, hybrid functions are obtained by combining various special polynomials, notably including block-pulse functions. With the aid of hybrid functions, it becomes feasible to construct a polynomial series for a function over a specific interval. Hence, the hybrid of various polynomials and their properties has recently become a topic of interest and research for many authors [10,23–25].

Razzaghi et al. [26] defined a hybrid of block-pulse and Bernoulli polynomials and presented a direct method for solving variational problems using a hybrid approach combining block-pulse functions and Bernoulli polynomials, thus simplifying the computational processes of complex variational problems.

Hashemizadeh and Mohsenyzadeh [27] obtained a hybrid of block-pulse and orthonormal Bernoulli polynomials and developed a method that allows solving systems of linear Volterra integral equations directly by means of operational matrices of Bernoulli polynomials.

The main goal of this part of our work is to contribute to the computation of integrals in a more understandable matrix form by using a hybrid function that combines the properties of block-pulse functions and polynomials $\mathrm{OB}_n(x, q)$.

In this section of our study, we construct a new function of orthogonal polynomials $\mathrm{OB}_n(x, q)$ obtained by using the Gram-Schmidt orthogonalization method, together with the block-pulse function in the interval $[0, t_f]$. We call these functions hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$.

Hybrid functions $\mathrm{OBH}_{nm}(t, q)$, where $n = 1, 2, \dots, N$, $m = 0, 1, \dots, M$, are defined over the interval $[0, t_f]$ as:

$$\mathrm{OBH}_{nm}(t, q) = \begin{cases} \mathrm{OB}_m\left(\frac{N}{t_f}t - n + 1, q\right) & t \in \left[\frac{n-1}{N}t_f\ ,\ \ \frac{n}{N}t_f\right), \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

where n and m are the order of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$ in equation (7), respectively.

For example, when $N = 3$, $M = 2$, $t_f = 1$ in equation (10), a hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$ is given by

$$\mathrm{OBH}_{nm}(t, q) = \begin{cases} \mathrm{OB}_m(3t - n + 1, q) & t \in \left[\frac{n-1}{3}, \frac{n}{3}\right), \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

In equation (11), we have

$$\left.\begin{aligned} &\mathrm{OBH}_{10}(t, q) = 1 \\ &\mathrm{OBH}_{11}(t, q) = 3t - \frac{1}{[2]_q} \\ &\mathrm{OBH}_{12}(t, q) = 9t^2 - \frac{3[2]_q}{q^2 + 1}t + \frac{q}{(q^2 + 1)[3]_q} \end{aligned}\right\} \quad t \in \left[0, \frac{1}{3}\right),$$

$$\left.\begin{aligned} &\mathrm{OBH}_{20}(t, q) = 1 \\ &\mathrm{OBH}_{21}(t, q) = 3t - \frac{1}{[2]_q} - 1 \\ &\mathrm{OBH}_{22}(t, q) = 9t^2 - 3\left([2]_q + \frac{[2]_q}{q^2 + 1}\right)t + \frac{[2]_q}{q^2 + 1} + \frac{q}{(q^2 + 1)[3]_q} + q \end{aligned}\right\} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right),$$

$$\left.\begin{aligned} &\mathrm{OBH}_{30}(t, q) = 1 \\ &\mathrm{OBH}_{31}(t, q) = 3t - \frac{1}{[2]_q} - 2 \\ &\mathrm{OBH}_{32}(t, q) = 9t^2 - \left(6[2]_q + \frac{3[2]_q}{q^2 + 1}\right)t + \frac{2[2]_q}{q^2 + 1} + \frac{q}{(q^2 + 1)[3]_q} + 4q \end{aligned}\right\} \quad t \in \left[\frac{2}{3}, 1\right].$$

