

A PERSPECTIVE ON AMALGAMATED RINGS VIA SYMMETRICITY

H. KOSE, B. UNGOR, Y. KURTULMAZ, AND A. HARMANCI

ABSTRACT. In this paper, we deal with some versions of reversibility and symmetry on amalgamated rings along an ideal.

2010 AMS Mathematics Subject Classification: 16N40, 16N80, 16U99.

Key words and phrases: Reversible ring, weakly reversible ring, symmetric ring, weak symmetric ring, GWS ring, weakly symmetric ring, amalgamated ring along an ideal.

1. INTRODUCTION

In [3], if R is a local Cohen-Macaulay ring, it is given necessary and sufficient conditions for $R \bowtie I$ to be a Gorenstein ring, where I is an ideal of R . The amalgamated duplication $R \bowtie E$ of a commutative ring R along an R -module E (that it is assumed to be an ideal in some overring of R and so E is an R -submodule of the total ring of fractions K of R), introduced by D'Anna and Fontana [4], is a subring $R \bowtie E = \{(r, r + e) \mid r \in R \text{ and } e \in E\}$ of $R \times K$ (endowed with the usual componentwise operations).

Let A and B be arbitrary rings with unity and I be a proper ideal of B and $f : A \rightarrow B$ be a ring homomorphism. We consider the following subring of $A \times B$:

$$A \bowtie^f I := \{(a, f(a) + x) \mid a \in A, x \in I\}$$

called *the amalgamation of A with B along I with respect to f* . This ring is a generalization of the amalgamated duplication of a commutative ring along an ideal which is introduced and studied by D'Anna and Fontana in [4, 5].

In [5], the authors studied characterizations for $A \bowtie^f I$ to be a Noetherian ring, an integral domain, a reduced ring and they characterized those distinguished pullbacks that can be expressed as an amalgamation provided the rings are commutative. Moreover, in [4], they investigated also the prime spectrum, chain properties and Krull dimension of $A \bowtie^f I$. This paper aims at studying the transfer of the notion of reversible rings, weakly reversible rings, symmetric rings, weak symmetric rings, generalized weakly symmetric rings and weakly symmetric rings to the amalgamation of rings along ideals. In addition to the Introduction, it is divided sections

and each section is devoted to a notion related to the versions of reversibility or symmetricity.

In the sequel, $\text{nil}(A)$ will denote the set of nilpotent elements of a ring A , also by \mathbb{Z} and \mathbb{Z}_n , we denote the integers and the ring of integers modulo n for a positive integer n .

2. REVERSIBILITY OF AMALGAMATED RINGS

A ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. In [2], Cohn studied reversible rings. Further studies are done in [1], [7], [9] and [12]. In this section, we investigate the conditions on the reversibility of the rings of the form $A \bowtie^f I$. We start with some examples to illustrate the definition.

Example 2.1. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ the ideal of B and $f : A \rightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then A is a reversible ring, B is not reversible. For $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in B$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$ but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$. Also $f(A) + I$ is reversible, and

$$A \bowtie^f I = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \right\}$$

is reversible.

Theorem 2.2. Let A and B be a pair of rings, $f : A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.

- (1) If $A \bowtie^f I$ is reversible, then A is reversible.
- (2) If A and $f(A) + I$ are reversible, so is $A \bowtie^f I$.
- (3) Assume that f is injective. If $f(A) + I$ is reversible, then so are A and $A \bowtie^f I$.
- (4) Let $f^{-1}(I) = \{0\}$.
 - (a) If B is reversible, then $A \bowtie^f I$ is reversible.
 - (b) If $f(A) + I$ is reversible, then $A \bowtie^f I$ is reversible.

Proof. (1) Let $a, b \in A$ with $ab = 0$. Then $(a, f(a))(b, f(b)) = 0$ in $A \bowtie^f I$. By hypothesis, $(b, f(b))(a, f(a)) = 0$. So $ba = 0$.

(2) Let $(a, f(a) + x), (b, f(b) + y) \in A \bowtie^f I$ with $(a, f(a) + x)(b, f(b) + y) = 0$. Then $ab = 0$ and $(f(a) + x)(f(b) + y) = 0$. By hypothesis, $ba = 0$ and $(f(b) + y)(f(a) + x) = 0$. It follows that $A \bowtie^f I$ is reversible.

(3) We first show that A is reversible. Let $a, b \in A$ with $ab = 0$. Then $f(a) + 0, f(b) + 0 \in f(A) + I$ and $(f(a) + 0)(f(b) + 0) = f(ab) = 0$. By hypothesis, $(f(b) + 0)(f(a) + 0) = 0 = f(ba)$. By the injectivity of f , $ba = 0$. So A is reversible. By (2), $A \bowtie^f I$ is reversible.

(4) (a) Note that $f(A) + I$ is reversible as a subring of the reversible ring B . It is clear that $f(A) + I$ is isomorphic to $A \bowtie^f I$ by the homomorphism α defined by $\alpha(a, f(a) + x) = f(a) + x$ where $(a, f(a) + x) \in A \bowtie^f I$.

