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RESEARCH

Hardy-Littlewood-Sobolev inequality in total Morrey spaces

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Abstract

We study some necessary and sufficient conditions for the boundedness of the Riesz potential operator I_α and its commutator on the total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$. We characterize the strong and weak Spanne type and Adams type boundedness of I_α on $L_{p,\lambda,\mu}(\mathbb{R}^n)$, respectively. We also give necessary and sufficient conditions for the boundedness of the commutator of the Riesz potential operator $[b, I_\alpha]$ on $L_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to the spaces $BMO(\mathbb{R}^n)$. As applications, we obtain some estimates for the Marcinkiewicz operator and fractional powers of some analytic semigroups on the total Morrey spaces.

Keywords Total Morrey spaces · Riesz potential · Maximal operator · Commutator · BMO space · Marcinkiewicz operator · Fractional powers of some analytic semigroups

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Introduction

Morrey spaces, denoted as $L_{p,\lambda}(\mathbb{R}^n)$ and named after Charles Morrey [20], generalize Lebesgue spaces by incorporating a parameter λ that captures the local behavior of functions within a given region, whereas Lebesgue spaces typically consider the global properties of a function. By adding this new parameter, Morrey spaces provide a more detailed analysis of functions, which is particularly useful in the study of partial differential equations (PDEs).

The total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by the author in [14], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. The space $L_{p,\lambda,\mu}(\mathbb{R}^n)$ is defined as the set of functions whose

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norm is finite, where the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$. Total Morrey spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to a classical Morrey space, and the case where $\mu = 0$ corresponds to a modified Morrey space, see [1, 4, 8, 12, 15–18, 21, 22].

The Hardy-Littlewood-Sobolev (HLS) inequality, which bounds the Riesz potential operator in Lebesgue spaces, has been extended to total Morrey spaces. These inequalities specify conditions under which the Riesz potential maps functions from a total Morrey space to another total Morrey space with potentially different parameters. In particular, the HLS inequality in total Morrey spaces investigates how applying the Riesz potential affects the integrability and decay of a function, as measured by the norms of the total Morrey space.

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,t) = \mathbb{R}^n \setminus B(x,t)$.

One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)| dy, \quad 0 \le \alpha < n,$$

where $|B(x,t)| = v_n t^n$ is the Lebesgue measure of the ball $B(x,t)$ and v_n is the volume of the unit ball in $\mathbb{R}^n$. It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ and is intimately related to the Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \qquad 0 < \alpha < n$$

(see, for example, [2] and [3]). The operators M_α and I_α play important role in real and harmonic analysis (see, for example [10] and [26]).

The aim of this paper is to establish necessary and sufficient conditions for the boundedness of the Riesz potential operator I_α and its commutator $[b, I_\alpha]$ on $L_{p,\lambda,\mu}(\mathbb{R}^n)$, when b belongs to the spaces $BMO(\mathbb{R}^n)$.

The structure of the paper is as follows. Section 1 presents definitions, auxiliary results, and some embeddings into the total Morrey space $L_{p,\lambda,\mu}(\mathbb{R}^n)$. Section 2 characterizes the strong and weak boundedness of the Spanne and Adams types for the Riesz potential operator I_α on the spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, respectively. Section 3 provides necessary and sufficient conditions for the boundedness of the commutator of the Riesz potential operator $[b, I_\alpha]$ on the spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$. Section 4 provides sufficient conditions for the boundedness of the modified Riesz potential operator $\widetilde{I_\alpha}$ on the spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 5, as an application, estimates are obtained for the Marcinkiewicz operator and fractional powers of some analytic semigroups in total Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

1 Basic properties of total Morrey spaces

At first, define the total Morrey space $L_{p,\lambda,\mu}(\mathbb{R}^n)$. In [14] the author introduced a variant of Morrey spaces on n-dimensional Euclidean space $\mathbb{R}^n$ called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, see also [1, 15–17, 21, 22].

Definition 1.1 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the classical Morrey space, by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space [12], and by $L_{p,\lambda,\mu}(\mathbb{R}^n)$ the total Morrey space the set of all classes of locally integrable functions f with the finite quasi-norms

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))}, \quad \|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))},$$

$$\|f\|_{L_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,t))},$$

respectively.

Definition 1.2 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $WL_{p,\lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ [12] and the weak total Morrey space $WL_{p,\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite quasi-norms

$$\|f\|_{WL_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t))}, \quad \|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t))},$$

$$\|f\|_{WL_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(B(x,t))},$$

respectively.

Lemma 1.3 ([14, Lemma 2]) *If $0 < p < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$L_{p,\lambda,\mu}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) \cap L_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\lambda,\mu}} = \max\left\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_{p,\mu}}\right\}.$$

Lemma 1.4 ([14, Lemma 3]) *If $0 < p < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$WL_{p,\lambda,\mu}(\mathbb{R}^n) = WL_{p,\lambda}(\mathbb{R}^n) \cap WL_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL_{p,\lambda}}, \|f\|_{WL_{p,\mu}}\right\}.$$

Remark 1.5 Let $0 < p < \infty$. If $\mu < 0$ or $\lambda > n$, then

$$L_{p,\lambda,\mu}(\mathbb{R}^n) = WL_{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n),$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 1.6 *If* $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$ *and* $0 \le \mu_1 \le \mu_2 \le n$, *then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}^n) \hookrightarrow L_{p,\lambda_2,\mu_2}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}} \le \|f\|_{L_{p,\lambda_1,\mu_1}}.$$

Lemma 1.7 *If* $0 < p < \infty$, $0 \le \lambda \le n$ *and* $0 \le \mu \le n$, *then*

$$L_{p,n,\mu}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \hookrightarrow L_{p,\lambda,n}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\lambda,n}} \le v_n^{1/p} \|f\|_{L_\infty} \le \|f\|_{L_{p,n,\mu}}.$$

Lemma 1.8 *If* $0 \le \lambda < n$, $0 \le \mu < n$, $0 \le \alpha < n - \lambda$ *and* $0 \le \beta < n - \mu$, *then for* $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$

$$L_{p,\lambda,\mu}(\mathbb{R}^n) \hookrightarrow L_{1,n-\alpha,n-\beta}(\mathbb{R}^n)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ *the following inequality*

$$\|f\|_{L_{1,n-\alpha,n-\beta}} \le v_n^{1/p'} \|f\|_{L_{p,\lambda,\mu}}$$

is valid.