3.1 Function approximation

Let $H = L_q^2[0,1] = \left\{ f = [0,1] \to [0,1] : \int_0^1 f_q^2(t) d_q t < \infty \right\}$ be the Hilbert space and

$$\{\mathrm{OBH}_{10}(t,q), \mathrm{OBH}_{20}(t,q), \ldots, \mathrm{OBH}_{NM}(t,q)\} \subset H$$

be the set of hybrid of block-pulse and orthogonal polynomials $\mathrm{OB}_n(x,q)$ and

$$\begin{aligned} Y = \operatorname{span}\{&\mathrm{OBH}_{10}(t,q), \mathrm{OBH}_{20}(t,q), \ldots, \mathrm{OBH}_{N0}(t,q), \mathrm{OBH}_{11}(t,q), \mathrm{OBH}_{21}(t,q), \\ &\ldots, \mathrm{OBH}_{N1}(t,q), \ldots, \mathrm{OBH}_{1M}(t,q), \mathrm{OBH}_{2M}(t,q), \ldots, \mathrm{OBH}_{NM}(t,q)\} \end{aligned}$$

is finite-dimensional vector subspace of H [28,29]. Unmatched best approach for any arbitrary elements of H like f is $f_0 \in Y$ such that for any $y \in Y$, the inequality

$$\|f - f_0\|_{2,q} \le \|f - y\|_{2,q}$$

where the norm is defined by

$$\|f\|_{2,q} = \left(\int_0^1 |f_q^2(t)| d_q t \right)^{\frac{1}{2}}$$

in [28]. Since $f_0 \in Y$, there exist the unique coefficients $c_{10,q}, c_{20,q}, \ldots, c_{NM,q}$ such that

$$f \simeq f_0 = \sum_{m=0}^{M} \sum_{n=1}^{N} c_{nm,q} \mathrm{OBH}_{nm}(t,q) = C^T(q)\mathrm{OBH}(t,q), \tag{12}$$

where

$$\begin{aligned} \mathrm{OBH}^T(t,q) = [&\mathrm{OBH}_{10}(t,q), \mathrm{OBH}_{20}(t,q), \ldots, \mathrm{OBH}_{N0}(t,q), \mathrm{OBH}_{11}(t,q), \mathrm{OBH}_{21}(t,q), \\ &\ldots, \mathrm{OBH}_{N1}(t,q), \ldots, \mathrm{OBH}_{1M}(t,q), \mathrm{OBH}_{2M}(t,q), \ldots, \mathrm{OBH}_{NM}(t,q)] \end{aligned}$$

and

$$C^T(q) = [c_{10,q}, c_{20,q}, \ldots, c_{N0,q}, c_{11,q}, c_{21,q}, \ldots, c_{N1,q}, \ldots, c_{1M,q}, c_{2M,q}, \ldots, c_{NM,q}].$$

For example, $N = 3, M = 2$, these are

$$\mathrm{OBH}^T(t,q) = [\mathrm{OBH}_{10}, \mathrm{OBH}_{20}, \mathrm{OBH}_{30}, \mathrm{OBH}_{11}, \mathrm{OBH}_{21}, \mathrm{OBH}_{31}, \mathrm{OBH}_{12}, \mathrm{OBH}_{22}, \mathrm{OBH}_{32}]$$

and

$$C^T(q) = [c_{10,q}, c_{20,q}, c_{30,q}, c_{11,q}, c_{21,q}, c_{31,q}, c_{12,q}, c_{22,q}, c_{32,q}].$$

Thus,

$$f_0 = \sum_{m=0}^{2} \sum_{n=1}^{3} c_{nm,q} \mathrm{OBH}_{nm}(t,q).$$

Using equation (12) we obtain

$$f_{ij} = \left\langle \sum_{m=0}^{M} \sum_{n=1}^{N} c_{nm,q} \mathrm{OBH}_{nm}(t,q), \mathrm{OBH}_{ij}(t,q) \right\rangle = \sum_{m=0}^{M} \sum_{n=1}^{N} c_{nm,q} d_{nm,q}^{ij}.$$

For $i = 1, 2, \ldots, N$ and $j = 0, 1, \ldots, M$ we have

$$f_{ij} = \langle f, b_{ij}(t,q) \rangle$$

and

$$d_{nm,q}^{ij} = \langle \mathrm{OBH}_{nm}(t,q), \mathrm{OBH}_{ij}(t,q) \rangle.$$

We obtain

$$\Phi(q) = D^T(q)C(q),$$

with

$$\Phi(q) = [f_{10}, f_{20}, \dots, f_{N0}, f_{11}, f_{21}, \dots, f_{N1}, \dots, f_{1M}, f_{2M}, \dots, f_{NM}]^T.$$