(b) Clear by the property that reversibility is preserved under isomorphism. \square

Example 2.3. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I =$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the ideal of B and $f : A \rightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$. Then A is reversible and B is not reversible. Also f is injective and

$$f(A) + I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

is reversible. By Theorem 2.2 (3), $A \bowtie^f I$ is a reversible ring.

The following example shows that the reverse implication of (1) in Theorem 2.2 need not be true in general.

Example 2.4. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$

the ideal of B and $f : A \rightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Let

$u = (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), v = (1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) \in A \bowtie^f I$. Then $uv = 0$ and $vu \neq 0$. Hence

$A \bowtie^f I$ is not reversible. Similarly, let $x = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in f(A) + I$. Then $xy = 0$ and $yx \neq 0$. Hence $f(A) + I$ is not reversible.

The following lemma is clear from definitions.

Lemma 2.5. Let A and B be a pair of rings, $f : A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.

- (1) If A and B are reversible rings, then $A \bowtie^f I$ is reversible.
- (2) Let $B = A$, $f = id_A$ and $I = (0)$. In this situation, $A \bowtie^f I$ is reversible if and only if A and B are reversible.

- (3) If $A \bowtie^f I$ is reversible, then every subring is reversible, in particular, the ideals $(0) \bowtie^f I$, $A \bowtie^f (0)$ and $I_1 \bowtie^f I$ are reversible where I_1 is any ideal of A .

Definition 2.6. We recall that, if $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β (see [5]).

Theorem 2.7. Let A and B be rings and I an ideal of B and $f : A \rightarrow B$ a homomorphism. Let $u : A \rightarrow A \times ((f(A) + I)/I)$ defined by $u(a) = (a, f(a) + I)$ and $v : A \times (f(A) + I) \rightarrow A \times ((f(A) + I)/I)$ defined by $v(a, f(b) + x) = (a, f(b) + I)$, where $a, b \in A$ and $x \in I$. Then $A \bowtie^f I$ is the pullback of the maps u and v .

Proof. Consider the following diagram with $\alpha(a, f(a) + t) = a$ and $\beta(a, f(a) + t) = (a, f(a) + t) \in A \times (f(A) + I)$ where $(a, f(a) + t) \in A \bowtie^f I$.

$$\begin{array}{ccccc}
 X & & & & \\
 \swarrow \theta & & \searrow \gamma & & \\
 & A \bowtie^f I & \xrightarrow{\alpha} & A & \\
 \delta \searrow & \downarrow \beta & & \downarrow u & \\
 & A \times (f(A) + I) & \xrightarrow{v} & A \times [(f(A) + I)/I] &
 \end{array}$$

Then $u\alpha = v\beta$. For if $(a, f(a) + t) \in A \bowtie^f I$, then $u\alpha(a, f(a) + t) = u(a) = (a, f(a) + I)$, $v\beta(a, f(a) + t) = (a, f(a) + I)$. Let X be any ring and γ and δ ring homomorphisms such that $v\delta = u\gamma$. For any $x \in X$, set $\gamma(x) = a \in A$. Then $u(a) = (a, f(a) + I)$ and $\delta(x) = (b, f(c) + i)$ for some $b, c \in A$ and $i \in I$. Hence $v\delta = u\gamma$ implies $a = b$ and $f(c) - f(a) \in I$. Define $\theta(x) = (a, f(a) + i)$. Then $\beta\theta = \delta$ and $\alpha\theta = \gamma$. On the other hand, θ is unique, for if ν satisfies $\alpha\nu = \gamma$ and $\beta\nu = \delta$, then $\nu = \theta$ since β is a monomorphism. This completes the proof. \square

Proposition 2.8. With the notation of Definition 2.6, we have:

- (1) If A is reversible and β is injective, then D is reversible.
- (2) If at least one of the following conditions holds
 - (a) A is reversible and $\text{nil}(B) \cap \text{Ker}\beta = \{0\}$,
 - (b) B is reversible and $\text{nil}(A) \cap \text{Ker}\alpha = \{0\}$,
 then D is reversible.

Proof. (1) Assume that A is reversible. Let (a, b) and (c, d) be in D with $(a, b)(c, d) = 0$, then $ac = 0$, $bd = 0$. We have $ca = 0$ and $\alpha(ca) = \alpha(c)\alpha(a) = \beta(d)\beta(b) = \beta(db) = 0$, then $db = 0$ since β is injective. So D is reversible.