Theorem 1.9 [14, Theorem 1], [17, Corollary 3.1] *Let* $1 \le p < \infty$, $0 \le \lambda < n$ *and* $0 \le \mu < n$.

1. If $p > 1$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $Mf \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{L_{p,\lambda,\mu}} \le C_{p,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

2. If $f \in L_{1,\lambda,\mu}(\mathbb{R}^n)$, *then* $Mf \in WL_{1,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{WL_{1,\lambda,\mu}} \le C_{1,\lambda,\mu} \|f\|_{L_{1,\lambda,\mu}},$$

where $C_{1,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

Theorem 1.10 [14, Theorem 3], [17, Corollary 3.1] *Let* $1 < p < \infty$, $0 \le \lambda < n$, $0 \le \mu < n$ *and* $b \in BMO(\mathbb{R}^n)$. *If* $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $M_b f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|M_b f\|_{L_{p,\lambda,\mu}} \le C_{p,\lambda,\mu} \|b\|_* \|f\|_{L_{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu}$ *depends only on* p, λ, μ *and* n.

Theorem 1.11 [23, Theorem 5.4] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 \le \lambda < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. *Then operator* I_α *is bounded from* $L_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\lambda\frac{q}{p}}(\mathbb{R}^n)$.

Theorem 1.12 [2] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 \le \lambda < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Then operator* I_α *is bounded from* $L_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\lambda}(\mathbb{R}^n)$.

Theorem 1.13 [24, Theorem 3.1] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 \le \lambda < n - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. *Then the following conditions are equivalent:*

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\lambda\frac{q}{p}}(\mathbb{R}^n)$.

Theorem 1.14 [19, Theorem 1] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 \le \lambda < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Then the following conditions are equivalent:*

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\lambda}(\mathbb{R}^n)$.

2 Hardy-Littlewood-Sobolev inequality in total Morrey spaces

The classical by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{q}\right)$.

The following local estimate is valid (see also [11]).

Lemma 2.1 [11, Theorem 5.1] *Let* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. *Then, for* $p > 1$ *the inequality*

$$\|I_\alpha f\|_{L_q(B(x,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}} \|f\|_{L_p(B(x,t))} \frac{dt}{t} \tag{2.1}$$

holds for all $B(x,r)$ *and for all* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover if $p = 1$, *then the inequality*

$$\|I_\alpha f\|_{WL_q(B(x,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}} \|f\|_{L_1(B(x,t))} \frac{dt}{t} \tag{2.2}$$

holds for all $B(x,r)$ *and for all* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Below is a Spanne type result for the Riesz potential in total Morrey spaces (see, for example, [11]).

Theorem 2.2 (Spanne type result) *Let* $1 \le p < \infty$, $0 \le \lambda, \mu < n$, $0 < \alpha < \min\left\{\frac{n-\lambda}{p}, \frac{n-\mu}{p}\right\}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

1. If $p > 1$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $I_\alpha f \in L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ *and*

$$\|I_\alpha f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}}, \tag{2.3}$$

where $C_{p,\lambda,\mu}$ *depends only on* p,λ,μ *and* n.

2. If $p=1$, $f\in L_{1,\lambda,\mu}(\mathbb{R}^n)$, then $I_\alpha f\in WWL_{q,\lambda q,\mu q}(\mathbb{R}^n)$ and

$$\|I_\alpha f\|_{WL_{q,\lambda q,\mu q}} \le C_{1,\lambda,\mu}\,\|f\|_{L_{1,\lambda,\mu}}, \tag{2.4}$$

where $C_{1,\lambda,\mu}$ is independent of f.

Proof Let $1<p<\infty$. From the inequality (2.1) (see Lemma 2.1) we get

$$\begin{aligned}
\|I_\alpha f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} &= \sup_{x\in\mathbb{R}^n,\,r>0} [r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,\|I_\alpha f\|_{L_q(B(x,r))}\\
&\lesssim \sup_{x\in\mathbb{R}^n,\,r>0} [r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,r^{\frac{n}{q}}\int_{2r}^\infty t^{-\frac{n}{q}}\,\|f\|_{L_p(B(x,t))}\frac{dt}{t}\\
&\lesssim \|f\|_{L_{p,\lambda,\mu}}\sup_{r>0}[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,r^{-\alpha+\frac{n}{p}}\int_r^\infty t^{\alpha-\frac{n}{p}}\,[t]_1^{\frac{\lambda}{p}}\,[1/t]_1^{-\frac{\mu}{p}}\frac{dt}{t}\\
&= \|f\|_{L_{p,\lambda,\mu}}\sup_{r>0}[r]_1^{-\alpha+\frac{n-\lambda}{p}}\,[1/r]_1^{\alpha-\frac{n-\mu}{p}}\int_r^\infty [t]_1^{\alpha-\frac{n-\lambda}{p}}\,[1/t]_1^{-\alpha+\frac{n-\mu}{p}}\frac{dt}{t}\\
&\approx \|f\|_{L_{p,\lambda,\mu}}\int_1^\infty [t]_1^{\alpha-\frac{n-\lambda}{p}}\,[1/t]_1^{-\alpha+\frac{n-\mu}{p}}\frac{dt}{t}\\
&\lesssim \|f\|_{L_{p,\lambda,\mu}},
\end{aligned}$$

which implies that the operator $I_\alpha f$ is bounded from $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$.