Additionally, we obtain $D(q) = [d^{ij}_{NM,q}]$, which is a matrix of order $N(M+1) \times N(M+1)$ and

$$D(q) = \int_0^1 \mathrm{OBH}(t,q)\mathrm{OBH}^T(t,q)\mathrm{d}t.$$

For $N = 3$ and $M = 2$, $D(q)$ matrix is

$$D(q) = \frac{1}{3}\begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{q-1}{[2]_q} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{2(q-1)}{[2]_q} & 0 \\ 0 & 0 & 0 & \frac{q}{[2]_q^2[3]_q} & 0 & 0 & 0 \\ 0 & -\frac{q-1}{[2]_q} & 0 & 0 & \frac{q^4-q^3-2q^2+4q-1}{[2]_q^2[3]_q} & 0 & 0 \\ 0 & 0 & -\frac{2(q-1)}{[2]_q} & 0 & 0 & \frac{4q^4-4q^3-6q^2+9q-2}{[2]_q^2[3]_q} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{q^4}{(q^2+1)^2[3]_q^2[5]_q} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ \frac{(q-1)(q^4-q^3-q^2-2)}{(q^2+1)[3]_q} & 0 \\ 0 & -\frac{2(2q^5-8q^4-5q^3-9q^2-8q-2)}{(q^2+1)[3]_q} \\ 0 & 0 \\ -\frac{q(q-1)(q^4-2q^3-2q^2+4q+1)}{(q^2+1)[2]_q[3]_q} & 0 \\ 0 & -\frac{4(2q^6-10q^5-3q^4+9q^3-5q^2+6q+2)}{(q^2+1)[2]_q[3]_q} \\ 0 & 0 \end{bmatrix},$$

and

$$A_3 = \begin{bmatrix} \frac{e_1}{(q^2+1)^2[3]_q^2[5]_q} & 0 \\ 0 & \frac{e_2}{(q^2+1)^2[3]_q^2[5]_q} \end{bmatrix},$$

where

$$\begin{aligned} e_1 &= q^{14} - 3q^{13} - q^{12} + 4q^{11} + q^{10} + 9q^9 - 5q^7 - 11q^6 - 9q^5 + 15q^4 - 3q^3 + q + 2, \\ e_2 &= 16q^{14} - 112q^{13} - 32q^{12} + 256q^{11} + 820q^{10} + 1756q^9 + 2560q^8 + 3172q^7 + 3170q^6 \\ &\quad + 2770q^5 + 2045q^4 + 1158q^3 + 466q^2 + 180q + 16. \end{aligned}$$

3.2 Operational matrix of integration

Let $K(q)$ be the $N(M+1)\times N(M+1)$ operational matrix of integration [26,30]; thus, the integration of the $\mathrm{OBH}_{nm}(t,q)$ is given by

$$\int_0^t \mathrm{OBH}(t',q)d_qt' \simeq K(q)\mathrm{OBH}(t,q).$$

For example with $M=2, N=2$, and $t_f=1$, we have

$$\mathrm{OBH}(t,q)=[\mathrm{OBH}_{10}(t,q),\mathrm{OBH}_{20}(t,q),\mathrm{OBH}_{11}(t,q),\mathrm{OBH}_{21}(t,q),\mathrm{OBH}_{12}(t,q),\mathrm{OBH}_{22}(t,q)]^T,$$

$$\left.\begin{aligned}\mathrm{OBH}_{10}(t,q)&=1\\ \mathrm{OBH}_{11}(t,q)&=2t-\frac{1}{[2]_q}\\ \mathrm{OBH}_{12}(t,q)&=4t^2-\frac{2[2]_q}{q^2+1}t+\frac{q}{(q^2+1)[3]_q}\end{aligned}\right\}\ t\in\left[0,\frac{1}{2}\right),$$

$$\left.\begin{aligned}\mathrm{OBH}_{20}(t,q)&=1\\ \mathrm{OBH}_{21}(t,q)&=2t-\frac{1}{[2]_q}-1\\ \mathrm{OBH}_{22}(t,q)&=4t^2-2\left([2]_q+\frac{[2]_q}{q^2+1}\right)t+\frac{[2]_q}{q^2+1}+\frac{q}{(q^2+1)[3]_q}+q\end{aligned}\right\}\ t\in\left[\frac{1}{2},1\right).$$