(2) By the symmetry of conditions (a) and (b), it is enough to show that condition

(a) holds. Let (a, b) and (c, d) be in D with $(a, b)(c, d) = 0$. Then $ac = 0$ and $bd = 0$. Since A is reversible, we have $ca = 0$. Also, $\beta(db) = \beta(d)\beta(b) = \alpha(c)\alpha(a) = 0$, so $db \in \text{Ker}\beta$. Also $(db)^2 = dbdb = 0$, this implies that $db \in \text{nil}(B) \cap \text{Ker}\beta = \{0\}$. Therefore D is reversible. \square

3. WEAKLY REVERSIBILITY OF AMALGAMATED RINGS

Reversible rings are generalized by Liang-Gang in [11], that is, a ring R is called *weakly reversible* if for all $a, b, r \in R$ such that $ab = 0$, $Rbra$ is a nil left ideal of R (equivalently, $braR$ is a nil right ideal of R). Weakly reversible rings are also studied in [7]. Reversible rings are weakly reversible. There are weakly reversible rings that are not reversible as the following example shows.

Example 3.1. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ the ideal of B and $f : A \rightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then f is injective, $f(A) \cap I = \{0\}$ and A is weakly reversible. By [11, Proposition 2.3], B is weakly reversible. Note that $A \bowtie^f I$ has the following elements $(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix})$. It is easy to check that $A \bowtie^f I$ is weakly reversible. Now $(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})(1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) = 0$ in $A \bowtie^f I$, but $(1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix})(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \neq 0$. So $A \bowtie^f I$ is not reversible.

Theorem 3.2. *Let A and B be rings, $f : A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.*

- (1) *Assume that f is injective. If $A \bowtie^f I$ is weakly reversible and $f(A) \cap I = \{0\}$, then both A and $f(A) + I$ are weakly reversible.*
- (2) *If A and $f(A) + I$ are weakly reversible, so is $A \bowtie^f I$.*
- (3) *Assume that f is injective. If $f(A) + I$ is weakly reversible, then so are A and $A \bowtie^f I$.*
- (4) *Let $f^{-1}(I) = \{0\}$.*
 - (a) *If B is weakly reversible, then $A \bowtie^f I$ is weakly reversible.*
 - (b) *If $f(A) + I$ is weakly reversible, then $A \bowtie^f I$ is weakly reversible.*

Proof. (1) Let $a, b \in A$ with $ab = 0$. Then $(a, f(a) + 0)(b, f(b) + 0) = 0$ in $A \bowtie^f I$. By hypothesis, $(A \bowtie^f I)(b, f(b) + 0)(c, f(c) + z)(a, f(a) + 0)$ is a nil left ideal of $A \bowtie^f I$ for all $(c, f(c) + z) \in A \bowtie^f I$. Since operations are componentwise, $Abca$ is

a nil left ideal of A , and so A is weakly reversible. Let $f(a) + x, f(b) + y \in f(A) + I$ with $(f(a) + x)(f(b) + y) = 0$. Then $(a, f(a) + x)(b, f(b) + y) = 0$. By hypothesis, $(A \bowtie^f I)(b, f(b) + y)(c, f(c) + z)(a, f(a) + x)$ is a nil left ideal of $A \bowtie^f I$. Then $(f(A) + I)(f(b) + y)(f(c) + z)(f(a) + x)$ is a nil left ideal of $f(A) + I$.

(2) Let $(a, f(a) + x), (b, f(b) + y) \in A \bowtie^f I$ with $(a, f(a) + x)(b, f(b) + y) = 0$. Hence $ab = 0$ and $(f(a) + x)(f(b) + y) = 0$. Since A and $f(A) + I$ are weakly reversible, we have $Abca$ is a nil left ideal of A for all $c \in A$ and $(f(A) + I)(f(b) + y)(f(c) + z)(f(a) + x)$ is a nil left ideal of $f(A) + I$ for all $f(c) + z \in f(A) + I$. Thus there exist $m > 0$ and $n > 0$ such that $(rbca)^m = 0$ for all $r \in A$ and $[(f(r) + s)(f(b) + y)(f(c) + z)(f(a) + x)]^n = 0$. For $k = \max\{n, m\}$, $[(r, f(r) + s)(b, f(b) + y)(c, f(c) + z)(a, f(a) + x)]^k = 0$. We have $(A \bowtie^f I)(b, f(b) + y)(c, f(c) + z)(a, f(a) + x)$ is a nil left ideal of $A \bowtie^f I$ for all $(c, f(c) + z) \in A \bowtie^f I$.

(3) It is enough to show that A is weakly reversible. Let $a, b \in A$ with $ab = 0$. Since $f(A) + I$ is weakly reversible, $(f(A) + I)f(b)(f(c) + x)f(a)$ is a nil left ideal of $f(A) + I$. That is, $(f(r) + s)f(b)(f(c) + x)f(a)$ is a nilpotent element of $f(A) + I$ for all $r, c \in A$ and $s, x \in I$. In particular, for $s = x = 0$ we have $f(r)f(b)f(c)f(a)$ is nilpotent. So there exists $m > 0$ such that $f((rbca)^m) = 0$. By hypothesis, $(rbca)^m = 0$ in A . So A is weakly reversible. By (2), $A \bowtie^f I$ is also weakly reversible.