Let $p=1$. From the inequality (2.2) (see Lemma 2.1) we get

$$\begin{aligned}
\|I_\alpha f\|_{WL_{q,\lambda q,\mu q}} &= \sup_{x\in\mathbb{R}^n,\,r>0} [r]_1^{-\lambda}\,[1/r]_1^{\mu}\,\|I_\alpha f\|_{WL_q(B(x,r))}\\
&\lesssim \sup_{x\in\mathbb{R}^n,\,r>0} [r]_1^{-\lambda}\,[1/r]_1^{\mu}\,r^{\frac{n}{q}}\int_{2r}^\infty t^{-\frac{n}{q}}\,\|f\|_{L^1(B(x,t))}\frac{dt}{t}\\
&\lesssim \|f\|_{L_{1,\lambda,\mu}}\sup_{r>0}[r]_1^{-\lambda}\,[1/r]_1^{\mu}\,r^{-\alpha+n}\int_r^\infty t^{\alpha-n}\,[t]_1^{\lambda}\,[1/t]_1^{-\mu}\frac{dt}{t}\\
&= \|f\|_{L^{1,\lambda,\mu}}\sup_{r>0}[r]_1^{-\alpha+n-\lambda}\,[1/r]_1^{\alpha-(n-\mu)}\int_r^\infty [t]_1^{\alpha-(n-\lambda)}\,[1/t]_1^{-\alpha+(n-\mu)}\frac{dt}{t}\\
&\approx \|f\|_{L_{1,\lambda,\mu}}\int_1^\infty [t]_1^{\alpha-(n-\lambda)}\,[1/t]_1^{-\alpha+(n-\mu)}\frac{dt}{t}\\
&\lesssim \|f\|_{L_{1,\lambda,\mu}},
\end{aligned}$$

which implies that the operator $I_\alpha f$ is bounded from $L_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL_{q,\lambda q,\mu q}(\mathbb{R}^n)$. □

From Theorem 2.2 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 2.3 [23, Theorem 5.4] *Let $1\le p<\infty$, $0<\lambda<n$, $0<\alpha<\frac{n-\lambda}{p}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$.*

1. If $p > 1$, $f \in L_{p,\lambda}(\mathbb{R}^n)$*, then* $I_\alpha f \in L_{q,\frac{\lambda q}{p}}(\mathbb{R}^n)$ *and*

$$\|I_\alpha f\|_{L_{q,\frac{\lambda q}{p}}} \leq C_{p,\lambda} \, \|f\|_{L_{p,\lambda}}, \tag{2.5}$$

where $C_{p,\lambda}$ *depends only on* p, λ *and* n.

2. If $p = 1$, $f \in L_{1,\lambda}(\mathbb{R}^n)$*, then* $I_\alpha f \in WL_{q,\lambda}(\mathbb{R}^n)$ *and*

$$\|I_\alpha f\|_{WL_{q,\lambda q}} \leq C_{1,\lambda} \, \|f\|_{L_{1,\lambda}}, \tag{2.6}$$

where $C_{1,\lambda}$ *is independent of* f.

Remark 2.4 Note that in the case of

Corollary 2.5 *Let* $1 \leq p < \infty$, $0 < \lambda < n$, $0 < \alpha < \frac{n-\lambda}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

1. If $p > 1$, $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$*, then* $I_\alpha f \in \widetilde{L}_{q,\frac{\lambda q}{p}}(\mathbb{R}^n)$ *and*

$$\|I_\alpha f\|_{\widetilde{L}_{q,\frac{\lambda q}{p}}} \leq C_{p,\lambda} \, \|f\|_{\widetilde{L}_{p,\lambda}}, \tag{2.7}$$

where $C_{p,\lambda}$ *depends only on* p, λ *and* n.

2. If $p = 1$, $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$*, then* $I_\alpha f \in W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ *and*

$$\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda q}} \leq C_{1,\lambda} \, \|f\|_{\widetilde{L}_{1,\lambda}}, \tag{2.8}$$

where $C_{1,\lambda}$ *is independent of* f.

Below is a Adams type result for the Riesz potential in total Morrey spaces, see, for example, [11, 13].

Theorem 2.6 (Adams type result) *Let* $1 \leq p < \infty$, $0 \leq \mu \leq \lambda < n$, $0 < \alpha < \frac{n-\lambda}{p}$.

1) If $1 < p < \frac{n-\lambda}{\alpha}$*, then condition* $\frac{\alpha}{n-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L_{q,\lambda,\mu}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{n-\lambda}{\alpha}$*, then condition* $\frac{\alpha}{n-\mu} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{1,\lambda,\mu}(\mathbb{R}^n)$ *to* $WL_{q,\lambda,\mu}(\mathbb{R}^n)$.

Proof *Sufficiency.* Let $1 \leq p < \frac{n-\lambda}{\alpha}$, $0 < \alpha < \frac{n-\lambda}{p}$, $\frac{\alpha}{n-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n-\lambda}$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ and r arbitrary positive number.

$$
\begin{aligned}
I_\alpha|f|(x) &= \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \approx \int_{\mathbb{R}^n} \Big(\int_{|x-y|}^{\infty} t^{\alpha-n-1}\,dt\Big)\,|f(y)|\,dy \\
&= \int_{\mathbb{R}^n}\int_0^{\infty} \chi_{(|x-y|,\infty)}(t)\,t^{\alpha-n-1}\,|f(y)|dtdy \\
&= \int_0^{\infty}\Big(\int_{\mathbb{R}^n} \chi_{(|x-y|,\infty)}(t)\,|f(y)|dy\Big)\,t^{\alpha-n-1}dt = \int_0^{\infty} t^{\alpha-n}\,\|f\|_{L_1(B(x,t))}\,\frac{dt}{t} \\
&= \int_0^{r} t^{\alpha-n}\,\|f\|_{L_1(B(x,t))}\,\frac{dt}{t} + \int_r^{\infty} t^{\alpha-n}\,\|f\|_{L_1(B(x,t))}\,\frac{dt}{t} \\
&\lesssim r^{\alpha}\,Mf(x) + r^{\alpha-\frac{n}{p}}\,\|f\|_{L_p(B(x,r))} \\
&\le r^{\alpha}\,Mf(x) + r^{\alpha-\frac{n}{p}}\,[r]_1^{\frac{\lambda}{p}}\,[1/r]_1^{-\frac{\mu}{p}}\,\|f\|_{L_{p,\lambda,\mu}} \\
&= r^{\alpha}\,Mf(x) + [r]_1^{\alpha-\frac{n-\lambda}{p}}\,[1/r]_1^{-\alpha+\frac{n-\mu}{p}}\,\|f\|_{L_{p,\lambda,\mu}} \\
&\le \min\Big\{r^{\alpha}\,Mf(x) + r^{\alpha-\frac{n-\lambda}{p}}\,\|f\|_{L_{p,\lambda,\mu}},\, r^{\alpha}\,Mf(x),\, r^{\alpha-\frac{n-\mu}{p}}\,\|f\|_{L_{p,\lambda,\mu}}\Big\}.
\end{aligned}
$$