Thus, we obtain

$$\begin{aligned}\int_0^t \mathrm{OBH}_{10}(t',q)d_qt'&=\begin{cases}t, & t\in\left[0,\frac{1}{2}\right)\\ \frac{1}{2}, & t\in\left[\frac{1}{2},1\right)\end{cases}\\ &=\frac{1}{2[2]_q}\mathrm{OBH}_{10}(t,q)+\frac{1}{2}\mathrm{OBH}_{20}(t,q)+\frac{1}{2}\mathrm{OBH}_{11}(t,q),\end{aligned}\tag{13}$$

$$\begin{aligned}\int_0^t \mathrm{OBH}_{20}(t',q)d_qt'&=\begin{cases}0, & t\in\left[0,\frac{1}{2}\right)\\ t-\frac{1}{2}, & t\in\left[\frac{1}{2},1\right)\end{cases}\\ &=\frac{1}{2[2]_q}\mathrm{OBH}_{20}(t,q)+\frac{1}{2}\mathrm{OBH}_{21}(t,q),\end{aligned}\tag{14}$$

$$\begin{aligned}\int_0^t \mathrm{OBH}_{11}(t',q)d_qt'&=\begin{cases}\frac{2}{[2]_q}t^2-\frac{1}{[2]_q}t, & t\in\left[0,\frac{1}{2}\right)\\ 0, & t\in\left[\frac{1}{2},1\right)\end{cases}\\ &=-\frac{q^2}{2[3]_q[2]_q^2}\mathrm{OBH}_{10}(t,q)+\left(-\frac{q(q-1)}{2[4]_q}\right)\mathrm{OBH}_{11}(t,q)+\frac{1}{2[2]_q}\mathrm{OBH}_{12}(t,q),\end{aligned}\tag{15}$$

$$\int_0^t \mathrm{OBH}_{21}(t',q)d_qt' = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right) \\ \frac{2}{[2]_q}t^2 - \left(\frac{1}{[2]_q} + 1\right)t + \frac{1}{2}, & t \in \left[\frac{1}{2}, 1\right) \end{cases} \tag{16}$$
$$= -\frac{q^2}{2[3]_q[2]_q^2}\mathrm{OBH}_{20}(t,q) + \left(-\frac{q(q-1)}{2[4]_q}\right)\mathrm{OBH}_{21}(t,q) + \frac{1}{2[2]_q}\mathrm{OBH}_{22}(t,q),$$

$$\int_0^t \mathrm{OBH}_{12}(t',q)d_qt' = \begin{cases} \frac{4}{[3]_q}t^3 - \frac{2}{q^2+1}t^2 + \frac{q}{(q^2+1)[3]_q}t, & t \in \left[0, \frac{1}{2}\right) \\ 0, & t \in \left[\frac{1}{2}, 1\right) \end{cases} \tag{17}$$
$$= \ldots + \frac{1}{2[3]_q}\mathrm{OBH}_{13}(t,q),$$

and

$$\int_0^t \mathrm{OBH}_{22}(t',q)d_qt' = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right) \\ \frac{4}{[3]_q}t^3 - 2\left(\frac{q^2+2}{q^2+1}\right)t^2 + \left(\frac{[2]_q}{q^2+1} + \frac{q}{(q^2+1)[3]_q} + q\right)t - \frac{q^3-q^2+2q}{2(q^2+1)}, & t \in \left[\frac{1}{2}, 1\right) \end{cases} \tag{18}$$
$$= \ldots + \frac{1}{2[3]_q}\mathrm{OBH}_{23}(t,q).$$

Using equations (13)–(18) and disregarding $\mathrm{OBH}_{13}(t,q)$ and $\mathrm{OBH}_{23}(t,q)$, we obtain

$$\int_0^t \mathrm{OBH}(t',q)d_qt' = \begin{bmatrix} \int_0^t \mathrm{OBH}_{10}(t',q)d_qt' \\ \int_0^t \mathrm{OBH}_{20}(t',q)d_qt' \\ \int_0^t \mathrm{OBH}_{11}(t',q)d_qt' \\ \int_0^t \mathrm{OBH}_{21}(t',q)d_qt' \\ \int_0^t \mathrm{OBH}_{12}(t',q)d_qt' \\ \int_0^t \mathrm{OBH}_{22}(t',q)d_qt' \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2[2]_q} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2[2]_q} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{q^2}{2[3]_q[2]_q^2} & 0 & -\frac{q(q-1)}{2[4]_q} & 0 & \frac{1}{2[2]_q} & 0 \\ 0 & \frac{q^2}{2[3]_q[2]_q^2} & 0 & -\frac{q(q-1)}{2[4]_q} & 0 & \frac{1}{2[2]_q} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathrm{OBH}(t,q).$$