(4)(a) Let $a, b \in A$ with $ab = 0$. Then $f(a)f(b) = 0$ in B . Since the subring $f(A)$ of B is weakly reversible, we have $f(A)f(b)f(c)f(a)$ is a nil left ideal of $f(A)$ for all $c \in A$. So there exists $m > 0$ such that $f((Abca)^m) = 0 \in I$. Thus $(Abca)^m \subseteq f^{-1}(I) = \{0\}$, we have $(Abca)^m = 0$. That is, $Abca$ is a nilpotent left ideal, hence it is a nil left ideal of A . So A is weakly reversible. By (2), we have that $A \bowtie^f I$ is weakly reversible.

(b) It is clear from (a). □

There are rings A and B with an injective homomorphism $f : A \rightarrow B$ and an ideal I of B with $f(A) \cap I = 0$ but $A \bowtie^f I$ is not weakly reversible.

Example 3.3. Let $A = \mathbb{Z}_2$ and $X = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $Y = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $B = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ the ring direct sum of X and Y . Let $I = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ be the ideal of B and $f(a) = a(e_{11} + e_{22} + e_{33} + e_{44})$ for $a \in A$ where e_{ij} denotes the 4×4 matrix units in B for each i, j . Let $U = (1, e_{11} + e_{12} + e_{33} + e_{44})$, $V = (0, e_{12} + e_{22})$ and $T = (0, e_{12} + e_{21})$. Then $UV = 0$. But $VTU = (0, e_{11} + e_{12} + e_{21} + e_{22})$ is a nilpotent element. Then f is injective and $f(A) \cap I = \{0\}$. On the other hand, $(A \bowtie^f I)VTU$ is not a nil left ideal. For if $Z = (0, e_{11})$, then $ZVTU = (0, e_{11} + e_{12})$ and $(ZVTU)^2 = ZVTU$. Hence $A \bowtie^f I$ is not weakly reversible.

4. SYMMETRICITY OF AMALGAMATED RINGS

Symmetric rings are defined by Lambek in [10]. A ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. In [6], this concept is extended to the central symmetric ring, that is, if $abc = 0$ implies acb is central in R for $a, b, c \in R$. Clearly symmetric property of rings are preserved under isomorphisms and under subrings. In this section we study necessary and sufficient conditions for $A \bowtie^f I$ to be symmetric.

Theorem 4.1. *Let A and B be a pair of rings, $f : A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.*

- (1) *If $A \bowtie^f I$ is symmetric, then so is A .*
- (2) *If A and $f(A) + I$ are symmetric, then so is $A \bowtie^f I$.*
- (3) *Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B . Then $A \bowtie^f I$ is a symmetric ring if and only if $f(A) + I$ and A are symmetric rings.*
- (4) *Assume that f is injective and $f(A) + I$ is a symmetric ring. Then $A \bowtie^f I$ is a symmetric ring.*
- (5) *Assume that $f^{-1}(I) = \{0\}$. If $f(A) + I$ is a symmetric ring, then $A \bowtie^f I$ is a symmetric ring.*

Proof. (1) Assume that $A \bowtie^f I$ is symmetric. Let $a, b, c \in A$ with $abc = 0$. Then $(a, f(a))(b, f(b))(c, f(c)) = 0$ in $A \bowtie^f I$. By assumption, $(a, f(a))(c, f(c))(b, f(b)) = 0$ from which we have $acb = 0$. So A is symmetric.

(2) Assume that A and $f(A) + I$ are symmetric. Let $(a, f(a) + x), (b, f(b) + y), (c, f(c) + z) \in A \bowtie^f I$ with $(a, f(a) + x)(b, f(b) + y)(c, f(c) + z) = 0$. Then $abc = 0$ and $(f(a) + x)(f(b) + y)(f(c) + z) = 0$. By assumption, $acb = 0$ and $(f(a) + x)(f(c) + z)(f(b) + y) = 0$. Hence $(a, f(a) + x)(c, f(c) + z)(b, f(b) + y) = 0$. So $A \bowtie^f I$ is symmetric.

(3) It is enough to show that if $A \bowtie^f I$ is symmetric, then $f(A) + I$ is symmetric. For if $s \in S \cap I$ and $f(a) + x, f(b) + y, f(c) + z \in f(A) + I$ such that $(f(a) + x)(f(b) + y)(f(c) + z) = 0$, then $(0, s(f(a) + x))(0, s(f(b) + y))(0, s(f(c) + z)) = 0$. By hypothesis, $(0, s(f(a) + x))(0, s(f(c) + z))(0, s(f(b) + y)) = 0$. Hence $s^3((f(a) + x)(f(c) + z)(f(b) + y)) = 0$. By the regularity of s , we have $(f(a) + x)(f(c) + z)(f(b) + y) = 0$.