Minimizing with respect to r, at

$$
r = \Big(\frac{\|f\|_{L_{p,\lambda,\mu}}}{Mf(x)}\Big)^{\frac{p}{n-\lambda}} \quad \text{and} \quad r = \Big(\frac{\|f\|_{L_{p,\lambda,\mu}}}{Mf(x)}\Big)^{\frac{p}{n-\mu}}
$$

we have

$$
I_\alpha\|f\|(x) \le \min\Big\{\big(Mf(x)\big)^{1-\frac{\alpha p}{n-\lambda}}\,\|f\|_{L_{p,\lambda,\mu}}^{\frac{\alpha p}{n-\lambda}},\, \big(Mf(x)\big)^{1-\frac{\alpha p}{n-\mu}}\,\|f\|_{L_{p,\lambda,\mu}}^{\frac{\alpha p}{n-\mu}}\Big\}, \tag{2.9}
$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 1.9 and inequality (2.9), we get

$$
\begin{aligned}
\|I_\alpha f\|_{L_{q,\lambda,\mu}} &\lesssim \|f\|_{L_{p,\lambda,\mu}}^{1-\frac{p}{q}}\,\|(Mf)^{\frac{p}{q}}\|_{L_{q,\lambda,\mu}} \\
&= \|f\|_{L_{p,\lambda,\mu}}^{1-\frac{p}{q}}\,\|Mf\|_{L_{p,\lambda,\mu}}^{\frac{p}{q}} \lesssim \|f\|_{L_{p,\lambda,\mu}},
\end{aligned}
$$

if $1 < p < q < \infty$ and

$$
\|I_\alpha f\|_{WL_{q,\lambda,\mu}} \lesssim \|f\|_{L_{1,\lambda,\mu}}^{1-\frac{1}{q}}\,\|Mf\|_{WL_{1,\lambda,\mu}}^{\frac{1}{q}} \lesssim \|f\|_{L_{1,\lambda,\mu}},
$$

if $p = 1 < q < \infty$.

Necessity. Let $1 < p < \frac{n-\lambda}{\alpha}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that I_α is bounded from $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1, t\}$. Then

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f_t\|_{L_p(B(x,r))} \\
&= t^{-\frac{n}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,tr))} \\
&= t^{-\frac{n}{p}} \sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\frac{\lambda}{p}} \sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\frac{\mu}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} [1/(tr)]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,tr))} \\
&= t^{-\frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p}} [1/t]_{1,+}^{-\frac{\mu}{p}} \|f\|_{L_{p,\lambda,\mu}},
\end{aligned}$$

and

$$I_\alpha f_t(x) = t^{-\alpha} I_\alpha f(tx),$$

$$\begin{aligned}
\left\|I_\alpha f_t\right\|_{L_{q,\lambda,\mu}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|I_\alpha f(t\,\cdot)\|_{L_q(B(x,r))} \\
&= t^{-\alpha-\frac{n}{q}} \sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} [1/(tr)]_1^{\frac{\mu}{p}} \|I_\alpha f\|_{L_q(B(tx,tr))} \\
&= t^{-\alpha-\frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{q}} \left\|I_\alpha f\right\|_{L_{q,\lambda,\mu}}.
\end{aligned}$$

By the boundedness of I_α from $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\left\|I_\alpha f\right\|_{L_{q,\lambda,\mu}} &= t^{\alpha+\frac{n}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \left\|I_\alpha f_t\right\|_{L_{q,\lambda,\mu}} \\
&\lesssim t^{\alpha+\frac{n}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \left\|f_t\right\|_{L_{p,\lambda,\mu}} \\
&= t^{\alpha+\frac{n}{q}-\frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{p}+\frac{\mu}{q}} \|f\|_{L_{p,\lambda,\mu}} \\
&= t^{\alpha} [t]_{1,+}^{-\frac{n-\lambda}{p}+\frac{n-\lambda}{q}} [1/t]_{1,+}^{\frac{n-\mu}{p}-\frac{n-\mu}{q}} \|f\|_{L_{p,\lambda,\mu}}.
\end{aligned}$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n-\mu}$, then by letting $t \to 0$ we have $\|I_\alpha f\|_{L_{q,\lambda,\mu}} = 0$ for all $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|I_\alpha f\|_{L_{q,\lambda,\mu}} = 0$ for all $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$.

Let $p = 1 < \frac{n-\lambda}{\alpha}$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that I_α is bounded from $L_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL_{q,\lambda,\mu}(\mathbb{R}^n)$. Then

$$\left\|f_t\right\|_{L_{1,\lambda,\mu}} = t^{-n} [t]_{1,+}^{\lambda} [1/t]_{1,+}^{-\mu} \|f\|_{L_{1,\lambda,\mu}}$$

and

$$\begin{aligned}\left\|I_\alpha f_t\right\|_{WL_{q,\lambda,\mu}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|I_\alpha f(t\,\cdot)\|_{WL_q(B(x,r))} \\ &= t^{-\alpha-\frac{n}{q}} \sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} [1/(tr)]_1^{\frac{\mu}{p}} \|I_\alpha f\|_{WL_q(B(tx,tr))} \\ &= t^{-\alpha-\frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} [1/t]_{1,+}^{-\frac{\mu}{q}} \left\|I_\alpha f\right\|_{WL_{q,\lambda,\mu}}.\end{aligned}$$