Therefore, 6 × 6 the $K(q)$ matrix is

$$K(q) = \frac{1}{2}\begin{bmatrix} \frac{1}{[2]_q} & 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{[2]_q} & 0 & 1 & 0 & 0 \\ -\frac{q^2}{[3]_q[2]_q^2} & 0 & -\frac{q(q-1)}{[4]_q} & 0 & \frac{1}{[2]_q} & 0 \\ 0 & -\frac{q^2}{[3]_q[2]_q^2} & 0 & -\frac{q(q-1)}{[4]_q} & 0 & \frac{1}{[2]_q} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let I and O be $N \times N$ identity and zero matrices, respectively, and we have

$$K(q) = \frac{1}{2}\begin{bmatrix} K_0(q) & I & O \\ \frac{q^2}{[3]_q[2]_q^2}I & -\frac{q(q-1)}{[2]_q(q^2+1)}I & \frac{1}{[2]_q}I \\ O & O & O \end{bmatrix},$$

where the $K_0(q)$ is

$$K_0(q) = \begin{bmatrix} -\mathrm{OB}_1(0,q) & 1 & \dots & 1 & 1 \\ 0 & -\mathrm{OB}_1(0,q) & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\mathrm{OB}_1(0,q) & 1 \\ 0 & 0 & \dots & 0 & -\mathrm{OB}_1(0,q) \end{bmatrix}.$$

3.3 q-Variational problems

In this chapter, we focus on the following problems:

In [31], for $0 \le \alpha < \beta \le +\infty$ and $k \in \mathbb{Z}^+$, q-variational problem is defined as follows:

$$J(y(x)) = \int_{q^\alpha}^{q^\beta} F(x, y(x), D_q y(x), \dots, D_q^k y(x)) d_q x, \tag{19}$$

$$\overset{\text{def}}{=} (1-q)\sum_{q^\alpha}^{q^\beta} xF(x, y(x), D_q y(x), \dots, D_q^k y(x)) \tag{20}$$

under the boundary constraints

$$\begin{aligned} y(q^\alpha) = y(q^{\beta+1}) &= c_0, \\ D_q y(q^\alpha) = D_q y(q^{\beta+1}) &= c_1, \\ &\vdots \\ D_q^{k-1} y(q^\alpha) = D_q^{k-1} y(q^{\beta+1}) &= c_{k-1}, \end{aligned}$$

where

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x},$$

for any function f, while the summation is performed by x on the set

$$L = \{q^{\beta}, q^{\beta-1}, \ldots, q^{\alpha+1}, q^{\alpha}\}.$$

In equations (19) and (20), for $\alpha \to 0$ and $\beta \to +\infty$, we have the q-variational problem

$$J(y(x)) = \int_0^1 F(x, y(x), D_q y(x), \ldots, D_q^k y(x)) d_q x \tag{21}$$

and $D_q^i y(0) = D_q^i y(1)$, $i = 0, 1, \ldots, k-1$.

In equation (21), for $q \to 1$, we can easily see the variational problem given in [10,26].

Now, we define the q-variational problem similar to Example 2 in [10] and show that we can solve this problem with the help of a hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$.

Example 1. We have the q-variational problem given in the following:

$$J = \int_0^1 \left(\frac{1}{2} (D_q y(t))^2 - t D_q^2 y(t) \right) d_q t, \tag{22}$$

where

$$D_q^2 y(t) = \begin{cases} -1, & 0 \le y(t) \le \frac{1}{4}, \\ 3, & \frac{1}{4} \le y(t) \le \frac{1}{2}, \\ -1, & \frac{1}{2} \le y(t) \le 1, \end{cases} \tag{23}$$

with the boundary conditions

$$y(0) = 0, \quad D_q y(0) = 0 \quad \text{and} \quad D_q y(1) = 0.$$

The exact solution of the q-variational problem in equation (22) is

$$y(t) = \begin{cases} \frac{1}{[2]_q} t^2, & 0 \le y(t) \le \frac{1}{4}, \\ -\frac{3}{[2]_q} t^2 + t + \frac{1}{4[2]_q} - \frac{1}{4}, & \frac{1}{4} \le y(t) \le \frac{1}{2}, \\ \frac{1}{[2]_q} t^2 - t - \frac{3}{4[2]_q} + \frac{3}{4}, & \frac{1}{2} \le y(t) \le 1. \end{cases}$$