(4) Assume that f is injective and $f(A) + I$ is symmetric. Let $a, b, c \in A$ with $abc = 0$. Hence $f(a)f(b)f(c) = 0$ in $f(A) + I$. Since $f(A) + I$ is symmetric, $f(a)f(c)f(b) = 0$. Then $f(acb) = 0$. By hypothesis, $acb = 0$ in A . So A is a symmetric ring. By (2), we have that $A \bowtie^f I$ is symmetric.

(5) Assume that $f^{-1}(I) = \{0\}$ and $f(A) + I$ is a symmetric ring. It is enough to show

that A is symmetric. Let $abc = 0$ for $a, b, c \in A$. Then $f(a)f(b)f(c) = 0$ in $f(A) + I$. Since the subring $f(A)$ of $f(A) + I$ is symmetric, we have $f(a)f(c)f(b) = 0 \in I$. Thus $acb \in f^{-1}(I) = \{0\}$. Hence $acb = 0$. That is, A is symmetric. By (2), $A \bowtie^f I$ is symmetric. \square

Example 4.2. Let x, y and z be indeterminates and consider the set

$$R = \{a_0 + a_1x + a_2y + a_3z \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$$

with componentwise addition and defining multiplication

$$(a_0 + a_1x + a_2y + a_3z)(b_0 + b_1x + b_2y + b_3z) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_2b_0)y + (a_0b_3 + a_3b_0 + a_1b_2)z.$$

Then R is a ring with identity. From this multiplication all products are zero except that $xy = z$ and that 1 acts as an identity. Let $A = B = R$. For every ring homomorphism $f: A \rightarrow B$ and every proper ideal I of B , $A \bowtie^f I$ is not symmetric because A is not symmetric [6, Example 2.2].

5. WEAK SYMMETRICITY OF AMALGAMATED RINGS

In this section we study weak symmetric rings. In [13], Ouyang and Chen discussed weak symmetric rings. A ring R is called *weak symmetric* if $abc \in \text{nil}(R)$ implies $acb \in \text{nil}(R)$ for all $a, b, c \in R$. In [13], it is proved that all symmetric rings are weak symmetric. Clearly weak symmetric property of rings are preserved under isomorphisms and under subrings. In this section we study necessary and sufficient conditions for $A \bowtie^f I$ to be weak symmetric.

Theorem 5.1. Let A and B be a pair of rings, $f: A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.

- (1) If $A \bowtie^f I$ is weak symmetric, then so is A .
- (2) If A and $f(A) + I$ are weak symmetric, then so is $A \bowtie^f I$.
- (3) Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B . Then $A \bowtie^f I$ is a weak symmetric ring if and only if $f(A) + I$ and A are weak symmetric rings.
- (4) Assume that $f^{-1}(I) \subseteq \text{nil}(A)$. If $f(A) + I$ is a weak symmetric ring, then $A \bowtie^f I$ is a weak symmetric ring.
- (5) If $f(A) + I$ is a weak symmetric ring and f is injective, then A and I are weak symmetric.

Proof. (1) Assume that $A \bowtie^f I$ is weak symmetric. Let $a, b, c \in A$ such that abc is nilpotent. Then $(a, f(a))(b, f(b))(c, f(c)) = (abc, f(abc))$ is nilpotent in $A \bowtie^f I$. By assumption $(a, f(a))(c, f(c))(b, f(b)) = (acb, f(acb))$ is nilpotent. So acb is nilpotent. It follows that A is weak symmetric.

(2) Assume that A and $f(A) + I$ are weak symmetric. Let $(a, f(a) + x)(b, f(b) + y)(c, f(c) + z)$ be nilpotent in $A \bowtie^f I$. Then $abc \in A$ and $(f(a) + x)(f(b) + y)(f(c) + z) \in f(A) + I$ are nilpotent. By assumption, acb and $(f(a) + x)(f(c) + z)(f(b) + y)$ are nilpotent respectively, in A and $f(A) + I$. Hence $(a, f(a) + x)(c, f(c) + z)(b, f(b) + y)$ is nilpotent in $A \bowtie^f I$ and so $A \bowtie^f I$ is weak symmetric.

(3) Necessity. By (1), A is weak symmetric. To prove $f(A) + I$ is weak symmetric, let $f(a) + x, f(b) + y, f(c) + z \in f(A) + I$ with $(f(a) + x)(f(b) + y)(f(c) + z)$ nilpotent and $s \in I \cap S$. Then

$$(0, f(0) + s(f(a) + x))(0, f(0) + s(f(b) + y))(0, f(0) + s(f(c) + z))$$

is nilpotent in $A \bowtie^f I$. By hypothesis,

$$(0, f(0) + s(f(a) + x))(0, f(0) + s(f(c) + z))(0, f(0) + s(f(b) + y))$$

is nilpotent in $A \bowtie^f I$. Then $s^3(f(a) + x)(f(c) + z)(f(b) + y)$ is nilpotent and $(f(a) + x)(f(c) + z)(f(b) + y)$ is nilpotent since s is central regular. Hence $f(A) + I$ is weak symmetric.