By the boundedness of I_α from $L_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL_{q,\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}\left\|I_\alpha f\right\|_{WL_{q,\lambda,\mu}} &= t^{\alpha+\frac{n}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \left\|I_\alpha f_t\right\|_{WL_{q,\lambda,\mu}} \\ &\lesssim t^{\alpha+\frac{n}{q}} [t]_{1,+}^{-\frac{\lambda}{q}} [1/t]_{1,+}^{\frac{\mu}{q}} \left\|f_t\right\|_{L_{1,\lambda,\mu}} \\ &= t^{\alpha+\frac{n}{q}-n} [t]_{1,+}^{\lambda-\frac{\lambda}{q}} [1/t]_{1,+}^{-\mu+\frac{\mu}{q}} \|f\|_{L_{1,\lambda,\mu}} \\ &= t^{\alpha} [t]_{1,+}^{-n+\lambda+\frac{n-\lambda}{q}} [1/t]_{1,+}^{n-\mu-\frac{n-\mu}{q}} \|f\|_{L_{1,\lambda,\mu}}.\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{n-\mu}$, then by letting $t \to 0$ we have $\|I_\alpha f\|_{WL_{q,\lambda,\mu}} = 0$ for all $f \in L_{1,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $1 > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|I_\alpha f\|_{WL_{q,\lambda,\mu}} = 0$ for all $f \in L_{1,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\mu} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. □

From Theorem 2.6 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 2.7 [2] *Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha$ and $1 \le p < \frac{n-\lambda}{\alpha}$.*

1) If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{n-\lambda}{\alpha}$, then condition $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\lambda}(\mathbb{R}^n)$.

Corollary 2.8 [12, Theorem 2] *Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha$ and $1 \le p < \frac{n-\lambda}{\alpha}$.*

1) If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

3 Commutator of the Riesz potential in total Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutator of Riesz potential $[b, I_\alpha]$ in the $L_{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

Definition 3.1 We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy < \infty,$$

where $f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

The following local estimate is valid (see also [13]).

Lemma 3.2 [13, Lemma 7.4] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then the inequality*

$$\|[b, I_\alpha]f\|_{L_q(B(x,r))} \lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \log\left(e + \frac{t}{r}\right) t^{-\frac{n}{q}} \, \|f\|_{L_p(B(x,t))} \frac{dt}{t} \tag{3.1}$$

holds for all $B(x,r)$ *and for all* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$.

The following is Spanne type result for commutator of Riesz potential in total Morrey spaces.

Theorem 3.3 (Spanne's type result) *Let* $1 < p < \infty$, $0 < \lambda, \mu < n$, $0 < \alpha < \min\left\{\frac{n-\lambda}{p}, \frac{n-\mu}{p}\right\}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$.

If $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $[b, I_\alpha]f \in L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ *and*

$$\|[b, I_\alpha]f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,\lambda,\mu} \, \|b\|_* \, \|f\|_{L_{p,\lambda,\mu}}, \tag{3.2}$$

where $C_{p,\lambda,\mu}$ *depends only on* p,λ,μ *and* n.

Proof Let $1 < p < \infty$. From the inequality (3.1) we get

$$\begin{aligned}
\|[b, I_\alpha]f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \, \|[b, I_\alpha]f\|_{L_q(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \, r^{\frac{n}{q}} \int_{2r}^{\infty} \log\left(e + \frac{t}{r}\right) t^{-\frac{n}{q}} \, \|f\|_{L_p(B(x,t))} \frac{dt}{t} \\
&\lesssim \|f\|_{L_{p,\lambda,\mu}} \sup_{r>0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \, r^{-\alpha+\frac{n}{p}} \int_{r}^{\infty} \log\left(e + \frac{t}{r}\right) t^{\alpha-\frac{n}{p}} \, [t]_1^{\frac{\lambda}{p}} \, [1/t]_1^{-\frac{\mu}{p}} \frac{dt}{t} \\
&= \|f\|_{L_{p,\lambda,\mu}} \sup_{r>0} [r]_1^{-\alpha+\frac{n-\lambda}{p}} \, [1/r]_1^{\alpha-\frac{n-\mu}{p}} \int_{r}^{\infty} \log\left(e + \frac{t}{r}\right) [t]_1^{\alpha-\frac{n-\lambda}{p}} \, [1/t]_1^{-\alpha+\frac{n-\mu}{p}} \frac{dt}{t} \\
&\approx \|f\|_{L_{p,\lambda,\mu}} \int_{1}^{\infty} \log(e + t) \, [t]_1^{\alpha-\frac{n-\lambda}{p}} \, [1/t]_1^{-\alpha+\frac{n-\mu}{p}} \frac{dt}{t} \\
&= \|f\|_{L_{p,\lambda,\mu}},
\end{aligned}$$

which implies that the operator $[b, I_\alpha]$ is bounded from $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$. □

Remark 3.4 We note that in the case $\lambda = \mu$ from Theorem 3.3 we obtain the Corollary 1.13.

The following theorem is one of our main s, in which we obtain conditions that guarantee the boundedness of the commutator of Riesz potential $[b, I_\alpha]$ from the space $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{p,\lambda,\mu}(\mathbb{R}^n)$ for $b \in BMO(\mathbb{R}^n)$.

Theorem 3.5 *Let $0 < \alpha < n$, $0 \le \mu \le \lambda < n$, $1 < p < \frac{n-\lambda}{\alpha}$ and $b \in BMO(\mathbb{R}^n)$. Then the condition $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha]$ from $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}(\mathbb{R}^n)$.*

Proof Let $0 < \alpha < n$, $0 \le \mu \le \lambda < n$, $1 < p < \frac{n-\lambda}{\alpha}$ and $b \in BMO(\mathbb{R}^n)$. Let also $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ and $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{2B}}$.

$$\big|[b, I_\alpha]f_1(x)\big| \le \int_{2B} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} |f(y)| dy \lesssim r^\alpha M_b f(x). \tag{3.3}$$

For $x \in B$ we have

$$\begin{aligned}\big|[b, I_\alpha]f_2(x)\big| &\le \int_{\complement_{2B}} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} |f(y)| dy \\ &\approx \int_{\complement_{2B}} \frac{|b(x) - b(y)|}{|x_0-y|^{n-\alpha}} |f(y)| dy\end{aligned}$$

Analogously to [13, Section 7.1], for all $p \in (1, \infty)$ and $x \in B$ we get

$$\big|[b, I_\alpha]f_2(x)\big| \lesssim \|b\|_* \int_{2r}^\infty \log\Big(e + \frac{t}{r}\Big) t^{\alpha - \frac{n}{p}} \|f\|_{L_p(B(x,t))} \frac{dt}{t} \tag{3.4}$$