Now, we solve the q-variational problem in equation (22) using the hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$. Let

$$D_q y(t) = C^T(q)\mathrm{OBH}(t, q).$$

By writing equation (23) in equation (22), we obtain

$$J = \frac{1}{2} \int_0^1 (D_q y(t))^2 d_q t + 4 \int_0^{\frac{1}{4}} y(t) d_q t - 4 \int_0^{\frac{1}{2}} y(t) d_q t + \int_0^1 y(t) d_q t.$$

Thus, we have

$$J = \frac{1}{2} \int_0^1 C^T(q)\mathrm{OBH}(t, q)\mathrm{OBH}^T(t, q)C(q) d_q t + 4C^T(q)K(q) \int_0^{\frac{1}{4}} \mathrm{OBH}(t, q) d_q t$$
$$- 4C^T(q)K(q) \int_0^{\frac{1}{2}} \mathrm{OBH}(t, q) d_q t + C^T(q)K(q) \int_0^1 \mathrm{OBH}(t, q) d_q t.$$

For

$$W(t) = \int_0^t \mathrm{OBH}(t', q) d_q t',$$

using the equation

$$D(q) = \int_0^1 \mathrm{OBH}(t, q)\mathrm{OBH}^T(t, q) d_q t,$$

we obtain

$$J = \frac{1}{2} C^T(q) D(q) C(q) + C^T(q) K(q) \left(4W\left(\frac{1}{4}\right) - 4W\left(\frac{1}{2}\right) + W(1)\right). \tag{24}$$

The boundary conditions in equation (22) can be represented using hybrid functions as

$$C^T(q)\mathrm{OBH}(0, q) = 0, \quad C^T(q)\mathrm{OBH}(1, q) = 0. \tag{25}$$

Now, we obtain the extremum of equation (24) subject to equation (25) using the Lagrange multiplier technique. For two multipliers $\lambda_1(q)$ and $\lambda_2(q)$, let us assume

$$J^* = J + \lambda_1(q) C^T(q)\mathrm{OBH}(0, q) + \lambda_2(q) C^T(q)\mathrm{OBH}(1, q).$$

We obtain the following necessary conditions:

$$D_{q,C^T} J^* = \frac{1}{2} D(q) C(q) + K(q)\left(4W\left(\frac{1}{4}\right) - 4W\left(\frac{1}{2}\right) + W(1)\right) + \lambda_1(q)\mathrm{OBH}(0, q) + \lambda_2(q)\mathrm{OBH}(1, q) = 0. \tag{26}$$

By choosing $M = 2$ and $N = 4$, equations (25) and (26) define a set of simultaneous linear algebraic equations from which the coefficient vector $C(q)$ and the multipliers $\lambda_1(q)$ and $\lambda_2(q)$ can be found.

4 Conclusion

In this article, the polynomials $\mathrm{OB}_n(x, q)$ obtained by applying the Gram-Schmidt orthogonalization method to q-Bernoulli polynomials are introduced and various properties are mentioned. In particular, the relations of polynomials $\mathrm{OB}_n(x, q)$ to little q-Legendre polynomials are shown. Then, the hybrid of block-pulse function and orthogonal polynomials $\mathrm{OB}_n(x, q)$ is defined. In addition, a special q-variational problem is defined and the functionality of the hybrid function is demonstrated to solve this problem. Thus, the importance of the practical and functional contributions that hybrid functions provide in future research is emphasized.

Acknowledgment: The authors thank the reviewers for giving valuable comments to improve the manuscript.

Funding information: The authors state no funding involved.

Author contributions: All authors contributed equally to this work.

Conflict of interest: The authors state no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Data availability statement: No data were used to support this study.

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