Sufficiency. It is clear from (2).

(4) Assume that $f^{-1}(I) \subseteq \text{nil}(A)$. It is sufficient to prove that A is weak symmetric. Let $a, b, c \in A$ with $abc \in \text{nil}(A)$. So there exists $m > 0$ such that $(abc)^m = 0$. Hence $f(a)f(b)f(c)$ is a nilpotent element of $f(A) + I$. Since $f(A) + I$ is weak symmetric, $f(a)f(c)f(b)$ is nilpotent. So there exists $n > 0$ such that $f(acb)^n = 0 \in I$. Thus $(acb)^n \in f^{-1}(I)$. By assumption $(acb)^n \in \text{nil}(A)$. So acb is a nilpotent element of A . By (2), we have that $A \bowtie^f I$ is weak symmetric.

(5) Let $a, b, c \in A$ with abc nilpotent. So $f(a)f(b)f(c)$ is nilpotent in $f(A) + I$. Since $f(A) + I$ is weak symmetric, $f(a)f(c)f(b)$ is nilpotent. So there exists $m > 0$ such that $(f(a)f(c)f(b))^m = 0 = f((acb)^m)$. By hypothesis $(acb)^m = 0$. We have that A is weak symmetric. Since weak symmetric property of rings are preserved under subrings, I is weak symmetric. This completes the proof. \square

Example 5.2. Let R be a domain and $T_3(R)$ the ring of 3×3 upper triangular

matrices. Consider the ideal $I = \begin{bmatrix} R & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of $T_3(R)$. Let $A = T_3(R)$ and

$B = T_3(R)/I$ and $f: A \rightarrow B$ be the natural projection. By [13, Example 2.2.], A is weak symmetric. Since B is isomorphic to the ring of 2×2 upper triangular matrices, B is also weak symmetric by [13, Proposition 2.3]. Being $f(A) + I = B$ and Theorem 5.1(2) imply $A \bowtie^f I$ is weak symmetric.

6. GENERALIZED WEAKLY SYMMETRICITY OF AMALGAMATED RINGS

In [14], generalized weakly symmetric rings (or GWS, for short) are discussed. A ring R is called *GWS* if $abc = 0$ implies bac is nilpotent for $a, b, c \in R$. Clearly, $abc = 0$ implies bac is nilpotent for $a, b, c \in R$ if and only if $abc = 0$ implies acb is nilpotent for $a, b, c \in R$ if and only if $abc = 0$ implies $Racrb$ is a nil left ideal of R for any $r \in R$ if and only if $abc = 0$ implies $brcaR$ is a nil right ideal of R for any $r \in R$. The class of GWS rings is a proper generalization of that of central symmetric rings.

Theorem 6.1. *Let A and B be a pair of rings, $f : A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.*

- (1) *If $A \bowtie^f I$ is GWS, then A is GWS.*
- (2) *If A and $f(A) + I$ are GWS, so is $A \bowtie^f I$.*
- (3) *Assume that f is injective. If $f(A) + I$ is GWS, then so are A and $A \bowtie^f I$.*
- (4) *Let $f^{-1}(I) = \{0\}$.*
 - (a) *If B is GWS, then $A \bowtie^f I$ is GWS.*
 - (b) *If $f(A) + I$ is GWS, then $A \bowtie^f I$ is GWS.*

Proof. (1) Assume that $A \bowtie^f I$ is GWS. Let $a, b, c \in A$ with $abc = 0$. Then $(a, f(a) + 0)(b, f(b) + 0)(c, f(c) + 0) = 0$. By assumption, $(a, f(a) + 0)(c, f(c) + 0)(b, f(b) + 0)$ is nilpotent. Then acb is nilpotent.

(2) Suppose that A and $f(A) + I$ are GWS and $(a, f(a) + x)(b, f(b) + y)(c, f(c) + z) = 0$ for some $(a, f(a) + x), (b, f(b) + y), (c, f(c) + z) \in A \bowtie^f I$. Then $abc = 0$ and $(f(a) + x)(f(b) + y)(f(c) + z) = 0$. By supposition, acb and $(f(a) + x)(f(c) + z)(f(b) + y)$ are nilpotent with $(acb)^n = 0$ and $((f(a) + x)(f(c) + z)(f(b) + y))^t = 0$ for some positive integers n and t . Let $m = \max\{n, t\}$. Then $((a, f(a) + x)(c, f(c) + z)(b, f(b) + y))^m = 0$. Hence $(a, f(a) + x)(c, f(c) + z)(b, f(b) + y)$ is nilpotent, and so $A \bowtie^f I$ is GWS.

(3) Let $a, b, c \in A$ with $abc = 0$. Then $(f(a) + 0)(f(b) + 0)(f(c) + 0) = 0$. By hypothesis, $(f(a) + 0)(f(c) + 0)(f(b) + 0)$ is nilpotent and so acb is nilpotent since f is injective. Then A is GWS. By (2), $A \bowtie^f I$ is GWS.