Then from inequalities (3.3) and (3.4) we get

$$\begin{aligned}\big|[b, I_\alpha]f(x)\big| &\lesssim r^\alpha M_b f(x) + \|b\|_* \int_{2r}^\infty \log\Big(e + \frac{t}{r}\Big) t^{\alpha-\frac{n}{p}} \|f\|_{L_p(B(x,t))} \frac{dt}{t} \\ &\le r^\alpha M_b f(x) + \|b\|_* \|f\|_{L_{p,\lambda,\mu}} \int_r^\infty \log\Big(e + \frac{t}{r}\Big) t^{\alpha-\frac{n}{p}} [t]_1^{\frac{\lambda}{p}} [1/t]_1^{-\frac{\mu}{p}} \frac{dt}{t} \\ &\le r^\alpha M_b f(x) + \|b\|_* \|f\|_{L_{p,\lambda,\mu}} r^{\alpha-\frac{n}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \int_1^\infty \log(e+t)\, t^{\alpha-\frac{n}{p}} [t]_1^{\frac{\lambda}{p}} [1/t]_1^{-\frac{\mu}{p}} \frac{dt}{t} \\ &\approx r^\alpha M_b f(x) + \|b\|_* \|f\|_{L_{p,\lambda,\mu}} r^{\alpha-\frac{n}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \\ &= r^\alpha M_b f(x) + \|b\|_* \|f\|_{L_{p,\lambda,\mu}} [r]_1^{\alpha-\frac{n-\lambda}{p}} [1/r]_1^{-\alpha+\frac{n-\mu}{p}} \\ &\le \min\Big\{r^\alpha M_b f(x) + r^{\alpha-\frac{n-\lambda}{p}} \|f\|_{L_{p,\lambda,\mu}}, r^\alpha M_b f(x), r^{\alpha-\frac{n-\mu}{p}} \|f\|_{L_{p,\lambda,\mu}}\Big\}.\end{aligned} \tag{3.5}$$

Minimizing with respect to r, at

$$r = \bigg(\frac{\|f\|_{L_{p,\lambda,\mu}}}{M_b f(x)}\bigg)^{\frac{p}{n-\lambda}} \quad \text{and} \quad r = \bigg(\frac{\|f\|_{L_{p,\lambda,\mu}}}{M_b f(x)}\bigg)^{\frac{p}{n-\mu}}$$

we have

$$\left|[b, I_\alpha] f(x)\right| \le \min\left\{\left(M_b f(x)\right)^{1-\frac{\alpha p}{n-\lambda}} \|f\|_{L_{p,\lambda,\mu}}^{\frac{\alpha p}{n-\lambda}}, \left(M_b f(x)\right)^{1-\frac{\alpha p}{n-\mu}} \|f\|_{L_{p,\lambda,\mu}}^{\frac{\alpha p}{n-\mu}}\right\}, \tag{3.6}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 1.10 and inequality (3.6), we get

$$\begin{aligned}\|[b, I_\alpha] f\|_{L_{q,\lambda,\mu}} &\lesssim \|f\|_{L_{p,\lambda,\mu}}^{1-\frac{p}{q}} \|(M_b f)^{\frac{p}{q}}\|_{L_{q,\lambda,\mu}} \\ &= \|f\|_{L_{p,\lambda,\mu}}^{1-\frac{p}{q}} \|M_b f\|_{L_{p,\lambda,\mu}}^{\frac{p}{q}} \lesssim \|f\|_{L_{p,\lambda,\mu}}.\end{aligned}$$

□

4 Modified Riesz potential in total Morrey spaces

We consider the modified Riesz potential

$$\widetilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} \left(|x-y|^{\alpha-n} - |y|^{\alpha-n} \chi_{\complement_{B(0,1)}}(y)\right) f(y) dy.$$

Note that in the limit case $\frac{n-\lambda}{\alpha} \le p \le \frac{n}{\alpha}$, assertion 1) of Theorem 2.6 does not hold. Moreover, there exists $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ such that $I_\alpha f(x) = \infty$ for all $x \in \mathbb{R}^n$. In [17, Theorem 3.2] we showed that if $\frac{n-\lambda}{\alpha} \le p \le \frac{n-\mu}{\alpha}$, then the operator M_α is bounded from $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$. However, as will be shown, statement 1) is valid for the modified Riesz potential $\widetilde{I}_\alpha$ if the space $L_\infty(\mathbb{R}^n)$ is replaced by the wider space $BMO(\mathbb{R}^n)$.

The following theorem is one of our main results, in which we obtain conditions that guarantee that the modified Riesz potential operator $\widetilde{I}_\alpha$ is bounded from the space $L_{p,\lambda,\mu}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

Theorem 4.1 *Let* $0 < \alpha < n$, $0 \le \mu \le \lambda < n$ *and* $\frac{n-\lambda}{\alpha} \le p \le \frac{n-\mu}{\alpha}$, *then the operator* $\widetilde{I}_\alpha$ *is bounded from* $L_{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $BMO(\mathbb{R}^n)$. *Moreover, if the integral* $I_\alpha f$ *exists almost everywhere for* $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, $\frac{n-\lambda}{\alpha} \le p \le \frac{n-\mu}{\alpha}$, *then* $I_\alpha f \in BMO(\mathbb{R}^n)$ *and the following inequality is valid*

$$\|I_\alpha f\|_* \le C\|f\|_{L_{p,\lambda,\mu}},$$

where $C > 0$ *is independent of* f.

Proof For given $x \in \mathbb{R}^n$, $y \in B(x,t)$ and $t > 0$ we denote

$$f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y) - f_1(y), \tag{4.1}$$

where $\chi_{B(x,2t)}$ is the characteristic function of the set $B(x, 2t)$. Then

$$\widetilde{I}_\alpha f(y) = \widetilde{I}_\alpha f_1(y) + \widetilde{I}_\alpha f_2(y) = F_1(y) + F_2(y), \tag{4.2}$$

where

$$F_1(y) = \int_{B(x,2t)} \left(|y - z|^{\alpha-n} - |x - z|^{\alpha-n} \chi_{\complement_{B(x,1)}}(z)\right) f(z)dz,$$

$$F_2(y) = \int_{\complement_{B(x,2t)}} \left(|y - z|^{\alpha-n} - |x - z|^{\alpha-n} \chi_{\complement_{B(x,1)}}(z)\right) f(z)dz.$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1 = -\int_{B(x,2t)\setminus B(x,\min\{1,2t\})} |x - z|^{\alpha-n} f(z)dz$$

is finite.