(4) Let $f^{-1}(I) = \{0\}$. (a) Assume that B is GWS, then $f(A) + I$ is GWS since GWS property is preserved under subrings. Let $a, b, c \in A$ with $abc = 0$. Then $f(a)f(b)f(c) = f(abc)$ is zero. Hence $f(A) + I$ being GWS implies that $f(acb)$ is nilpotent. So there exists $n > 0$ such that $f(acb)^n = 0 \in I$. Thus $(acb)^n \in f^{-1}(I)$. Therefore $(acb)^n = 0$ and so A is GWS. By (2), $A \bowtie^f I$ is GWS.

(b) It is clear from (4)(a). □

Example 6.2. Let F be a field and consider the ring $R = \begin{bmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{bmatrix}$ and the

ideal $I = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & F \end{bmatrix}$. Let $A = R$ and $B = R/I$ and f be the natural projection

$\pi: A \rightarrow B$. By [14, Theorem 2.7] A is GWS. Since B is isomorphic to the ring of 2×2 upper triangular matrices over F , B is GWS. Note that $f(A) + I = B$. Then $A \bowtie^f I$ is GWS by Theorem 6.1(2).

7. WEAKLY SYMMETRICITY OF AMALGAMATED RINGS

The present authors discussed weakly symmetric rings in [8]. A ring R is *weakly symmetric* if for all $a, b, c, r \in R$, abc is nilpotent implies $Racrb$ is a nil left ideal of R (equivalently, $acrbR$ is a nil right ideal of R). Every symmetric ring is reversible, similarly every weakly symmetric ring is weakly reversible. In fact, if R is a weakly symmetric ring and $ab = 0$, then $1ab = 0$ and so $R1bra$ is a nil left ideal for all $r \in R$. Hence $Rbra$ is a nil left ideal of R .

Theorem 7.1. Let A and B be rings, $f: A \rightarrow B$ a ring homomorphism and I a proper ideal of B . Then the following hold.

- (1) If $A \bowtie^f I$ is weakly symmetric, then A is weakly symmetric.
- (2) If A and $f(A) + I$ are weakly symmetric, so is $A \bowtie^f I$.
- (3) Assume that f is injective. If $f(A) + I$ is weakly symmetric, then so are A and $A \bowtie^f I$.
- (4) Let $f^{-1}(I) = \{0\}$.
 - (a) If B is weakly symmetric, then $A \bowtie^f I$ is weakly symmetric.
 - (b) If $f(A) + I$ is weakly symmetric, then $A \bowtie^f I$ is weakly symmetric.

Proof. (1) Assume that $A \bowtie^f I$ is weakly symmetric. Let $a, b, c \in R$. If abc is nilpotent, then $(a, f(a) + 0)(b, f(b) + 0)(c, f(c) + 0)$ is nilpotent in $A \bowtie^f I$. So $(A \bowtie^f I)(a, f(a) + 0)(c, f(c) + 0)(x, f(x) + t)(b, f(b) + 0)$ is a nil left ideal of $A \bowtie^f I$ for all $(x, f(x) + t) \in A \bowtie^f I$. Then $Aacxb$ is a nil left ideal for all $x \in A$.

(2) Suppose that A and $f(A) + I$ are weakly symmetric. Let $(a, f(a) + x), (b, f(b) + y), (c, f(c) + z) \in A \bowtie^f I$. If $(a, f(a) + x)(b, f(b) + y)(c, f(c) + z)$ is nilpotent, then abc and $(f(a) + x)(f(b) + y)(f(c) + z)$ are nilpotent in A and $f(A) + I$, respectively. Hence $Aacxb$ is a nil left ideal of A for all $x \in A$ and $(f(A) + I)(f(a) + x)(f(c) + z)(f(d) + v)(f(b) + y)$ is a nil left ideal of $f(A) + I$ for all $f(d) + v \in f(A) + I$. Thus $(A \bowtie^f I)(a, f(a) + x)(c, f(c) + z)(d, f(d) + v)(b, f(b) + y)$ is a nil left ideal of $A \bowtie^f I$.

(3) Assume that f is injective and $f(A) + I$ is weakly symmetric. Let $a, b, c \in A$. If abc is nilpotent, then $(f(a) + 0)(f(b) + 0)(f(c) + 0)$ is nilpotent in $f(A) + I$. By assumption, $(f(A) + I)(f(a) + 0)(f(c) + 0)(f(d) + z)(f(b) + 0)$ is a nil left ideal of $f(A) + I$ for all $f(d) + z \in f(A) + I$. It implies that $f(Aacdb)$ is a nil subset of $f(A) + I$. Since f is injective, $Aacdb$ is a nil left ideal of A for all $d \in A$. Since A and $f(A) + I$ are weakly symmetric, by (2), $A \bowtie^f I$ is weakly symmetric.