Note also that

$$\begin{aligned} F_1(y) - a_1 = &\int_{B(x,2t)} |y - z|^{\alpha-n} f(z)dz \\ &- \int_{B(x,2t)\setminus B(x,\min\{1,2t\})} |x - z|^{\alpha-n} f(z)dz \\ &+ \int_{B(x,2t)\setminus B(x,\min\{1,2t\})} |x - z|^{\alpha-n} f(z)dz \\ = &\int_{\mathbb{R}^n} |y - z|^{\alpha-n} f_1(z)dz = I_\alpha f_1(y). \end{aligned}$$

Therefore

$$|F_1(y) - a_1| = |I_\alpha f_1(y)| \le \int_{B(0,3t)} |z|^{\alpha-n} |f(y - z)|dz.$$

Then

$$\begin{aligned} &|B(x, t)|^{-1} \int_{B(x,t)} |F_1(y) - a_1|\, dy \\ &\le |B(x, t)|^{-1} \int_{B(x,t)} \left(\int_{B(0,3t)} |z|^{\alpha-n} |f(y - z)|dz\right) dy \\ &= |B(x, t)|^{-1} \int_{B(0,3t)} \left(\int_{B(x,t)} |f(y - z)|dy\right) |z|^{\alpha-n} dz \\ &\le C_1 t^{-n} [t]_1^{n-\alpha} \|f\|_{\widetilde{L}_{1,n-\alpha}} \int_{B(0,3t)} |z|^{\alpha-n} dz \\ &\le C_2 \left(t^{-1}[t]_1\right)^{n-\alpha} \|f\|_{\widetilde{L}_{1,n-\alpha}} \le C_2 \|f\|_{\widetilde{L}_{1,n-\alpha}}. \end{aligned} \tag{4.3}$$

Denote

$$a_2 = \int_{B(x,\max\{1,2t\})\setminus B(x,2t)} |x-z|^{\alpha-n} f(z)dz.$$

If $2|x-y| \leq |x-z|$, then

$$||y-z|^{\alpha-n} - |x-z|^{\alpha-n}| \leq C|x-y||x-z|^{\alpha-n-1}.$$

By the Hölder's inequality we have

$$\begin{aligned}
|F_2(y) - a_2| &\leq C|x-y| \int_{\complement_{B(x,2t)}} |f(z)||x-z|^{\alpha-n-1} dz \\
&\leq C|x-y| \sum_{j=0}^{\infty} \int_{B(x,2^{j+2}t)\setminus B(x,2^{j+1}t)} |f(z)|\,|x-z|^{\alpha-n-1}\, dz \\
&\leq C|x-y| \sum_{j=0}^{\infty} (2^{j+1}t)^{\alpha-n-1} \int_{B(x,2^{j+2}t)} |f(z)|\, dz \\
&\leq C|x-y|\, \|f\|_{L_{1,n-\alpha}} \sum_{j=0}^{\infty} (2^{j+2}t)^{\alpha-\frac{n}{p}-1} (2^{j+2}t)^{\frac{\lambda}{p}} \\
&\leq C|x-y|\, t^{\alpha-\frac{n-\lambda}{p}-1}\, \|f\|_{L_{1,n-\alpha}} = \|f\|_{L_{1,n-\alpha,n-\alpha}}. \qquad (4.4)
\end{aligned}$$

Therefore, from (4.3) and (4.4) we have

$$\sup_{x,t} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left|\widetilde{I}^{\alpha} f(y) - a_f\right| dy \lesssim \|f\|_{L_{1,n-\alpha,n-\alpha}}. \qquad (4.5)$$

Finally let $\frac{n-\lambda}{\alpha} \leq p \leq \frac{n-\mu}{\alpha}$ and $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, then from (4.5) and Lemma 1.8 we get

$$\left\|\widetilde{I}_{\alpha} f\right\|_{*} \leq 2 \sup_{x,t} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left|\widetilde{I}_{\alpha} f(y) - a_f\right| dy \leq C\|f\|_{L_{p,\lambda,\mu}}.$$

The Theorem 4.1 is proved. □

Corollary 4.2 *Let $0 < \alpha < n$, $0 \leq \lambda < n$ and $p = \frac{n}{\alpha}$, then the operator $\widetilde{I}_{\alpha}$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. Moreover, if the integral $I_{\alpha} f$ exists almost everywhere for $f \in L_{p,\lambda}(\mathbb{R}^n)$, $p = \frac{n}{\alpha}$, then $I_{\alpha} f \in BMO(\mathbb{R}^n)$ and the following inequality is valid*

$$\|I_{\alpha} f\|_{*} \leq C\|f\|_{L_{p,\lambda}},$$

where $C > 0$ is independent of f.

Corollary 4.3 *Let $0 < \alpha < n$, $0 \leq \lambda < n$ and $\frac{n-\lambda}{\alpha} \leq p \leq \frac{n}{\alpha}$, then the operator $\widetilde{I}_{\alpha}$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. Moreover, if the integral $I_{\alpha} f$ exists almost* *everywhere for* $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, $\frac{n-\lambda}{\alpha} \le p \le \frac{n}{\alpha}$, *then* $I_\alpha f \in BMO(\mathbb{R}^n)$ *and the following inequality is valid*

$$\|I_\alpha f\|_* \le C\|f\|_{\widetilde{L}_{p,\lambda}},$$

where $C > 0$ *is independent of* f.

5 Some applications

In this section, we shall apply Theorems 2.2, 2.6, 3.3 and 3.5 to several particular operators such as the Marcinkiewicz operator and fractional powers of the some analytic semigroups.

5.1 Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$, equipped with Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbb{R}^n$, has zero mean on S^{n-1}, and satisfies the condition $\Omega \in L_\infty(S^{n-1})$.

The Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

As is known, the Marcinkiewicz integral is one of the classical operators of harmonic analysis, belonging to a wide class of Littlewood-Paley g-functions and playing an important role in harmonic analysis and the theory of partial differential equations. Research into the mapping properties of the Marcinkiewicz integral and its commutators in various functional spaces is a topical issue. In 1958, Stein [25] first introduced the operator μ_Ω, which is the higher dimensional generalization of Marcinkiewicz integral in one-dimension, and showed that μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p \le 2$ and weak type (1.1), provided $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$.

For $0 \le \alpha < n$ the fractional Marcinkiewicz operator $\mu_{\Omega,\alpha}$ is defined by (see [27])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

The sublinear commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$[b, \mu_{\Omega,\alpha}](f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} [b(x) - b(y)] f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

By Minkowski inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

It is known that for $b \in BMO(\mathbb{R}^n)$ the operators $\mu_{\Omega,\alpha}$ and $[b, \mu_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ (see [27, 28]), then from Theorems 2.2, 2.6, 3.3 and 3.5 we get

Corollary 5.1 *Let $1 \le p < \infty$, $0 \le \lambda, \mu < n$, $0 < \alpha < \min\left\{\frac{n-\lambda}{p}, \frac{n-\mu}{p}\right\}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $\mu_{\Omega,\alpha}$ is bounded from $L_{p,\lambda,\mu}$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}$ for $p > 1$ and from $L_{1,\lambda,\mu}$ to $WL_{q,\lambda q,\mu q}$ for $p = 1$.*

Corollary 5.2 *Let $1 \le p < \infty$, $0 \le \mu \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{p}$, and $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. Then $\mu_{\Omega,\alpha}$ is bounded from $L_{p,\lambda,\mu}$ to $L_{q,\lambda,\mu}$ for $p > 1$ and from $L_{1,\lambda,\mu}$ to $WL_{q,\lambda,\mu}$ for $p = 1$.*

Corollary 5.3 *Let $1 < p < \infty$, $0 \le \lambda, \mu < n$, $0 < \alpha < \min\left\{\frac{n-\lambda}{p}, \frac{n-\mu}{p}\right\}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. Then $[b, \mu_{\Omega,\alpha}]$ is bounded from $L_{p,\lambda,\mu}$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}$*

Corollary 5.4 *Let $1 < p < \infty$, $0 \le \mu \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{p}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ and $b \in BMO(\mathbb{R}^n)$. Then $[b, \mu_{\Omega,\alpha}]$ is bounded from $L_{p,\lambda,\mu}$ to $L_{q,\lambda,\mu}$.*

5.2 Fractional powers of the some analytic semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{5.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, Chapter 5 in [26].

Property (5.1) is satisfied for large classes of differential operators (see, for example [5]). In [5] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorems 2.2 and 2.6 are also applicable for proving boundedness of those operators and commutators from $L_{p,\lambda,\mu}$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}$ and from $L_{p,\lambda,\mu}$ to $L_{q,\lambda,\mu}$.

Corollary 5.5 *Let condition (5.1) be satisfied. Moreover, let* $1 \le p < \infty$, $0 \le \lambda, \mu < n$, $0 < \alpha < \min\left\{\frac{n-\lambda}{p}, \frac{n-\mu}{p}\right\}$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $L^{-\alpha/2}$ *is bounded from* $L_{p,\lambda,\mu}$ *to* $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}$ *for* $p > 1$ *and from* $L_{1,\lambda,\mu}$ *to* $WL_{q,\lambda q,\mu q}$ *for* $p = 1$.

Proof Since the semigroup e^{-tL} has the kernel $p_t(x, y)$ which satisfies condition (5.1), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [9]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \lesssim \|I_\alpha(|f|)\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \lesssim \|f\|_{L_{p,\lambda,\mu}},$$

if $1 < p < q < \infty$ and

$$\|L^{-\alpha/2} f\|_{WL_{q,\lambda q,\mu q}} \lesssim \|I_\alpha(|f|)\|_{WL_{q,\lambda q,\mu q}} \lesssim \|f\|_{L_{1,\lambda,\mu}},$$

if $p = 1 < q < \infty$. □

Corollary 5.6 *Let condition (5.1) be satisfied. Moreover, let* $1 \le p < \infty$, $0 \le \mu \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{p}$, *and* $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. *Then* $L^{-\alpha/2}$ *is bounded from* $L_{p,\lambda,\mu}$ *to* $L_{q,\lambda,\mu}$ *for* $p > 1$ *and from* $L_{1,\lambda,\mu}$ *to* $WL_{q,\lambda,\mu}$ *for* $p = 1$.

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and $L^{-\alpha/2}$ is defined as follows

$$[b, L^{-\alpha/2}]f(x) = b(x)L^{-\alpha/2} f(x) - L^{-\alpha/2}(bf)(x).$$

In [9], Chanillo's result was generalized from $(-\Delta)$ to the more general operator L defined above. More precisely, they showed that if $b \in BMO(\mathbb{R}^n)$, then the commutator operator $[b, L^{-\alpha/2}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then, from theorems 3.3 and 3.5 it follows

Corollary 5.7 *Let condition (5.1) be satisfied. Moreover, let* $1 < p < \infty$, $0 \le \lambda, \mu < n$, $0 < \alpha < \min\left\{\frac{n-\lambda}{p}, \frac{n-\mu}{p}\right\}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then* $[b, L^{-\alpha/2}]$ *is bounded from* $L_{p,\lambda,\mu}$ *to* $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}$.

Corollary 5.8 *Let condition (5.1) be satisfied. Moreover, let* $1 < p < \infty$, $0 \le \mu \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{p}$, $\frac{\alpha}{n-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then* $[b, L^{-\alpha/2}]$ *is bounded from* $L_{p,\lambda,\mu}$ *to* $L_{q,\lambda,\mu}$.

6 Conclusion

In this paper, we present necessary and sufficient conditions for the boundedness of the Riesz potential I_α and the commutator of the Riesz potential $[b, I_\alpha]$ in the total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, when b belongs to the BMO spaces $BMO(\mathbb{R}^n)$. As an application, we obtain estimates for the Marcinkiewicz operator and fractional powers of some analytic semigroups in the total Morrey spaces.

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Declarations

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