(4) (a) Suppose that $f^{-1}(I) = \{0\}$ and B is weakly symmetric. Let $(a, f(a) + x)$, $(b, f(b) + y)$, $(c, f(c) + z) \in A \bowtie^f I$. If $(a, f(a) + x)(b, f(b) + y)(c, f(c) + z)$ is nilpotent, then abc and $(f(a) + x)(f(b) + y)(f(c) + z)$ are nilpotent in A and $f(A) + I$, respectively. The ring B being weakly symmetric, $Bf(a)f(c)xf(b)$ is a nil left ideal of B for all $x \in B$. For any $t, d \in A$, $f(tacdb)$ is nilpotent, say $f((tacdb)^n) = 0$ for some positive integer n . Then $(tacdb)^n \in f^{-1}(I)$. Hence $(tacdb)^n = 0$. Thus $Aacdb$ is a nil left ideal of A . On the other hand, any subring of a weakly symmetric ring is weakly symmetric. Therefore $f(A) + I$ is weakly symmetric. By (2), $A \bowtie^f I$ is weakly symmetric.

(b) Suppose that $f^{-1}(I) = \{0\}$ and $f(A) + I$ is weakly symmetric. Let $(a, f(a) + x)$, $(b, f(b) + y)$, $(c, f(c) + z) \in A \bowtie^f I$. Assume that $(a, f(a) + x)(b, f(b) + y)(c, f(c) + z)$ is nilpotent in $A \bowtie^f I$. Then abc , $f(a)f(b)f(c)$ and $(f(a) + x)(f(b) + y)(f(c) + z)$ are nilpotent. Hence $f(A)f(a)f(c)f(d)f(b)$ is a nil left ideal and $(f(A) + I)(f(a) + x)(f(c) + z)(f(d) + t)(f(b) + y)$ is a nil left ideal of $f(A) + I$ for all $f(d) + t \in f(A) + I$. It follows that $(A \bowtie^f I)(a, f(a) + x)(c, f(c) + z)(d, f(d) + t)(b, f(b) + y)$ is a nil left ideal of $A \bowtie^f I$ for all $(d, f(d) + t) \in A \bowtie^f I$. \square

Example 7.2. Let F be a field and consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and the ideal

$I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. Let $A = B = R$ and $f: A \rightarrow B$ be a homomorphism defined by $f\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}$. Note that $f(A) + I = B$. It is easy to check that A is weakly symmetric. Therefore $A \bowtie^f I$ is weakly symmetric by Theorem 7.1(2).

REFERENCES

- [1] D. D. Anderson and V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra, 27(6)(1999), 2847-2852.
- [2] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc., 31(6)(1999), 641-648.
- [3] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra, 306(2)(2006), 507-519.
- [4] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl., 6(3)(2007), 443-459.
- [5] M. D'Anna, C. A. Finocchiaro and M. Fontana *Amalgamated algebras along an ideal*, Commutative Algebra and its Applications, Walter de Gruyter, Berlin, (2009), 155-172.

- [6] G. Kafkas, B. Ungor, S. Halicioglu and A. Harmanci, *Generalized symmetric rings*, Algebra Discrete Math., 12(2)(2011), 72-84.
- [7] H. Kose, B. Ungor, S. Halicioglu and A. Harmanci, *A generalization of reversible rings*, Iran. J. Sci. Technol. Trans. A Sci., 38(1)(2014), 43-48.
- [8] H. Kose, B. Ungor, Y. Kurtulmaz and A. Harmanci, *A nil approach to symmetricity of rings*, submitted for publication.
- [9] J. Krempa and D. Niewieczerzal, *Rings in which annihilators are ideals and their application to semi-group rings*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 25(9)(1977), 851-856.
- [10] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull., 14(1971), 359-368.
- [11] Z. Liang and Y. Gang, *On weakly reversible rings*, Acta Math. Univ. Comenian., 76(2)(2007), 189-192.
- [12] G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra, 174(3)(2002), 311-318.
- [13] L. Ouyang and H. Chen, *On weak symmetric rings*, Comm. Algebra, 38(2)(2010), 697-713.
- [14] J. Wei, *Generalized weakly symmetric rings*, J. Pure Appl. Algebra, 218(9)(2014), 1594-1603.

HANDAN KOSE, DEPARTMENT OF MATHEMATICS, AHI EVRAN UNIVERSITY, KIRSEHIR, TURKEY
Email address: `handan.kose@ahievran.edu.tr`

BURCU UNGOR, DEPARTMENT OF MATHEMATICS, ANKARA UNIVERSITY, ANKARA, TURKEY
Email address: `bungor@science.ankara.edu.tr`

YOSUM KURTULMAZ, DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, ANKARA, TURKEY
Email address: `yosum@fen.bilkent.edu.tr`

ABDULLAH HARMANCI, DEPARTMENT OF MATHEMATICS, HACETTEPE UNIVERSITY, ANKARA, TURKEY
Email address: `harmanci@hacettepe.edu.tr`